Scattering amplitudes for $e^+e^- \to 3$ jets at next-to-next-to-leading order QCD

S. Moch [a*], P. Uwer [b†], and S. Weinzierl [c‡]

[a]Deutsches Elektronensynchrotron DESY Platanenallee 6, D–15738 Zeuthen, Germany

[b]Institut für Theoretische Teilchenphysik, Universität Karlsruhe, D-76128 Karlsruhe, Germany

[c]Dipartimento di Fisica, Università di Parma, INFN Gruppo Collegato di Parma, 43100 Parma, Italy

We present the calculation of the fermionic contribution to the QCD two-loop amplitude for $e^+e^- \to q\bar{q}g$.

1. Introduction

Today's high energy collider experiments have reached such an accuracy that next-to-leading order (NLO) theoretical predictions are no longer sufficient. While for inclusive observables like for example the total hadronic cross section in e^+e^- annihilation the step from NLO to next-to-next-leading order (NNLO) is far less demanding – and has been taken a long time ago – the contrary is true for less inclusive quantities as for example jet rates. Given these difficulties it is clear that only for specific reactions NNLO calculations can be envisaged. Important reactions for which one should go beyond the NLO approximation are for example: Bhabha scattering, $pp, p\bar{p} \to 2$ jets, and $e^+e^- \to 3$ jets. Among these examples the e^+e^- annihilation into 3 jets is of particular interest. Historically this reaction was one of the first clean tests of Quantum Chromodynamics (QCD) as the theory of strong interaction. Today the 3-jet production in e^+e^- annihilation is still an excellent laboratory for precise tests of QCD and jet physics. If one studies how a jet is modeled in the perturbative prediction, it also becomes obvious that only in higher order perturbation theory will give a reliable description (c.f. fig. 1). Due to the fact that lepton colliders provide a very clean environment for QCD analyses $e^+e^- \to 3$ jets is also an excellent reaction for measureing the QCD coupling α_s with high accuracy. It is exactly here where the need for NNLO predictions for 3-jet production becomes most prominent: at present the measurements of α_s are plagued by scale uncertainties of the theoretical predictions [1] (see also H. Stenzel, these proceedings). These uncertainties arise from uncalculated higher order contributions. They can be reduced by a NNLO calculation. For a 3-jet prediction at NNLO accuracy several ingredients are needed: First the matrix elements for $e^+e^- \to q\bar{q}ggg$, $q\bar{q}q'\bar{q}'g$ have to be evaluated. They are obtained from a leading-order calculation and are known for a long time [2, 3]. Second the one-loop matrix elements for $e^+e^- \to q\bar{q}gg, q\bar{q}q'\bar{q}'$ are needed. They have been calculated in refs. [4–7]. Finally one needs the NNLO amplitudes for $e^+e^- \to q\bar{q}g$ [8–11]. In the following the calculation of the NNLO matrix elements is reported. One should keep in mind that the combination of these 3 ingredients into an infrared finite cross section is still a highly non-trivial – and so

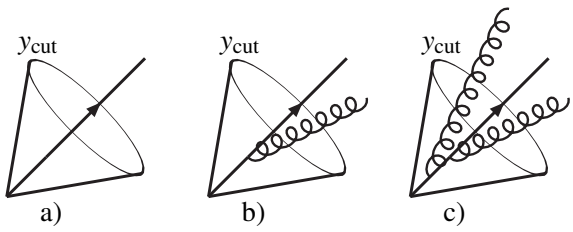

Figure 1. Modelling of a jet in leading a), next-to-leading b), and next-to-next-to-leading order c).

*moch@ifh.de
†uwer@particle.uni-karlsruhe.de (speaker)
‡stefanw@fis.unipr.it

far unsolved – problem.

