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Non-Singlet Structure Functions at Three Loops: Fermionic Contributions

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Abstract

We compute the fermionic (n_f) contributions to the flavour non-singlet structure functions in unpolarized electromagnetic deep-inelastic scattering at third order of massless perturbative QCD. Complete results are presented for the corresponding n_f-parts of the three-loop anomalous dimension and the three-loop coefficient functions for the structure functions F_2 and F_L. Our results agree with all partial and approximate results available in the literature. The present calculation also facilitates a complete determination of the threshold-resummation parameters B_2 and D_2^{DIS} of which only the sum was known so far, thus completing the information required for the next-to-next-to-leading logarithmic resummation. We find that D_2^{DIS} vanishes in the $\overline{\mathrm{MS}}$ scheme.

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1 Introduction

Structure functions in deep-inelastic scattering (DIS) form the backbone of our knowledge of the proton's parton densities, which are indispensable for analyses of hard scattering processes at proton–(anti-)proton colliders like the TEVATRON and the future LHC. Structure functions are also among the quantities best suited for precisely measuring the strong coupling constant α_s. Over the past twenty years DIS experiments have proceeded to a high (few-percent) accuracy and a wide kinematic coverage [1]. More results, especially at high scales Q^2, can be expected from the forthcoming high-luminosity phase of the electron–proton collider HERA at DESY. On the theoretical side, at least the next-to-next-to-leading order (NNLO) corrections of perturbative QCD need to be taken into account in order to make full use of these measurements and to make quantitatively reliable predictions for hard processes at hadron colliders.

The calculation of NNLO processes in perturbative QCD is far from easy. For deep-inelastic structure functions, in particular, the current situation is that, while the coefficient functions are known to two loops [2, 3, 4, 5, 6], only six/seven integer Mellin moments of the corresponding three-loop anomalous dimensions have been computed for lepton–hadron [7, 8, 9] and lepton–photon DIS [10], together with the same moments of the three-loop coefficient functions. The hadronic results have been employed, directly [11, 12, 13, 14] and indirectly [15, 16] via x-space approximations constructed from them [17, 18, 19], to improve the data analysis and some hadron-collider predictions. However, the number of available moments is rather limited, and hence these results cannot provide sufficient information at small values of the Bjorken variable x.

For the complete information one needs to obtain either all even or odd (depending on the quantity under consideration) Mellin moments, or do the complete calculation in Bjorken-x space. We have adopted the first approach. Following the formalism of ref. [20, 21, 7, 8] to obtain the lower fixed moments, we have used recursive methods to extend the calculation to all values of the Mellin moment N. This is by no means trivial, since before the start of the calculation the mathematics of the answer was still poorly understood [22]. Hence it was first necessary to develop the understanding of harmonic sums [23, 24, 25, 26] and harmonic polylogarithms [27, 28, 29]. In addition the Mellin transforms and the inverse Mellin transforms from Bjorken-x-space to Mellin space and back had to be solved [29]. These conceptual problems have been overcome and the method has been shown to work in a complete re-calculation of the two-loop coefficient functions [6].

The concept of working in Mellin space is not new. This method was already used in the early QCD papers [30, 31, 32]. But even in the case of the two-loop anomalous dimensions it was still possible to do the resulting sums in a rather direct manner [33, 23]. This changed with the two-loop evaluation of σ_L/σ_T in which Kazakov and Kotikov [34, 35] managed to obtain some of the integrals via recursion relations or first order difference equations.

In this article, we present the fermionic (n_f) corrections to the flavour non-singlet structure functions in electromagnetic DIS at the three-loop level. This includes the three-loop anomalous dimensions which are needed for the completion of the NNLO calculation, as well as the three-loop coefficient functions which are, at least at large x, the most important contribution to the next-to-next-to-next-to-leading order (N^{3}LO) correction [36]. Of course, the n_f-part is not the complete calculation. Yet we decided to present it already now, because the complete calculation (including the singlet part) will still take quite some (computer) time. Thus, for the time being, NNLO calculations of the Drell-Yan process [37, 38] (which are relevant for luminosity monitoring at TEVATRON and LHC [39, 40, 41]) and of Higgs production [38, 42] have to rely on parton distributions evolved with the approximate splitting functions of ref. [19]. Our present calculation provides a first check of the reliability of these approximations. More importantly, it turns out that already this calculation is sufficient to provide the last relevant missing information for the extension of the soft-gluon (threshold) resummation for DIS [43, 44, 45] to the next-to-next-to-leading logarithmic accuracy [46]. In fact, our result for the resummation parameter $D_2^{\rm DIS}$ is most intriguing and calls for further studies.

This article is organized as follows. In section 2 we outline those parts of the calculation, which differ from previous two-loop [6] and fixed-N three-loop [7, 8, 9] analyses. The present calculation does not yet involve the full complexity of the method, as the most difficult diagram topologies do not occur. Therefore we postpone a full account to a later publication. In section 3 we present our explicit even-N Mellin-space results, except for the rather lengthy expressions for the three-loop coefficient functions which are deferred to appendix A. The corresponding three-loop quantities in x-space can be found in section 4. Here, instead of writing down the cumbersome exact expressions for the coefficient functions, we follow the procedure applied in refs. [17, 18] to the two-loop coefficient functions, and provide approximate parametrizations which are compact and sufficiently accurate for all numerical applications. In section 5 we then discuss the implications of our calculation on the soft-gluon resummation, before we summarize our results in section 6.

