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The analytic value of a 3-loop sunrise graph in a particular kinematical configuration.

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Abstract

We consider the scalar integral associated to the 3-loop sunrise graph with a massless line, two massive lines of equal mass M, a fourth line of mass equal to Mx, and the external invariant timelike and equal to the square of the fourth mass. We write the differential equation in x satisfied by the integral, expand it in the continuous dimension d around $d=4$ and solve the system of the resulting chained differential equations in closed analytic form, expressing the solutions in terms of Harmonic Polylogarithms. As a byproduct, we give the limiting values of the coefficients of the $(d-4)$ expansion at $x=1$ and $x=0$.

1 Introduction

This paper is devoted to the analytic evaluation of the scalar integral associated to the 3-loop sunrise graph with a massless line, two massive lines of equal mass M, a fourth line of mass m, with $m \neq M$, and the external invariant timelike and equal to m^2, as depicted in Fig. 1. We use the same method as in [1], which

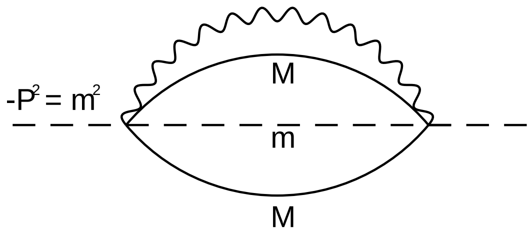

Figure 1: The considered 3-loop sunrise graph.

deals however with a 2-loop case. We start by writing the differential equation in $x = m/M$ satisfied by the considered integral (which turns out to be a third order differential equation), expand it in $(d-4)$, where d is the continuous dimension which regularizes the integral, and solve the resulting system of chained differential equations for the coefficients of the expansion in $(d-4)$ by repeated use of the variation of constants formula by Euler. The approach allows to obtain the coefficients of the expansion of the integral in $(d-4)$ up to virtually any order in $(d-4)$, expressing all of them in terms of Harmonic Polylogarithms (HPL's [2]) of argument x. At $x = 0, 1$ we recover the two quantities J_{13} and J_{11} introduced in [3] for the analytic evaluation of the $(g-2)$ of the electron at 3 loops in QED. J_{13}, which corresponds to the vacuum graph of Fig.2, and J_{11}, which is the on-mass-shell value of the graph of Fig.1 at $x = 1$, where evaluated analytically in [3] up to order $(d-4)^2$ and $(d-4)^3$ respectively. As a byproduct of the analytic result established for arbitrary x, by taking its limiting values at $x = 0$ and $x = 1$ we obtain two more terms of the expansion in $(d-4)$ of J_{13} (the analytic value of J_{13}, exact in d, can be easily obtained by direct integration of the corresponding loop integral, so that this result can be seen as a check of the approach) and of J_{11} (a result not yet known in the literature). Those higher order terms in the $(d-4)$ expansion are expected to be needed in the analytic evaluation of 4-loop QED static quantities (such as $(g-2)$ of the electron, charge slope and renormalization counterterms).

2 The differential equation

Our aim is to evaluate the scalar integral

$$F(d, M^2, m^2, P^2 = -m^2) = \frac{1}{(2\pi)^{3(d-2)}} \int \frac{d^d k_1 \; d^d k_2 \; d^d k_3}{k_1^2 (k_2^2 + M^2)(k_3^2 + M^2)[(P - k_1 - k_2 - k_3)^2 + m^2]} \; . \qquad (2.1)$$

In terms of the dimensionless variable $x = m/M$ and putting $P = Mp$ one can define the function $\Phi(d, x)$ as

$$F(d, M^2, m^2, P^2 = -m^2) = M^{(3d-8)} \; C^3(d) \; \Phi(d, x) \; , \qquad (2.2)$$

where $C(d) = (4\pi)^{\frac{4-d}{2}} \Gamma(3 - d/2)$ is an overall normalization factor, with the limiting value $C(4) = 1$ at $d = 4$. The previous equation can also be written as

$$\Phi(d, x) = \frac{C^{-3}(d)}{(2\pi)^{3(d-2)}} \int \frac{d^d k_1 \; d^d k_2 \; d^d k_3}{k_1^2 (k_2^2 + 1)(k_3^2 + 1)[(p - k_1 - k_2 - k_3)^2 + x^2]} \; , \quad (p^2 = -x^2) \; . \qquad (2.3)$$