2. Old and new approaches to calculate NNLO amplitudes

Due to the tremendous activities in the field enormous progress has been made in the past. All the relevant double box scalar integrals have been calculated [12–15]. For a few reactions these results have been used to obtain NNLO scattering amplitudes [8, 9, 16–25]. Today one can say that a *standard approach* to perform these calculations exist. Essentially it consists of three steps. First one generates automatically the relevant Feynman diagrams. In a second step the reduction of tensor integrals to scalar integrals is done via Schwinger parametrization [26,27]. The scalar integrals obtained in this step appear in shifted dimensions and with raised powers of the propagators. In a last step, equations derived from partial integration and Lorentz invariance are used to reduce the scalar integrals to a small set of master integrals [28–30]. The final result is than obtained in terms of these master integrals. They have to be calculated analytically. The bottleneck of this approach is that one has to create and to solve a huge system of equations to obtain the desired reduction to master integrals. In contrast to the naive expectation, topologies which at first glance look not so difficult may involve much more work than the more complicated once like for example the double box. In ref. [31] we have proposed a different method. The basic idea is that one applies only obvious reductions like for example the *triangle rule*. In particular one does not perform a complete reduction to master integrals. Instead one calculates the scalar integrals in higher dimensions and with raised powers of the propagators directly. To illustrate the method let us consider the so-called penta-box. A corresponding Feynman diagram is shown in fig. 2a. Applying Schwinger parametrization one arrives at the scalar topology shown schematically in fig. 2b. An obvious application of the triangle rule allows the elimination of two propagators [32]. The resulting topology is often called C-topology. This reduction is shown in fig. 3. As mentioned earlier the C-topology appears in higher dimensions ($d = 2m - 2\varepsilon$, $m \in N$) and with raised powers of the propagators $(\nu_1, \ldots, \nu_5)$. In the approach advocated here an analytic expression for

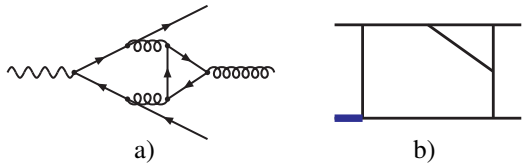

Figure 2. Tensor reduction of the general pentabox topology a) to scalar integrals in higher dimensions and raised powers of the propagators b).

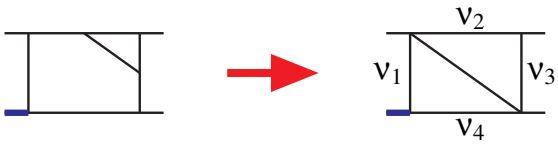

Figure 3. Application of the triangle rule yielding the C-topology $\mathrm{C}(m = \frac{d}{2} + \varepsilon, \nu_1, \nu_2, \nu_3, \nu_4, \nu_5)$

the C-topology is needed. Such an expression can be obtained in terms of infinite sums [31]. The generic structure is shown in eq. (1).

$$\mathrm{C}(m = \frac{d}{2} + \varepsilon, \nu_1, \ldots, \nu_5) \sim \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} \frac{x_1^{i_1}}{i_1!} \frac{x_2^{i_2}}{i_2!} \Bigg[x_1^{2m-2\varepsilon-\nu_{1235}} x_2^{2m-2\varepsilon-\nu_{2345}} \frac{\Gamma(i_1 + 2m - 2\varepsilon - \nu_{125})}{\Gamma(i_1 + 1 + 2m - 2\varepsilon - \nu_{1235})} \times \frac{\Gamma(i_1 + i_2 + 3m - 3\varepsilon - \nu_{12345})}{\Gamma(i_2 + 1 + 2m - 2\varepsilon - \nu_{2345})} \times \frac{\Gamma(i_1 + i_2 + 2m - 2\varepsilon - \nu_{235})\Gamma(i_2 + 2m - 2\varepsilon - \nu_{345})}{\Gamma(i_1 + i_2 + 4m - 4\varepsilon - \nu_{12345} - \nu_5)} + \ldots \Bigg] \tag{1}$$

It is obvious that such a representation is completely useless unless one is able to evaluate the ε-expansion in terms of known functions. Fortunately this is possible using a few basic algorithms for *nested sums* [31]. The basic idea here is that one rewrites the sums using

the relation

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_{ij} = a_{00} + \sum_{j=1}^{\infty} a_{0j} + \sum_{i=1}^{\infty} a_{i0} + \sum_{n=1}^{\infty}\sum_{j=1}^{n-1} a_{j(n-j)}. \quad (2)$$