2 Method

Because we are considering only the non-singlet structure functions in this paper, the method for the calculation of their moments closely follows ref. [7]. Hence there is not much need to explain the physics of the method here again. Thus we will discuss only the differences introduced by the fact that we now compute all moments simultaneously as a function of the moment number N. Since N is not a fixed constant, we cannot resort to the techniques of ref. [7], where the Mincer program [47, 48] was used as the tool to solve the integrals. Instead, we will have to introduce new techniques. However, we can give N a positive integer value at any point of the derivations and calculations, after which the Mincer program can be invoked to verify that the results are correct. From a practical point of view this is the most powerful feature of the Mellin-space approach, as it greatly simplifies the checking of all programs.

Similar to the fixed-N computations of refs. [8, 9], the diagrams are generated automatically with a special version of the diagram generator QGRAF [49]. For all the symbolic manipulations of the formulae we use the latest version of the program FORM [50]. The calculation is performed in dimensional regularization [51, 52, 53, 54] with $D = 4 - 2\varepsilon$. Hence the unrenormalized Mellin-space results will be functions of ε, N, and the values $\zeta_{3,\ldots,5}$ of the Riemann ζ-function. The renormalization is carried out in the $\overline{\mathrm{MS}}$-scheme [55, 56] as described in ref. [7].

We distinguish three categories of diagrams: complete diagrams, composite building blocks and basic building blocks. A complete diagram is a Feynman diagram with all its structure like traces and dotproducts in the numerator. Such a diagram may lead to a large number of more fundamental integrals that cannot be reduced by considerations of momentum conservation only. For the understanding of composite and basic building blocks, one has to realize that in the framework of the operator-product expansion we eventually have to take N derivatives with respect to the parton momentum P after which P is put equal to zero. This projects out the N-th Mellin moment [57] and it effectively amputates the legs of the parton, leaving us with propagator-type diagrams. Therefore, we define the topology of a diagram as the propagator topology when the P-momentum legs have been amputated. The three-loop propagator topologies of the BE (Benz) type and of the O4 type are shown in fig. 1. For the notation we refer to refs. [47, 48]. The external lines in the propagator topology are referred to as Q.

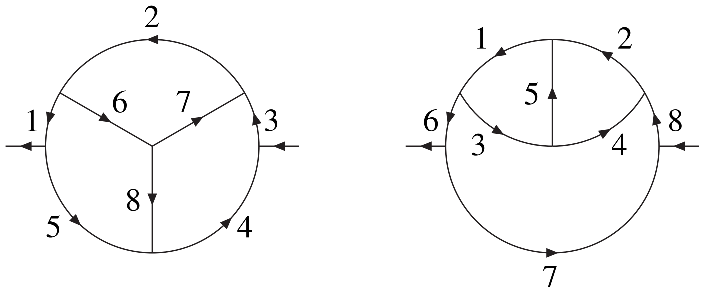

Figure 1: The topologies BE (left) and O4 (right) of propagator-type diagrams, with the line numbering as employed in figs. 2 and 3 below. The external lines carry the momentum Q.

When the numbers of the position of the incoming and outgoing momenta have been attached we are referring to subtopologies. For instance, BE_{13} is a subtopology of type BE in which the momentum P comes in in line 1 and leaves in line 3, assuming the numbering of the BE topology as in fig. 1. We define basic building blocks (BBB) as integrals in which both the incoming and the outgoing P-momentum are attached to the same line as, e.g., in BE_{22}. In composite building blocks (CBB), on the other hand, the incoming and the outgoing P-momentum are attached to different lines, as in the case of BE_{13} mentioned above. Of course we could have introduced names for all these three-loop four-point functions, but since eventually the P-momentum legs get amputated the above notation seems the clearest scheme. In this way, it is only a small step to an easy pictorial representation of the integrals as used in ref. [6].

For the calculation of the n_f-parts of the non-singlet structure functions F_2 and F_L, one does not need to consider all three-loop topologies. In fact, the only genuine three-loop subtopologies one has to solve are of the BE-type and of the O4-type, with two complete diagrams of either type entering the calculation. These diagrams are displayed in figs. 2 and 3.

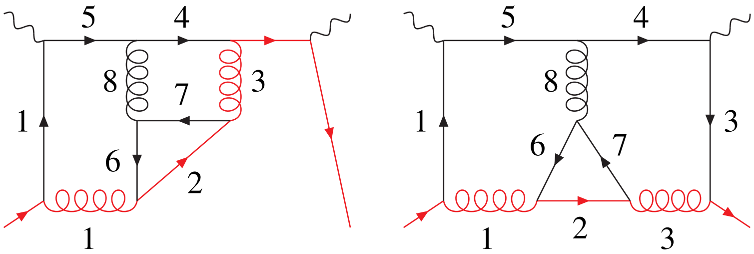

Figure 2: The diagrams of topology BE which contribute to the fermionic part of the non-singlet structure functions F_2 and F_L. The subtopologies are BE_{1Q} (left) and BE_{13} (right).