Following [1], for evaluating Eq.(2.3) we will first establish the differential equation in x satisfied by $\Phi(d,x)$. For writing the differential equation, we have to find the *Master Integrals* (MI's) of the graph of Fig.1 . That is done by solving the *Integration by Parts Identities* (IBP-id's) [4], for the integrals associated with the graph of Fig. 1. We generated an initial system of 492 equations involving 449 unknown associated integrals. By using the symbolical solutions [5] of the IBP-id's it becomes a system of 205 equations in 76 unknown integrals. By solving explicitly that last system, we were left with four MI's only, which we take to be the fully scalar integral of Eq.(2.1) and three more integrals, having as integrand the same denominator as in Eq.(2.1) and numerators respectively equal to the scalar products $(k_1 \cdot k_2), (k_2 \cdot k_3)$ and $(P \cdot k_2)$. Those four MI's satisfy a system of four 1^{st} Order Differential Equations in the x variable. We rewrite them in terms of a single higher order equation for $\Phi(d,x)$. In so doing, it turns out that the fourth MI decouples, and $\Phi(d,x)$ is found to satisfy the following 3^{rd} Order Differential Equation in x, exact in d,

$$
\begin{aligned}
& x^2(1-x^2)\frac{d^3}{dx^3}\Phi(d,x) + \Big\{ \big[2-(d-4)\big] + \big[2+5(d-4)\big]x^2 \Big\} x \frac{d^2}{dx^2}\Phi(d,x) \\
& + \Big\{ \big[-6-8(d-4)-2(d-4)^2\big] + \big[2-4(d-4)-6(d-4)^2\big]x^2 \Big\} \frac{d}{dx}\Phi(d,x) \\
& \qquad - \big[8+14(d-4)+6(d-4)^2\big]x\ \Phi(d,x) = \frac{x\ x^{(d-4)}}{(d-4)^3} \ . \qquad (2.4)
\end{aligned}
$$

Let us note that the above equation looks slightly simpler when written in terms of x^2; but as x is the proper variable for expressing the final results, we use x since the very beginning.

3 The $x \to 0$ behaviour.

For studying the $x \to 0$ behaviour, we can try to solve Eq. (2.4) with the ansatz

$$
\Phi(d,x) = \sum_i x^{\alpha_i} \left(\sum_{n=0}^{\infty} a_n^{(i)}(d)\ x^n \right) \ . \qquad (3.1)
$$

Four possible values for the exponents α_i are found:

$$
\begin{aligned}
\alpha_1 &= 0 \ , \\
\alpha_2 &= -(d-2) \ , \\
\alpha_3 &= (2d-5) \ , \\
\alpha_4 &= (d-2) \ , \qquad (3.2)
\end{aligned}
$$

where the last value is forced by the inhomogeneous term, while the first three behaviours are those of the solutions of the homogeneous part of the equation. By inspection of Eq.(2.3), we see that the aimed $\Phi(d,x)$ is finite in the $x \to 0$ limit when $d > 2$. That is true even when the continuous parameter d is in the range $2 < d < 2.5$; as in that range of d the exponents α_2, α_3 give terms which diverge for $x \to 0$, the corresponding behaviours are ruled out. In the $d \to 4$ limit, that implies the absence of terms which for $x \to 0$ diverge as $1/x^2$ (corresponding to α_2) or behave as odd powers of x (corresponding to α_3). That gives two conditions which can be used for determining two of the three arbitrary constants present in the solution of Eq.(2.4). Further, only even powers of x are found to appear inside the brackets of Eq.(3.1), so that the actual expansion of $\Phi(d,x)$ for $x \to 0$ has the form

$$
\begin{aligned}
\Phi(d,x) &= \sum_{k=0}^{\infty} a_{2k}(d)x^{2k} \\
&+ x^{(d-2)} \sum_{k=0}^{\infty} b_{2k}(d)x^{2k} \ . \qquad (3.3)
\end{aligned}
$$

The equation determines the coefficients $a_{2k}(d)$ in terms of the first coefficient $a_0(d)$ (which must then fixed by some different method), while all the coefficients $b_{2k}(d)$ are fully fixed by the inhomogeneous term of the equation. Explicitly, one finds

$$\begin{aligned} a_2(d) &= -\frac{(3d-8)(d-3)}{d\,(2d-7)}\,a_0(d)\ , \\ a_4(d) &= \frac{3}{2}\,\frac{(3d-10)(3d-8)(d-3)(d-4)}{d\,(d+2)(2d-7)(2d-9)}\,a_0(d)\ , \\ b_0(d) &= -\frac{1}{2}\,\frac{1}{(d-2)^2(d-3)(d-4)^3}\ , \\ b_2(d) &= \frac{1}{2}\,\frac{1}{d\,(d-2)^2(d-4)^2(d-5)}\ , \end{aligned} \tag{3.4}$$

etc.