Furthermore one can expand the Γ-functions by the means of

$$\Gamma(-n+\varepsilon) = \frac{(-1)^n}{\varepsilon n!}\Gamma(1+\varepsilon)(1 + S_1(n)\varepsilon + S_{1,1}(n)\varepsilon^2 + S_{1,1,1}(n)\varepsilon^3 + S_{1,1,1,1}(n)\varepsilon^4 + \ldots \quad (3)$$

and related identities [33]. Here $S_{1,\ldots}(n)$ denote the harmonic sums, see e.g. ref. [33]. Inspecting the sums which appear using eq. (2) and eq. (3) one observes that the algorithms needed are just a generalization of the algorithms studied in ref. [33]. In particular defining the nested sums as a generalization of the harmonic sums by

$$S(n;m_1,\ldots,m_k;x_1,\ldots,x_k) = \sum_{i_1=1}^{n}\sum_{i_2=1}^{i_1}\cdots\sum_{i_k=1}^{i_{k-1}} \frac{x_1^{i_1}}{i_1^{m_1}}\cdots\frac{x_k^{i_k}}{i_k^{m_k}} = \sum_{i=1}^{n}\frac{x_1^i}{i^{m_1}} S(i;m_2,\ldots,m_k;x_2,\ldots,x_k) \quad (4)$$

the following four basic operations are sufficient to evaluate the ε-expansion of the C-topology in terms of multiple polylogarithms [34]:

1. Multiplication

$$S(n;m_1,\ldots;x_1,\ldots) \times S(n;m_1',\ldots;x_1',\ldots) \quad (5)$$

2. Sums involving i and $n-i$:

$$\sum_{i=1}^{n-1}\frac{x_1^i}{i^{m_1}} S(i;m_2\ldots;x_2,\ldots) \times \frac{{x_1'}^{n-i}}{(n-i)^{m_1'}} S(n-i;m_2',\ldots;x_2',\ldots) \quad (6)$$

3. Conjugations:

$$\sum_{i=1}^{n}\binom{n}{i}(-1)^i\frac{x_0^i}{i^{m_0}} S(i;m_1,\ldots,m_k;x_1,\ldots,x_k) \quad (7)$$

4. Sums involving binomials and $n-i$:

$$\sum_{i=1}^{n-1}\binom{n}{i}(-1)^i\frac{x_1^i}{i^{m_1}} S(i;m_2\ldots;x_2,\ldots) \times \frac{{x_1'}^{n-i}}{(n-i)^{m_1'}} S(n-i;m_2',\ldots;x_2',\ldots) \quad (8)$$

In ref. [31] we have given explicit algorithms for the four basic operations. These algorithms are well suited for an implementation in a computer algebra system. For their implementation we used Form3 [35] and Ginac [36, 37]. Using these programs we calculated more than 300 C-topologies analytically. In particular we have checked that our results for the two master integrals $\mathrm{C}(2,1,1,1,1,1), \mathrm{C}(2,1,1,1,1,2)$ agree with those given by Gehrmann and Remiddi [38]. All the simpler topologies can be calculated in the same way.

3. Results

The amplitude $\mathcal{A}_\gamma$ for the process

$$e^+e^- \to \gamma^* \to q(p_1)g(p_2)\bar{q}(p_3) \quad (9)$$

can be written as a leptonic part L_μ multiplied by a hadronic part H_μ:

$$\mathcal{A}_\gamma \sim \frac{1}{s} H_\mu L^\mu = \frac{1}{s} g_s (H_\mu^{(0)} + \frac{\alpha_s}{2\pi} H_\mu^{(1)} + \left(\frac{\alpha_s}{2\pi}\right)^2 H_\mu^{(2)} + \ldots) L^\mu, \quad (10)$$

with $s = (p_1+p_2+p_3)^2$ and $g_s = \sqrt{4\pi\alpha_s}$. Furthermore the hadronic part can be decomposed according to the colour structure. In particular we have

$$H_\mu^{(2)} = T_{q\bar{q}}^a \Bigg(N^2 H_{2,\mu}^{(2)} + H_{0,\mu}^{(2)} + \frac{1}{N^2} H_{-2,\mu}^{(2)} + n_f N H_{n_f,1,\mu}^{(2)} + \frac{n_f}{N} H_{n_f,-1,\mu}^{(2)} + n_f^2 H_{n_f^2,\mu}^{(2)} + \Sigma_f \left(\frac{4}{N} - N\right) H_{\Sigma_f,\mu}^{(2)} \Bigg), \quad (11)$$