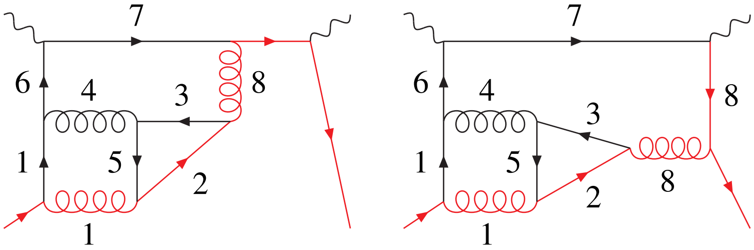

Figure 3: The diagrams of topology O4, which contribute to the fermionic part of the non-singlet structure functions F_2 and F_L. These diagrams are examples of the subtopologies $\mathrm{O4}_{1Q}$ and $\mathrm{O4}_{18}$.

The main problem we are faced with as compared to the corresponding two-loop calculation [6] is that the necessary reduction equations become much more complicated. In addition the set of equations that was available at that moment was not maximal, and the structure of the more complicated topologies needs the maximal set of reduction equations. This involves some equations in which the vanishing of $P \cdot P$ is an issue that should be considered with care, a point that will be explained in full detail in a later publication in which we have to deal with all topologies.

For this calculation we first studied the basic building blocks. Here it is easy to expand the single propagator that contains the momentum P to sufficient powers in P for the N-th moment. This number of powers can be less than N as there may be some powers of P in the numerator already. It should be noted that if we have a power of $P \cdot Q$ in the numerator we are effectively computing the moment $N-1$ of the integral. At this point we write down all equations based on integration by parts, all scaling relations and all form-factor equations that can be constructed. Next follows a potentially difficult process in which we have to combine these relations to construct equations that can systematically bring the powers of the denominators in the integral down, either reducing them to zero or leaving them at a fixed unique value. When a line is eliminated a simpler topology is reached and we can refer to the reduction equations for that topology. This will eventually lead to a topology that is simple enough so that we can calculate the integral.

A problem arises when a reduction equation cannot take the power n of a denominator beyond one, because it is of the type $nI(n+1) \to I(n)$. Also when the power of a propagator is not an integer, one can only try to reduce this power to a fixed value for which the line does not vanish. In these cases we leave the power either at one or at this fixed value (usually $1+\varepsilon$) and we continue with the next propagator. Eventually we will have integrals in which only the line with the power that involves N will not have a standard value. The N will occur both in the power of the denominator $p \cdot p$ and in the numerator $2P \cdot p$ when p is the momentum associated to this line. By this time our remaining equations which have also been treated to bring the powers of their progagators to standard values may have become rather lengthy. But we still need them to bring the difference of the powers in the numerator and the denominator to a fixed value, which can be either 0, 1 or 2, something we have to leave open for reasons explained below. Now there are several possibilities: If there is only one integral left of the type to be solved, the equation directly determines the solution. This is however rarely the case. Usually there are several terms left, each with a different power of $2P \cdot Q$, leaving us with an equation of the type

$$a_0(N)\, I(N) + a_1(N)\, I(N-1) + \ldots + a_m(N)\, I(N-m) \;=\; G(N) \tag{1}$$

in which the function G refers to a potentially horrendous combination of integrals of simpler topologies. Eq. (1) defines a recursion relation or difference equation of order m. For the present calculation we did not have to go beyond order 2. It should be noted that recently difference equations have been encountered by Laporta in refs. [58, 59].

These difference equations can be solved by making the ansatz that the solution will be a combination of harmonic sums. If the proper combination has been selected, the coefficients of the harmonic sums can be obtained by substituting the trial solution into the equation and solving the resulting, potentially large set of linear equations. There can be several thousand equations in such a system. Of course one has to have a solution for the function $G(N)$ in eq. (1) first, and $m-1$ boundary values are required. Because these boundary values are basically fixed integer-N moments, they can be obtained using the Mincer program. The need for knowing the function G puts a rigid hierarchy in the order in which we have to treat the topologies.

Solving the difference equations is rather slow work. Hence we compute their solutions only once and tabulate the results. We usually do this for several values of the difference of the powers of the numerator and the denominator as mentioned above eq. (1). The reason behind this is that the equations we use for either raising or lowering this difference may contain a so-called spurious pole in ε when we try to bring this value to one. The concept of spurious poles is rather important. The rule of the triangle [60, 61], for instance, can involve a factor proportional to $1/\varepsilon$ when an integral is being reduced. Close inspection reveals that when powers of the loop momentum are present in the numerator, it is possible that more than one of such poles is generated before a denominator is removed. The resolved triangle [62] shows, however, that it is possible to sum all contributions of such a reduction and that eventually there can be no more than one pole per eliminated line. This means that the extra poles should cancel between the many generated terms. But when we work only a to given cutoff in powers of ε (both for reasons of economy and because we cannot evaluate some integrals easily beyond certain powers in ε) these temporary poles could spoil the final result in the same way as such things can happen in numerical calculations at fixed precision. We call them spurious poles, because in principle they can be avoided. One of the greatest difficulties in deriving reduction equations is to indeed avoid such spurious poles. For some integrals, no spurious-pole free formula could be found for the last reduction when bringing the difference of the power of the denominator and the numerator either from 2 to 1 or from 0 to 1. This problem has been circumvented by solving the resulting difference equations for the three cases 0, 1 and 2 separately.

Since the evaluation of all the basic building block integrals that can occur requires very much computer time, we decided to tabulate all these integrals for the complicated topologies. This saves much computer time, because each integral is typically used many times. Also the more complicated topologies use many integrals of a less complicated type, hence their evaluation becomes much faster once the latter integrals have been tabulated.