4 The $x \to 1$ behaviour.

A similar discussion applies to the $x \to 1$ behaviour. When solving Eq. (2.4) with the ansatz

$$\Phi(d,x) = \sum_i (1-x)^{\beta_i} \left(\sum_{n=0}^{\infty} c_n^{(i)}(d)\ (1-x)^n \right)\ ,$$

one finds for the exponents β_i the values

$$\begin{aligned} \beta_1 &= 0\ , \\ \beta_2 &= 1\ , \\ \beta_3 &= 2(d-2)\ , \\ \beta_4 &= 2\ , \end{aligned} \tag{4.1}$$

the first three due to the homogeneous part of the equation, the fourth to the inhomogeneous term. By inspection of Eq.(2.3), one finds that the integral is regular at $x = 1$, so that only the behaviour corresponding to β_3 is ruled out. When expanded in powers of $(d-4)$ in the $d \to 4$ limit, it would generate for $x \to 1$ terms in $\ln(1-x)$, which are therefore forbidden. That information, coming from the study of the behaviour at $x = 1$, provides with a further condition for the determination of the three integration constants.

5 The expansion in $(d-4)$.

The 3-loop integral $\Phi(d,x)$, Eq.(2.3), develops a triple pole in $(d-4)$ when $d \to 4$, so that its Laurent's expansion reads

$$\Phi(x,d) = \sum_{n=-3}^{\infty} (d-4)^n\ \Phi^{(n)}(x)\ . \tag{5.1}$$

By inserting it into Eq.(2.4), the coefficients $\Phi^{(n)}(x)$ are found to satisfy the following system of chained differential equations

$$\left[\frac{d^3}{dx^3} + 2\left(\frac{1}{x} + \frac{1}{1-x} - \frac{1}{1+x}\right)\frac{d^2}{dx^2} - 2\left(\frac{3}{x^2} + \frac{1}{1-x} + \frac{1}{1+x}\right)\frac{d}{dx} - 4\left(\frac{2}{x} + \frac{1}{1-x} - \frac{1}{1+x}\right)\right] \Phi^{(n)}(x)$$

$$
\begin{aligned}
= & \left(\frac{1}{x}-\frac{2}{1-x}+\frac{2}{1+x}\right)\frac{d^2}{dx^2}\Phi^{(n-1)}(x) \\
& +2\left(\frac{4}{x^2}+\frac{3}{1-x}+\frac{3}{1+x}\right)\frac{d}{dx}\Phi^{(n-1)}(x)+7\left(\frac{2}{x}+\frac{1}{1-x}-\frac{1}{1+x}\right)\Phi^{(n-1)}(x) \\
& +2\left(\frac{1}{x^2}+\frac{2}{1-x}+\frac{2}{1+x}\right)\frac{d}{dx}\Phi^{(n-2)}(x)+3\left(\frac{2}{x}+\frac{1}{1-x}-\frac{1}{1+x}\right)\Phi^{(n-2)}(x) \\
& +\frac{1}{2}\left(\frac{2}{x}+\frac{1}{1-x}-\frac{1}{1+x}\right)\frac{1}{(n+3)!}\ln^{(n+3)}(x) \quad , \qquad (5.2)
\end{aligned}
$$

where $\Phi^{(n)}(x)=0$ when $n<-3$. Note that in the equation for $\Phi^{(-3)}(x)$ the inhomogeneous term is just the proper term of the expansion in $(d-4)$ of the inhomogeneous term of Eq.(2.4), which is known (as matter of fact, a rational factor), while in the equation for $\Phi^{(-2)}(x)$, $\Phi^{(-3)}(x)$ appears in the *r.h.s.* as a contribution to the inhomogeneous term and so on; given such a chained structure of the equations, it is natural to solve them recursively, starting from the equation for $\Phi^{(-3)}(x)$ and going up to the successive coefficients of the expansion in $(d-4)$ of $\Phi(d,x)$. Summarizing, all the equations (5.2) have the same form,