with $T_{q\bar{q}}^a$ the generator of the $SU(N)$ gauge group and n_f the number of massless quarks. The last contribution arise from a closed quark loop coupled directly to the photon. (The factor Σ_f involves a sum over the charges of the active quarks.) We note that the contribution proportional to n_f^2 is completly determined by the renormalization. It does not involve any non-trivial two-loop topology. In this talk we discuss only the contributions proportional to $n_f N$ and $\frac{n_f}{N}$. The hadronic contribution can also be decomposed according to the spinor structure:

$$H_\mu = c_1 \frac{1}{s}\langle p_1|\not{p}_2|p_3\rangle \varepsilon_{g_\mu}$$

$$
\begin{aligned}
&+c_2\frac{1}{s^2}\langle p_1|\not{p}_2|p_3\rangle(\varepsilon_g\cdot p_1)p_{3\mu}\\
&+c_3\frac{1}{s^2}\langle p_1|\not{p}_2|p_3\rangle(\varepsilon_g\cdot p_3)p_{1\mu}\\
&+c_4\frac{1}{s^2}\langle p_1|\not{p}_2|p_3\rangle(\varepsilon_g\cdot p_1)p_{1\mu}\\
&+\ldots+c_{13}\frac{1}{s^2}\langle p_1|\not{p}_2|p_3\rangle(\varepsilon_g\cdot p_3)p_{2\mu} \qquad (12)
\end{aligned}
$$

where the scalar functions c_1-c_{13} depend only on the ratios $x_1=2(p_1\cdot p_2)/s$, $x_2=2(p_2\cdot p_3)/s$. Due to various constraints arising from gauge invariance or quark anti-quark antisymmetry only 4 of them are needed. All the remaining ones can be expressed in terms of these 4 functions which we chose to be c_2, c_4, c_6 and c_{12}. The perturbative expansion in α_s of the functions c_i is defined through

$$
c_i=g_s\left(c_i^{(0)}+\left(\frac{\alpha_s}{2\pi}\right)c_i^{(1)}+\left(\frac{\alpha_s}{2\pi}\right)^2c_i^{(2)}+O(\alpha_s^3)\right). \qquad (13)
$$

Using the methods described in the previous section we calculated all the 13 functions and used the constraints as an additional check of our calculation. Furthermore we checked the structure of the ultraviolet as well as the infrared divergencies [39]. We have also compared our results with results published recently [8,9]. In refs. [8,9] the result is written in terms of 2-dimensional harmonic polylogarithms instead of the multiple polylogarithms used here. Expressing the 2-dimensional harmonic polylogarithms in terms of multiple polylogarithms we found complete agreement for the n_fN and $\frac{n_f}{N}$ contribution calculated here.

Following ref. [9] we used the general structure of the infrared divergencies as described by Catani [39] to define the finite parts of the coefficients c_i:

$$
c_i^{(2),\mathrm{fin}} \;=\; c_i^{(1),\mathrm{ren}}-\mathbf{I}^{(1)}(\varepsilon)c_i^{(1),\mathrm{ren}}-\mathbf{I}^{(2)}(\varepsilon)c_i^{(0)}, \qquad (14)
$$

with the one- and two-loop insertion operators $\mathbf{I}^{(1)}(\varepsilon)$ and $\mathbf{I}^{(2)}(\varepsilon)$ given in ref. [39]. As an example, we present the result for the n_fN-contribution to the finite part $c_4^{(2),\mathrm{fin}}$ at two loops,