The next step is the evaluation of the composite building blocks. Here again we first construct all possible equations. It turns out to be most economic to leave the propagators with the momentum P unexpanded. The fact that eventually an expansion to N powers of P will take place then requires some special calculational rules. If for instance an equation is multiplied by $P \cdot Q$, we have to replace N by $N-1$ at the same time. Such rules can be major sources of errors. Hence it is very fortunate that at any moment we can decide that N has a fixed value like four or five and then evaluate the integrals in the equation with the Mincer program to see whether it is still correct.

The equations are used to set up a reduction scheme that is similar to the one for the BBB's. If a line is eliminated we obtain either a simpler topology or a BBB. Otherwise we reduce the power of a denominator to a standard value and continue with the next line. In numerous cases a reduction can only be done by means of a difference equation (1). If this is a first order difference equation it can be solved directly, introducing one sum. Such sums are of a benign type and can be evaluated afterwards. If the difference equation is of a higher order we have to consider all more fundamental integrals first before we can solve the equation. The further reduction scheme becomes then rather complicated, but not impossible. On the average each subtopology can require several weeks of work before it has been completely solved by these methods.

Analogous to the BBB case discussed above, we have tabulated the more complicated CBB integrals entering the calculation. It is not only a matter of computer time that renders this necessary. Also the size of the intermediate expressions becomes a most relevant factor: if one is not careful, even a hard disk of 100 GBytes can become restrictive. In practice, already expressions significantly larger than 10 GBytes took too long for evaluation without further optimization. A careful hierarchy of tabulation managed to avoid these problems.

Having programs for all basic and composite building blocks renders the remainder of the calculation rather straightforward. The major difference to the fixed-moment calculations is now that we obtain much longer results due to the presence of the parameter N in the answer. We have checked the correctness of each individual diagram for several values of N, by comparing with the results of a Mincer calculation. In addition we have compared the complete renormalized results with the results in the literature for the available values of N.

3 Results in Mellin space

Here we present the N-space coefficient functions and the anomalous dimensions up to the third order in the renormalized coupling α_s. All results are given in the $\overline{\mathrm{MS}}$-scheme with the renormalization scales identified with the physical hard scale Q. Thus the perturbative expansion of the non-singlet coefficient functions and anomalous dimensions can be written as

$$
\begin{aligned}
C_{i,\mathrm{ns}}(\alpha_s, N) &= \sum_{n=0}^{\infty} \left(\frac{\alpha_s}{4\pi}\right)^n c_{i,\mathrm{ns}}^{(n)}(N) \, , && (2) \\
\gamma_{\mathrm{ns}}(\alpha_s, N) &= \sum_{n=0}^{\infty} \left(\frac{\alpha_s}{4\pi}\right)^{n+1} \gamma_{\mathrm{ns}}^{(n)}(N) && (3)
\end{aligned}
$$

with $i = 2, L$ in eq. (2). There is no need to consider different choices of the scales, as they do not require functions beyond those introduced in eqs. (2) and (3). Recall that for F_2 the $(n+1)$-loop anomalous dimensions and the n-loop coefficient functions together form the $\mathrm{N^nLO}$ approximation of (renormalization-group improved) perturbative QCD.

There are, actually, three different non-singlet combinations of coefficient functions and splitting functions. These combinations all coincide at order α_s, but they all differ beyond the second order. Only the so-called '+'-combinations (involving sums over quarks and antiquarks) are probed in electromagnetic DIS, hence only these quantities are addressed in the present article. Consequently our results below apply directly only to all even-integer values of N from which, however, the results for arbitrary N can be uniquely inferred by analytic continuation.

Our N-space results are expressed in terms of harmonic sums $S_{\vec{m}}(N)$. In the following all harmonic sums are understood to have the argument N, i.e., we employ the short-hand notation $S_{\vec{m}} \equiv S_{\vec{m}}(N)$. In addition we use operators $\mathbf{N}_{\pm}$ and $\mathbf{N}_{\pm\mathbf{i}}$ which shift the argument N of a given function by ± 1 or a larger integer i,

$$
\mathbf{N}_{\pm} f(N) = f(N \pm 1) \, , \qquad \mathbf{N}_{\pm\mathbf{i}} f(N) = f(N \pm i) \, . \qquad (4)
$$

We normalize the trivial leading-order (LO) coefficient function and recover, of course, the well-known result for the LO anomalous dimension [30, 31]

$$
\begin{aligned}
c_{2,\mathrm{ns}}^{(0)}(N) &= 1 \, , && (5) \\
\gamma_{\mathrm{ns}}^{(0)}(N) &= C_F (2(\mathbf{N}_{-} + \mathbf{N}_{+}) S_1 - 3) \, . && (6)
\end{aligned}
$$