$$
\left[\frac{d^3}{dx^3}+2\left(\frac{1}{x}+\frac{1}{1-x}-\frac{1}{1+x}\right)\frac{d^2}{dx^2}-2\left(\frac{3}{x^2}+\frac{1}{1-x}+\frac{1}{1+x}\right)\frac{d}{dx}-4\left(\frac{2}{x}+\frac{1}{1-x}-\frac{1}{1+x}\right)\right]\Phi^{(n)}(x) = K^{(n)}(x) \ , \qquad (5.3)
$$

where $K^{(n)}(x)$ stands for the inhomogeneous term, which can be taken as known when solving the equations starting from $n=-3$ and continuing in order of growing n.

6 The solution.

The homogenous equation associated to all the equations corresponding to different values of n is always the same, namely

$$
\left[\frac{d^3}{dx^3}+2\left(\frac{1}{x}+\frac{1}{1-x}-\frac{1}{1+x}\right)\frac{d^2}{dx^2}-2\left(\frac{3}{x^2}+\frac{1}{1-x}+\frac{1}{1+x}\right)\frac{d}{dx}-4\left(\frac{2}{x}+\frac{1}{1-x}-\frac{1}{1+x}\right)\right]\phi(x)=0 \ , \qquad (6.1)
$$

so that all the Eq.s(5.3) can be solved by Euler's method of the variation of the constants. In this case the general solution can be written as

$$
\begin{aligned}
\Phi^{(n)}(x) = & +\phi_1(x)\left[\Phi_1^{(n)}+\int^x dy\frac{\phi_2(y)\phi_3'(y)-\phi_2'(y)\phi_3(y)}{W(y)}K^{(n)}(y)\right] \\
& +\phi_2(x)\left[\Phi_2^{(n)}-\int^x dy\frac{\phi_1(y)\phi_3'(y)-\phi_1'(y)\phi_3(y)}{W(y)}K^{(n)}(y)\right] \\
& +\phi_3(x)\left[\Phi_3^{(n)}+\int^x dy\frac{\phi_1(y)\phi_2'(y)-\phi_1'(y)\phi_2(y)}{W(y)}K^{(n)}(y)\right] \ , \qquad (6.2)
\end{aligned}
$$

where the $\phi_i(x), i=1,2,3$ are three linearly independent solutions of the homogeneous equation Eq.(6.1), $W(x)$ their Wronskian and the $\Phi_i^{(n)}, i=1,2,3$ the integration constants.

Although algorithms for solving differential equations, even with simple rational coefficients, are not available in general, by trial and error it is not difficult to find (and in any case easy to verify) that Eq.(6.1) admits the three linearly independent solutions

$$
\begin{aligned}
\phi_1(x) &= (1-x^2) \ , \\
\phi_2(x) &= -\frac{1}{2}\frac{(1-x^2)(1-x^4)}{x^2}-2(1-x^2)H(0;x) \ ,
\end{aligned}
$$

$$
\begin{aligned}
\phi_3(x) \;=\;& +\frac{3}{512}\frac{(1-x^2)(1-x^4)}{x^2}\;[H(-1;x)+H(1;x)] \\
& +\frac{3}{128}(1-x^2)\;[H(0,-1;x)+H(0,1;x)] \\
& -\frac{1}{256}\frac{(x^2+3)(3x^2+1)}{x}\;,
\end{aligned} \tag{6.3}
$$

where the functions $H(0;x), H(-1;x), H(1;x), H(0,-1;x)$ etc. are harmonic polylogarithms [1], [2]. The Wronskian of the three above functions $\phi_i(x)$ is

$$
W(x) = -\frac{(1-x)^2(1+x)^2}{x^2}\;. \tag{6.4}
$$

When inserting Eq.s(6.3), Eq.(6.4) and $K^{(n)}(x)$, as defined by Eq.s(5.2,5.3), into Eq.(6.2), one sees that in analogy to previous work [1] all the integrations can be carried out in terms of harmonic polylogarithms, up to any desired value of n, and one is left only with the problem of fixing, for any order n, the three integration constants $\Phi_i^{(n)}$.