$$
\begin{aligned}
&c_4^{(2),\mathrm{fin}}(x_1,x_2)\Big|_{n_fN}=n_fN\Bigg(-\frac{19}{18}\frac{1}{x_1(1-x_2)}\\
&+\frac{1}{36}\frac{\ln(x_2)}{(1-x_2)}\left[36\frac{(1+x_1)}{(x_1+x_2)}-\frac{59}{x_1}-\frac{50}{x_1(1-x_2)}\right.\\
&\left.+12\frac{(1-3x_1)(1-x_1)}{x_1(1-x_1-x_2)}\right]+\frac{(x_1-x_2)}{(x_1+x_2)^3}\mathrm{R}_1(x_1,x_2)\\
&-\left[\frac{5}{12}\frac{(2-x_2)}{x_1(1-x_2)^2}+\frac{2}{(x_1+x_2)^2}\right]\mathrm{Li}_2(1-x_2)\\
&+\frac{1}{4}\frac{(2-x_2)}{x_1(1-x_2)^2}\left[\ln(x_2)^2+\frac{1}{3}\zeta_2\right]\\
&-\frac{1}{12}\frac{(7x_1-5x_2-6x_1x_2+5{x_2}^2+{x_1}^2)}{(1-x_2)x_1(1-x_1-x_2)(x_1+x_2)}\ln(x_1)\\
&-\frac{\mathrm{R}(x_1,x_2)}{(1-x_1-x_2)(1-x_2)(x_1+x_2)}\left[\frac{17}{6}x_2-\frac{35}{12}x_1\right.\\
&-\frac{19}{12}+\frac{1}{12}\frac{(1+x_1)}{(1-x_2)}-\frac{1}{3}\frac{(1-3x_1)(1-x_1)}{(1-x_1-x_2)}\\
&\left.+\frac{5{x_2}^2}{12x_1}-\frac{5}{6}\frac{x_2}{x_1}+2\frac{x_1(1+x_1)}{(x_1+x_2)}\right]\\
&-\left[\frac{1}{12}\frac{(2-x_2)}{x_1(1-x_2)^2}-\frac{2}{(x_1+x_2)^2}\right]\mathrm{Li}_2(1-x_1)\Bigg)\\
&-i\pi n_fN\left(\frac{(2-x_2)\ln(x_2)}{2x_1(1-x_2)^2}+\frac{1}{2}\frac{1}{x_1(1-x_2)}\right). \qquad (15)
\end{aligned}
$$

We have introduced the function $\mathrm{R}(x_1,x_2)$, which is well known from [40],

$$
\begin{aligned}
\mathrm{R}(x_1,x_2)=&\left(\frac{1}{2}\ln(x_1)\ln(x_2)-\ln(x_1)\ln(1-x_1)\right.\\
&\left.+\frac{1}{2}\zeta_2-\mathrm{Li}_2(x_1)\right)+(x_1\leftrightarrow x_2). \qquad (16)
\end{aligned}
$$

In addition, it is convenient, to define the symmetric function $\mathrm{R}_1(x_1,x_2)$, which contains a particular combination of multiple polylogarithms [34],

$$
\begin{aligned}
\mathrm{R}_1(x_1,x_2)=&\left(\ln(x_1)\mathrm{Li}_{1,1}\left(\frac{x_1}{x_1+x_2},x_1+x_2\right)\right.\\
&-\frac{1}{2}\zeta_2\ln(1-x_1-x_2)-\frac{1}{2}\ln(x_1)\ln(x_2)\ln(1-x_1-x_2)\\
&-\ln(x_1)\mathrm{Li}_2(x_1+x_2)-\mathrm{Li}_{1,2}\left(\frac{x_1}{x_1+x_2},x_1+x_2\right)\\
&\left.-\mathrm{Li}_{2,1}\left(\frac{x_1}{x_1+x_2},x_1+x_2\right)+\mathrm{Li}_3(x_1+x_2)\right)\\
&+(x_1\leftrightarrow x_2). \qquad (17)
\end{aligned}
$$

As one can see all multiple polylogarithms have simple arguments. As a consequence it is straightforward to obtain the analytic continuation, which can be used for the crossing of the amplitude.

4. Conclusion

In this talk we have presented the calculation of a specific colour structure of the QCD two-loop ampli-

tude $e^+e^- \to q\bar{q}g$. The computation has been done by a completely new method developed by the authors [31]. It has been shown that the approach which is based on nested sums and their algebraic properties provides a very efficient method for two-loop calculations. Furthermore the tools developed in this approach can also be used for a systematic expansion of generalized hypergeometric functions. In addition the presented results provide a very important cross check on recently obtained results [8, 9].

Acknowledgment: P.U. would like to thank the organizers for the pleasant atmosphere at QCD02.