In our notation, the next-to-leading order (NLO) non-singlet coefficient function for F_2 [56] and the corresponding anomalous dimension [33, 23] read

$$
\begin{aligned}
c_{2,\mathrm{ns}}^{(1)}(N) &= C_F (7\mathbf{N}_{+} S_1 + 2S_1 - 9 + (\mathbf{N}_{-} + \mathbf{N}_{+})[2S_{1,1} - 3S_1 - 2S_2]), && (7) \\
\gamma_{\mathrm{ns}}^{(1)}(N) &= 4C_A C_F \Big(2\mathbf{N}_{+} S_3 - \frac{17}{24} - 2S_{-3} - \frac{28}{3} S_1 + (\mathbf{N}_{-} + \mathbf{N}_{+}) \Big[\frac{151}{18} S_1 + 2S_{1,-2} - \frac{11}{6} S_2\Big]\Big)
\end{aligned}
$$

$$+4C_Fn_f\Big(\frac{1}{12}+\frac{4}{3}S_1-(\mathbf{N}_-+\mathbf{N}_+)\Big[\frac{11}{9}S_1-\frac{1}{3}S_2\Big]\Big)+4C_F{}^2\Big(4S_{-3}+2S_1+2S_2-\frac{3}{8}$$
$$+\mathbf{N}_-\Big[S_2+2S_3\Big]-(\mathbf{N}_-+\mathbf{N}_+)\Big[S_1+4S_{1,-2}+2S_{1,2}+2S_{2,1}+S_3\Big]\Big). \qquad (8)$$

At next-to-next-to-leading order (NNLO, N^2LO), we have re-calculated the coefficient function $c^{(2)}_{2,\rm ns}$ of refs. [2, 6], and we have computed all fermionic contributions to the splitting function $\gamma^{(2)}_{\rm ns}$, the n_f^1 terms being a new result of this article. The fermionic NNLO corrections — the complete expression for the N-space coefficient function can be found in ref. [6] — are given by

$$c^{(2)}_{2,\rm ns}(N) = 4C_Fn_f\Big((1-\mathbf{N}_+)\Big[\frac{122}{27}S_1+\frac{7}{6}S_{1,1}\Big]-(\mathbf{N}_--1)\Big[\frac{89}{108}S_1-S_2\Big]-(\mathbf{N}_-+\mathbf{N}_+)\Big[\frac{5}{6}S_3$$
$$+\frac{13}{18}S_{1,1}+\frac{1}{3}S_{1,1,1}-\frac{2}{3}S_{2,1}-\frac{1}{3}S_{1,2}\Big]-\frac{1}{6}S_{1,1}+\frac{457}{144}-\frac{247}{108}S_1+\frac{19}{6}\mathbf{N}_+S_2\Big), \qquad (9)$$

$$\gamma^{(2)}_{\rm ns}(N) = 16C_AC_Fn_f\Big(\frac{3}{2}\zeta_3-\frac{5}{4}+\frac{10}{9}S_{-3}-\frac{10}{9}S_3+\frac{4}{3}S_{1,-2}-\frac{2}{3}S_{-4}+2S_{1,1}-\frac{25}{9}S_2+\frac{257}{27}S_1$$
$$-\frac{2}{3}S_{-3,1}-\mathbf{N}_+\Big[S_{2,1}-\frac{2}{3}S_{3,1}-\frac{2}{3}S_4\Big]+(1-\mathbf{N}_+)\Big[\frac{23}{18}S_3-S_2\Big]-(\mathbf{N}_-+\mathbf{N}_+)\Big[S_{1,1}+\frac{1237}{216}S_1$$
$$+\frac{11}{18}S_3-\frac{317}{108}S_2+\frac{16}{9}S_{1,-2}-\frac{2}{3}S_{1,-2,1}-\frac{1}{3}S_{1,-3}-\frac{1}{2}S_{1,3}-\frac{1}{2}S_{2,1}-\frac{1}{3}S_{2,-2}+S_1\zeta_3+\frac{1}{2}S_{3,1}\Big]\Big)$$
$$+16C_Fn_f{}^2\Big(\frac{17}{144}-\frac{13}{27}S_1+\frac{2}{9}S_2+(\mathbf{N}_-+\mathbf{N}_+)\Big[\frac{2}{9}S_1-\frac{11}{54}S_2+\frac{1}{18}S_3\Big]\Big)+16C_F{}^2n_f\Big(\frac{23}{16}-\frac{3}{2}\zeta_3$$
$$+\frac{4}{3}S_{-3,1}-\frac{59}{36}S_2+\frac{4}{3}S_{-4}-\frac{20}{9}S_{-3}+\frac{20}{9}S_1-\frac{8}{3}S_{1,-2}-\frac{8}{3}S_{1,1}-\frac{4}{3}S_{1,2}+\mathbf{N}_+\Big[\frac{25}{9}S_3-\frac{4}{3}S_{3,1}$$
$$-\frac{1}{3}S_4\Big]+(1-\mathbf{N}_+)\Big[\frac{67}{36}S_2-\frac{4}{3}S_{2,1}+\frac{4}{3}S_3\Big]+(\mathbf{N}_-+\mathbf{N}_+)\Big[S_1\zeta_3-\frac{325}{144}S_1-\frac{2}{3}S_{1,-3}+\frac{32}{9}S_{1,-2}$$
$$-\frac{4}{3}S_{1,-2,1}+\frac{4}{3}S_{1,1}+\frac{16}{9}S_{1,2}-\frac{4}{3}S_{1,3}+\frac{11}{18}S_2-\frac{2}{3}S_{2,-2}+\frac{10}{9}S_{2,1}+\frac{1}{2}S_4-\frac{2}{3}S_{2,2}-\frac{8}{9}S_3\Big]\Big). \qquad (10)$$

The corresponding formulae for the longitudinal coefficient function $C_{L,\rm ns}$ are deferred to the appendix, together with the rather lengthy N-space results for the fermionic parts of the third-order coefficient functions $c^{(3)}_{i,\rm ns}$, $i=2,L$, which partly also represent new results of this article (the n_f^2 term for F_2 has already been presented in ref. [63]). Notice that $c^{(3)}_{L,\rm ns}$ can be considered a NNLO quantity, since C_L vanishes at order α_s^0. On the other hand $c^{(3)}_{2,\rm ns}$ represents, at least at large N, the dominant part of the N^3LO corrections to F_2 [36].