It is worth emphasizing that the *quantitative* determination of the integration constants can be obtained by simply imposing to the solution the *qualitative* behaviours for $x \to 0$, $x \to 1$ discussed in the previous sections Sec.(3,4). More exactly, by imposing the right qualitative behaviour at order n, one obtains the explicite, quantitative value of the integration constants up to order $n-2$ included (in other words: imposing the right qualitative behaviour to $\Phi^{(-1)}(x)$ one fixes the first three constants $\Phi_i^{(-3)}$, and so on). We have evaluated the solutions up to $n = 5$ (the results involves harmonic polylogarithms of weight 8), and fixed the integration constants up to $n = 3$ included. The integration constants were found to be

$$
\Phi_3^{(k)} = 0 \qquad \text{(for any } k\text{)}\;;
$$

$$
\begin{aligned}
\Phi_1^{(-3)} &= \frac{7}{192}\;; \quad \Phi_2^{(-3)} = -\frac{3}{128}\;; \quad \Phi_1^{(-2)} = -\frac{19}{256}\;; \quad \Phi_2^{(-2)} = \frac{37}{512}\;; \\
\Phi_1^{(-1)} &= \frac{7}{96}\;; \quad \Phi_2^{(-1)} = -\frac{233}{3072}\;; \quad \Phi_1^{(0)} = -\frac{259}{4608} + \frac{1}{24}\zeta(3)\;; \quad \Phi_2^{(0)} = \frac{1645}{36864}\;; \\
\Phi_1^{(1)} &= \frac{3395}{55296} - \frac{13}{128}\zeta(3) + \frac{1}{960}\pi^4\;; \quad \Phi_2^{(1)} = -\frac{15149}{442368} + \frac{9}{256}\zeta(3)\;; \\
\Phi_1^{(2)} &= -\frac{108691}{663552} + \frac{97}{768}\zeta(3) - \frac{13}{5120}\pi^4 + \frac{3}{16}\zeta(5)\;; \quad \Phi_2^{(2)} = \frac{1038085}{5308416} - \frac{95}{1536}\zeta(3) + \frac{9}{10240}\pi^4\;; \\
\Phi_1^{(3)} &= \frac{4181051}{7962624} - \frac{1225}{9216}\zeta(3) + \frac{97}{30720}\pi^4 - \frac{117}{256}\zeta(5) - \frac{1}{48}\zeta^2(3) + \frac{1}{2688}\pi^6\;; \\
\Phi_2^{(3)} &= -\frac{56294669}{63700992} - \frac{19}{12288}\pi^4 + \frac{2833}{18432}\zeta(3) + \frac{81}{512}\zeta(5)\;.
\end{aligned}
$$

The explicit expressions of all the $\Phi^{(n)}(x)$ which we have evaluated are too long to be listed here, and we give only the values up to $n = 0$:

$$
\Phi^{(-3)}(x) = -\frac{1}{24} - \frac{1}{12}x^2\;; \tag{6.5}
$$

$$
\Phi^{(-2)}(x) = +\frac{7}{96} + \frac{5}{32}x^2 - \frac{1}{96}x^4 - \frac{1}{8}x^2\;H(0;x)\;; \tag{6.6}
$$

$$
\Phi^{(-1)}(x) = -\frac{25}{384} - \frac{71}{384}x^2 + \frac{35}{1152}x^4 + \frac{1}{4}x^2\left(1-\frac{1}{8}x^2\right)\;H(0;x) - \frac{1}{16}x^2\;H(0;x)H(0;x)\;; \tag{6.7}
$$

$$
\begin{aligned}
\Phi^{(0)}(x) \;=\;& +\frac{19}{1536} + \frac{107}{512}x^2 - \frac{559}{13824}x^4 \\
& -\frac{1}{32}\left(1+\frac{25}{2}x^2-\frac{35}{12}x^4\right)\;H(0;x) + \frac{1}{8}x^2\left(1-\frac{1}{8}x^2\right)\;H(0;x)H(0;x) \\
& -\frac{1}{32}\frac{(1-x^2)(1-x^4)}{x^2}[H(-1;x)H(0;x) - H(0;x)H(1;x) + H(0,1;x) - H(0,-1;x)]
\end{aligned}
$$

$$-\frac{1}{48}x^2\ H(0;x)H(0;x)H(0;x) + \frac{1}{8}(1-x^2)[H(0,1;x)H(0;x) - H(0,-1;x)H(0;x)]$$
$$+\frac{1}{4}(1-x^2)\left[H(0,0,-1;x) - H(0,0,1;x) + \frac{1}{6}\zeta(3)\right] \ . \qquad (6.8)$$