REFERENCES

1. G. Dissertori, (2001), hep-ex/0105070,
2. F.A. Berends, W.T. Giele and H. Kuijf, Nucl. Phys. B321 (1989) 39,
3. K. Hagiwara and D. Zeppenfeld, Nucl. Phys. B313 (1989) 560,
4. Z. Bern et al., Nucl. Phys. B489 (1997) 3, hep-ph/9610370,
5. Z. Bern, L.J. Dixon and D.A. Kosower, Nucl. Phys. B513 (1998) 3, hep-ph/9708239,
6. E.W.N. Glover and D.J. Miller, Phys. Lett. B396 (1997) 257, hep-ph/9609474,
7. J.M. Campbell, E.W.N. Glover and D.J. Miller, Phys. Lett. B409 (1997) 503, hep-ph/9706297,
8. L.W. Garland et al., Nucl. Phys. B627 (2002) 107, hep-ph/0112081,
9. L.W. Garland et al., (2002), hep-ph/0206067,
10. S. Moch, P. Uwer and S. Weinzierl, (2002), hep-ph/0207167,
11. S. Moch, P. Uwer and S. Weinzierl, (2002), hep-ph/0207043,
12. V.A. Smirnov, Phys. Lett. B460 (1999) 397, hep-ph/9905323,
13. J.B. Tausk, Phys. Lett. B469 (1999) 225, hep-ph/9909506,
14. V.A. Smirnov, Phys. Lett. B491 (2000) 130, hep-ph/0007032,
15. V.A. Smirnov, Phys. Lett. B500 (2001) 330, hep-ph/0011056,
16. Z. Bern, L.J. Dixon and D.A. Kosower, JHEP 01 (2000) 027, hep-ph/0001001,
17. C. Anastasiou et al., Nucl. Phys. B601 (2001) 318, hep-ph/0010212,
18. C. Anastasiou et al., Nucl. Phys. B601 (2001) 341, hep-ph/0011094,
19. C. Anastasiou et al., Phys. Lett. B506 (2001) 59, hep-ph/0012007,
20. C. Anastasiou et al., Nucl. Phys. B605 (2001) 486, hep-ph/0101304,
21. E.W.N. Glover, C. Oleari and M.E. Tejeda-Yeomans, Nucl. Phys. B605 (2001) 467, hep-ph/0102201,
22. Z. Bern, L.J. Dixon and A. Ghinculov, Phys. Rev. D63 (2001) 053007, hep-ph/0010075,
23. Z. Bern, A. De Freitas and L.J. Dixon, JHEP 09 (2001) 037, hep-ph/0109078,
24. Z. Bern, A. De Freitas and L. Dixon, JHEP 03 (2002) 018, hep-ph/0201161,
25. C. Anastasiou, E.W.N. Glover and M.E. Tejeda-Yeomans, Nucl. Phys. B629 (2002) 255, hep-ph/0201274,
26. O.V. Tarasov, Phys. Rev. D54 (1996) 6479, hep-th/9606018,
27. O.V. Tarasov, Nucl. Phys. B502 (1997) 455, hep-ph/9703319,
28. G. 't Hooft and M.J.G. Veltman, Nucl. Phys. B44 (1972) 189,
29. K.G. Chetyrkin and F.V. Tkachov, Nucl. Phys. B192 (1981) 159,
30. T. Gehrmann and E. Remiddi, Nucl. Phys. B580 (2000) 485, hep-ph/9912329,
31. S. Moch, P. Uwer and S. Weinzierl, J. Math. Phys. 43 (2002) 3363, hep-ph/0110083,
32. C. Anastasiou, E.W.N. Glover and C. Oleari, Nucl. Phys. B575 (2000) 416, hep-ph/9912251,
33. J.A.M. Vermaseren, Int. J. Mod. Phys. A14 (1999) 2037, hep-ph/9806280,
34. A.B. Goncharov, Math. Res. Lett. 5 (1998) 497.
35. J.A.M. Vermaseren, (2000), math-ph/0010025,
36. C. Bauer, A. Frink and R. Kreckel, J. Symbolic Computation 33 (2002), cs/0004015,
37. S. Weinzierl, Comput. Phys. Commun. 145 (2002) 357, math-ph/0201011,
38. T. Gehrmann and E. Remiddi, Nucl. Phys. B601 (2001) 248, hep-ph/0008287,
39. S. Catani, Phys. Lett. B427 (1998) 161, hep-ph/9802439,
40. R.K. Ellis, D.A. Ross and A.E. Terrano, Nucl. Phys. B178 (1981) 421,