As briefly mentioned at the end of section 2, we have subjected our results to a number of checks. First of all, we have calculated some lower even moments in an arbitrary covariant gauge with the Mincer program [47, 48], keeping the gauge parameter ξ in the gluon propagator. All dependence on ξ does cancel in the final results. Secondly the n_f^2-contribution to $\gamma^{(2)}_{\rm ns}$ is known from the work of Gracey [64] and we agree with his result. Furthermore the coefficients of $\ln^k N$, $k=3,\ldots 5$, of $c^{(3)}_{2,\rm ns}(N)$ agree with the prediction of the soft-gluon resummation [65]. Finally, we have checked the result of each individual diagram for several integer values of N by comparing with the results of a Mincer calculation. Thus, as the strongest check, our results reproduce the fixed even moments $N=2,\ldots,14$ computed in refs. [7, 8, 9].

4 Third-order results in x-space

The x-space coefficient functions and the splitting functions are obtained from the results of the previous section by an inverse Mellin transformation, which maps the harmonic sums of moment space [23, 24, 25, 26] to harmonic polylogarithms in x-space [27, 28, 29]. This transformation can be performed by a completely algebraic procedure [29, 6] based on the fact that the harmonic sums also occur as coefficients of the Taylor expansion of harmonic polylogarithms.

Here we confine ourselves to the third-order results; for the two-loop non-singlet splitting functions and coefficient functions the reader is referred to refs. [67, 2, 6]. For brevity the exact results are written down only for the splitting function, conventionally related to the anomalous dimension (3) by

$$\gamma_{\rm ns}^{(n)}(N) \;=\; -\int_0^1 dx\, x^{N-1}\, P_{\rm ns}^{(n)}(x)\ . \tag{11}$$

The fermionic part of $P_{\rm ns}^{(n)}$ involves only simpler harmonic polylogarithms which can be expressed in terms of the usual (poly-)logarithms. The x-space analogue of eq. (10), graphically displayed in fig. 4, can thus be written as

$$\begin{aligned}
P_{\rm ns}^{(2)}(x) \;=\; & 16 C_A C_F n_f \Big(p_{\rm qq}(x) \Big[\frac{5}{9}\zeta_2 - \frac{209}{216} - \frac{3}{2}\zeta_3 - \frac{167}{108}\ln(x) + \frac{1}{3}\ln(x)\zeta_2 - \frac{1}{4}\ln^2(x)\ln(1-x) \\
& - \frac{7}{12}\ln^2(x) - \frac{1}{18}\ln^3(x) - \frac{1}{2}\ln(x){\rm Li}_2(x) + \frac{1}{3}{\rm Li}_3(x) \Big] + p_{\rm qq}(-x)\Big[\frac{1}{2}\zeta_3 - \frac{5}{9}\zeta_2 - \frac{2}{3}\ln(1+x)\zeta_2 \\
& + \frac{1}{6}\ln(x)\zeta_2 - \frac{10}{9}\ln(x)\ln(1+x) + \frac{5}{18}\ln^2(x) - \frac{1}{6}\ln^2(x)\ln(1+x) + \frac{1}{18}\ln^3(x) - \frac{10}{9}{\rm Li}_2(-x) \\
& - \frac{1}{3}{\rm Li}_3(-x) - \frac{1}{3}{\rm Li}_3(x) + \frac{2}{3}{\rm H}_{-1,0,1}(x) \Big] + (1+x)\Big[\frac{1}{6}\zeta_2 + \frac{1}{2}\ln(x) - \frac{1}{2}{\rm Li}_2(x) - \frac{2}{3}{\rm Li}_2(-x) \\
& - \frac{2}{3}\ln(x)\ln(1+x) + \frac{1}{24}\ln^2(x) \Big] + (1-x)\Big[\frac{1}{3}\zeta_2 - \frac{257}{54} + \ln(1-x) - \frac{17}{9}\ln(x) - \frac{1}{24}\ln^2(x) \Big] \\
& + \delta(1-x)\Big[\frac{5}{4} - \frac{167}{54}\zeta_2 + \frac{1}{20}{\zeta_2}^2 + \frac{25}{18}\zeta_3 \Big] \Big) + 16 C_F {n_f}^2 \Big(p_{\rm qq}(x)\Big[\frac{5}{54}\ln(x) - \frac{1}{54} + \frac{1}{36}\ln^2(x) \Big] \\
& + (1-x)\Big[\frac{13}{54} + \frac{1}{9}\ln(x) \Big] - \delta(1-x)\Big[\frac{17}{144} - \frac{5}{27}\zeta_2 + \frac{1}{9}\zeta_3 \Big] \Big) + 16 {C_F}^2 n_f \Big(p_{\rm qq}(x)\Big[\frac{5}{3}\zeta_3 - \frac{55}{48} \\
& + \frac{5}{24}\ln(x) + \frac{1}{3}\ln(x)\zeta_2 + \frac{10}{9}\ln(x)\ln(1-x) + \frac{1}{4}\ln^2(x) + \frac{2}{3}\ln^2(x)\ln(1-x) + \frac{2}{3}\ln(x){\rm Li}_2(x) \\
& - \frac{2}{3}{\rm Li}_3(x) - \frac{1}{18}\ln^3(x) \Big] + p_{\rm qq}(-x)\Big[\frac{10}{9}\zeta_2 - \zeta_3 + \frac{4}{3}\ln(1+x)\zeta_2 - \frac{1}{3}\ln(x)\zeta_2 - \frac{5}{9}\ln^2(x) \\
& + \frac{20}{9}\ln(x)\ln(1+x) - \frac{1}{9}\ln^3(x) + \frac{1}{3}\ln^2(x)\ln(1+x) + \frac{20}{9}{\rm Li}_2(-x) + \frac{2}{3}{\rm Li}_3(-x) + \frac{2}{3}{\rm Li}_3(x) \\
& - \frac{4}{3}{\rm H}_{-1,0,1}(x) \Big] + (1+x)\Big[\frac{7}{36}\ln^2(x) - \frac{67}{72}\ln(x) + \frac{4}{3}\ln(x)\ln(1+x) + \frac{1}{12}\ln^3(x) + \frac{2}{3}{\rm Li}_2(x) \\
& + \frac{4}{3}{\rm Li}_2(-x) \Big] + (1-x)\Big[\frac{1}{9}\ln(x) - \frac{10}{9} - \frac{4}{3}\ln(1-x) + \frac{2}{3}\ln(x)\ln(1-x) - \frac{1}{3}\ln^2(x) \Big] \\
& - \delta(1-x)\Big[\frac{23}{16} - \frac{5}{12}\zeta_2 - \frac{29}{30}{\zeta_2}^2 + \frac{17}{6}\zeta_3 \Big] \Big)\ ,
\end{aligned} \tag{12}$$