Having the explicit form of the solution for any x it is almost trivial to obtain its value at $x = 1$ and $x = 0$. It is to be observed, in this respect, that the integration constants $\Phi_k^{(n-1)}, \Phi_k^{(n)}$ do not contribute to the $x = 1$ value up to order n, so that with the explicit value of the integration constants up to $n = 3$ included we can evaluate $\Phi(d, x = 1)$ in Fig.(2) up to order $n = 5$ included. By using the tables of [6] we find the

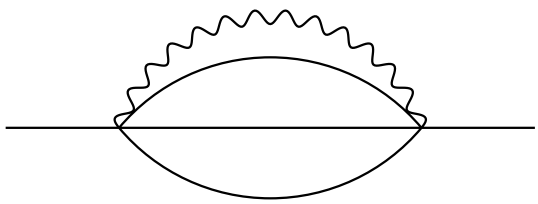

Figure 2: The considered 3-loop sunrise graph, in the case of equal masses, on mass-shell.

following expansion in $(d-4)$

$$\begin{aligned}
\Phi(d,x=1) \ &\stackrel{d\to 4}{=} \\
&-\frac{1}{8}\frac{1}{(d-4)^3} + \frac{7}{32}\frac{1}{(d-4)^2} - \frac{253}{1152}\frac{1}{(d-4)} + \frac{2501}{13824} \\
&+(d-4)\left(-\frac{59437}{165888} + \frac{1}{18}\pi^2\right) \\
&+(d-4)^2\left(+\frac{2831381}{1990656} - \frac{71}{216}\pi^2 + \frac{1}{3}\pi^2\ln 2 - \frac{7}{9}\zeta(3)\right) \\
&+(d-4)^3\left(-\frac{117529021}{23887872} + \frac{3115}{2592}\pi^2 - \frac{71}{36}\pi^2\ln 2 + \frac{497}{108}\zeta(3) + \frac{7}{9}\pi^2\ln^2 2 - \frac{43}{1080}\pi^4 + \frac{2}{9}\ln^4 2 + \frac{16}{3}a_4\right) \\
&+(d-4)^4\left(+\frac{4081770917}{286654464} - \frac{109403}{31104}\pi^2 + \frac{3115}{432}\pi^2\ln 2 - \frac{21805}{1296}\zeta(3) - \frac{497}{108}\pi^2\ln^2 2 + \frac{3053}{12960}\pi^4\right. \\
&\qquad \left. -\frac{71}{54}\ln^4 2 - \frac{284}{9}a_4 - \frac{43}{180}\pi^4\ln 2 + \frac{14}{9}\pi^2\ln^3 2 + \frac{7}{36}\pi^2\zeta(3) + \frac{4}{15}\ln^5 2 + \frac{341}{12}\zeta(5) - 32a_5\right) \\
&+(d-4)^5\left(-\frac{125873914573}{3439853568} + \frac{3386467}{373248}\pi^2 - \frac{109403}{5184}\pi^2\ln 2 + \frac{765821}{15552}\zeta(3) + \frac{21805}{1296}\pi^2\ln^2 2\right. \\
&\qquad -\frac{26789}{31104}\pi^4 + \frac{3115}{648}\ln^4 2 + \frac{3115}{27}a_4 - \frac{497}{54}\pi^2\ln^3 2 - \frac{497}{432}\pi^2\zeta(3) + \frac{3053}{2160}\pi^4\ln 2 \\
&\qquad -\frac{24211}{144}\zeta(5) + \frac{568}{3}a_5 - \frac{71}{45}\ln^5 2 + \frac{41}{810}\pi^6 + \frac{10}{9}\pi^2\ln^4 2 - \frac{151}{180}\pi^4\ln^2 2 + \frac{88}{3}\zeta(3)\ln^3 2 \\
&\qquad \left. -\frac{27}{2}\pi^2\zeta(3)\ln 2 + 176\zeta(5)\ln 2 - 176a_5\ln 2 + \ln^6 2 + \frac{2723}{36}\zeta(3)^2 - 160\ a_6 + 176\ b_6\right) \\
&+ \mathcal{O}(\ (d-4)^6\ )\ . \qquad (6.9)
\end{aligned}$$

(For the definition of the mathematical constants, see for instance Table 1 of [1]). The terms up to $(d-4)^3$ agree, within an obvious change of notation, with the formula for J_{11} given in [3]; the last two terms, of order $(d-4)^4, (d-4)^5$ are new in the literature, and are in full agreement with the numerical values for the same quantities given in [7].