where we have introduced

$$p_{\mathrm{qq}}(x) \;=\; 2\,(1-x)^{-1} - 1 - x \tag{13}$$

and all divergences for $x \to 1$ are understood in the sense of $+$-distributions. In eq. (12) we have left one particular harmonic polylogarithm, $\mathrm{H}_{-1,0,1}(x)$, unsubstituted. This function is given by

$$\mathrm{H}_{-1,0,1}(x) \;\equiv\; \int_0^x \frac{dz}{1+z}\,\mathrm{Li}_2(z) \;=\; \mathrm{Li}_2(x)\ln(1+x) + \frac{1}{2}S_{1,2}(x^2) - S_{1,2}(-x) - S_{1,2}(x)\;, \tag{14}$$

where the representation by the Nielsen function $S_{1,2}$ has been derived in ref. [66]. $\mathrm{H}_{-1,0,1}(x)$ can also be expressed in terms of trilogarithms, albeit with more complicated arguments [6].

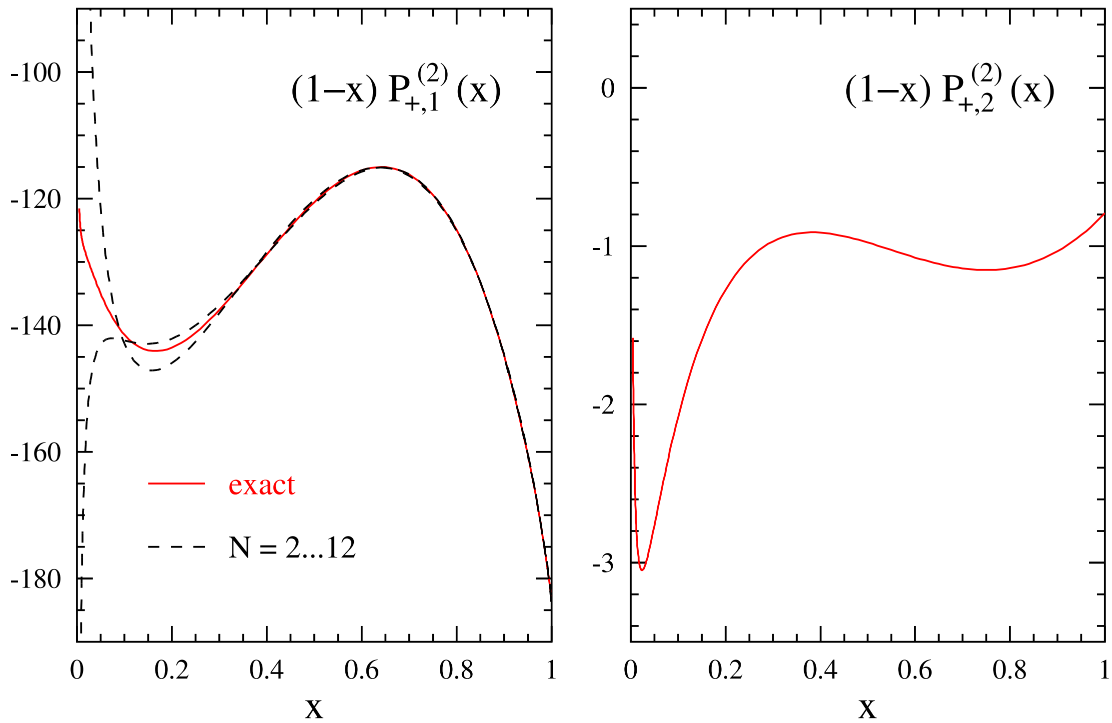

Figure 4: The n_f^1 and n_f^2 parts $P^{(2)}_{+,1}(x)$ and $P^{(2)}_{+,2}(x)$ of the three-loop non-singlet splitting function (12), multiplied by $(1-x)$ for display purposes. Also shown in the left part (dashed curve) is the uncertainty band derived in ref. [19] from the lowest six even-integer moments [7, 8, 9].