Somewhat surprisingly, the coefficients of the expansion of $\Phi(d, x=0)$ in $(d-4)$ at order n are harder to obtain in this way, as they involve all the integration constants up to n included. On the other hand,

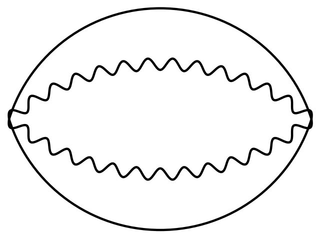

Figure 3: The vacuum 3-loop graph.

$\Phi(d, x)$ at $x=0$ corresponds to the vacuum amplitude of Fig.(3), which can be easily obtained by direct integration of the corresponding Feynman graph amplitude, so that its value can be used in practice as a check on $\Phi(d, x=0)$. Conversely, by using the explicit value of $\Phi(d, x=0)$ obtained by the direct integration, one can fix all the integration constants by imposing the proper behaviour at $x=0$ and the value at $x=0$, namely $a_0(d)$ in the notation of Eq.(3.3).

For completeness, we list here the exact value of $\Phi(d,0) = a_0(d)$

$$\begin{aligned}
\Phi(d, x=0) &= \frac{C^{-3}(d)}{(2\pi)^{3(d-2)}} \int \frac{d^dk_1\; d^dk_2\; d^dk_3}{k_1^2(k_2^2+1)(k_3^2+1)(k_1+k_2+k_3)^2} \\
&= -\frac{1}{32\,(d-4)} \frac{\Gamma^2(-1-(d-4))\;\Gamma^2(1+\frac{1}{2}(d-4))\;\Gamma(-2-\frac{3}{2}(d-4))}{\Gamma^2(1-\frac{1}{2}(d-4))\;\Gamma(-2-2(d-4))\;\Gamma(2+\frac{1}{2}(d-4))}\;; \qquad (6.10)
\end{aligned}$$

its expansion in $(d-4)$ is

$$\begin{aligned}
\Phi(d, x=0) = &-\frac{1}{24}\frac{1}{(d-4)^3} + \frac{7}{96}\frac{1}{(d-4)^2} - \frac{25}{384}\frac{1}{(d-4)} - \frac{5}{1536} + \frac{1}{24}\zeta(3) \\
&+(d-4)\left(+\frac{959}{6144} - \frac{7}{96}\zeta(3) + \frac{1}{960}\pi^4\right) \\
&+(d-4)^2\left(-\frac{10493}{24576} + \frac{25}{384}\zeta(3) - \frac{7}{3840}\pi^4 + \frac{3}{16}\zeta(5)\right) \\
&+(d-4)^3\left(+\frac{85175}{98304} + \frac{5}{1536}\zeta(3) + \frac{5}{3072}\pi^4 - \frac{21}{64}\zeta(5) + \frac{1}{2688}\pi^6 - \frac{1}{48}\zeta^2(3)\right) \\
&+(d-4)^4\left(-\frac{610085}{393216} - \frac{959}{6144}\zeta(3) + \frac{1}{12288}\pi^4 + \frac{75}{256}\zeta(5) - \frac{1}{1536}\pi^6 + \frac{7}{192}\zeta^2(3)\right. \\
&\qquad\left. - \frac{1}{960}\pi^4\zeta(3) + \frac{83}{128}\zeta(7)\right) \\
&+(d-4)^5\left(+\frac{4087919}{1572864} + \frac{10493}{24576}\zeta(3) - \frac{959}{245760}\pi^4 + \frac{15}{1024}\zeta(5) + \frac{25}{43008}\pi^6\right. \\
&\qquad\left. - \frac{25}{768}\zeta^2(3) + \frac{7}{3840}\pi^4\zeta(3) - \frac{581}{512}\zeta(7) + \frac{13}{115200}\pi^8 - \frac{3}{16}\zeta(3)\zeta(5)\right) \\
&+ \mathcal{O}(\,(d-4)^6\,)\;. \qquad (6.11)
\end{aligned}$$

The above expansion corresponds and agrees (again up to trivial changes of notation) with the quantities J_{13} of [3] and I_{14} of [7].

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