The x-space coefficient functions involve harmonic polylogarithms of weight four, which in general cannot be expressed in terms of standard polylogarithms and Nielsen functions anymore. Instead of writing down the cumbersome exact expressions, we prefer to present sufficiently accurate, compact parametrizations in terms of the $+$-distributions and end-point logarithms

$$\mathcal{D}_k \;=\; \left[\frac{\ln^k(1-x)}{1-x}\right]_+ , \quad L_1 \;=\; \ln(1-x)\,, \quad L_0 \;=\; \ln x\;. \tag{15}$$

It is convenient to apply this procedure (which has been employed in ref. [17] for the two-loop coefficient functions) also to the n_f^1 part of the splitting function (12). Inserting the numerical value of the QCD colour factors, this function can be approximated by

$$
\begin{aligned}
P_{\rm ns}^{(2)}(x) \;\cong\;& n_f\,(\,-\,183.187\,\mathcal{D}_0 - 173.927\,\delta(1-x) - 5120/81\,L_1 - 197.0 + 381.1\,x + 72.94\,x^2 \\
& \quad + 44.79\,x^3 - 1.497\,xL_0^3 - 56.66\,L_0L_1 - 152.6\,L_0 - 2608/81\,L_0^2 - 64/27\,L_0^3\,) \\
+ \;& n_f^2\,(\,-\,\mathcal{D}_0 - (51/16 + 3\,\zeta_3 - 5\,\zeta_2)\,\delta(1-x) + x\,(1-x)^{-1}L_0\,(3/2\,L_0 + 5) + 1 \\
& \quad + (1-x)\,(6 + 11/2\,L_0 + 3/4\,L_0^2)\,)\;64/81\ . \qquad (16)
\end{aligned}
$$

Corresponding parametrizations for the three-loop coefficient functions read

$$
\begin{aligned}
c_{2,\rm ns}^{(3)}(x) \;\cong\;& n_f\,(\,640/81\,\mathcal{D}_4 - 6592/81\,\mathcal{D}_3 + 220.573\,\mathcal{D}_2 + 294.906\,\mathcal{D}_1 - 729.359\,\mathcal{D}_0 \\
& \quad + 2572.597\,\delta(1-x) - 640/81\,L_1^4 + 167.2\,L_1^3 - 315.3\,L_1^2 + 4742\,L_1 \\
& \quad + 762.1 + 7020\,x + 989.4\,x^2 + L_0L_1\,(326.6 + 65.93\,L_0 + 1923\,L_1) \\
& \quad + 260.1\,L_0 + 186.5\,L_0^2 + 12224/243\,L_0^3 + 728/243\,L_0^4\,) \\
+ \;& n_f^2\,(\,64/81\,\mathcal{D}_3 - 464/81\,\mathcal{D}_2 + 7.67505\,\mathcal{D}_1 + 1.00830\,\mathcal{D}_0 - 103.2655\,\delta(1-x) \\
& \quad - 64/81\,L_1^3 + 15.46\,L_1^2 - 51.71\,L_1 + 59.00\,x + 70.66\,x^2 + L_0L_1\,(-80.05 \\
& \quad - 10.49\,L_0 + 41.67\,L_1) - 8.050\,L_0 - 1984/243\,L_0^2 - 368/243\,L_0^3\,)\ , \qquad (17)
\end{aligned}
$$

$$
\begin{aligned}
c_{L,\rm ns}^{(3)}(x) \;\cong\;& n_f\,(\,1024/81\,L_1^3 - 112.4\,L_1^2 + 340.3\,L_1 + 409 - 210\,x - 762.6\,x^2 - 1792/81\,xL_0^3 \\
& \quad + L_0L_1\,(969.2 + 304.8\,L_0 - 288.2\,L_1) + 200.8\,L_0 + 64/3\,L_0^2 + 0.046\,\delta(1-x)\,) \\
+ \;& n_f^2\,(\,3x\,L_1^2 + (6 - 25\,x)\,L_1 - 19 + (317/6 - 12\,\zeta_2)\,x - 6x\,L_0L_1 + 6x\,{\rm Li}_2(x) \\
& \quad + 9x\,L_0^2 - (6 - 50\,x)\,L_0\,)\;64/81\ . \qquad (18)
\end{aligned}
$$

The n_f^2 parts of $P_{\rm ns}^{(2)}$ and $c_{L,\rm ns}^{(3)}$, the +-distribution contributions (up to a numerical truncation of the coefficients involving ζ_i), and the rational coefficients of the (sub-)leading regular end-point terms are exact in eqs. (16) – (18). The remaining coefficients have been determined by fits to the exact results, for which we have used the Fortran package of ref. [68]. The above parametrizations deviate from the exact expressions by one part in thousand or less, an accuracy which should be amply sufficient for foreseeable numerical applications.

5 Implications for the threshold resummation

The large-N/large-x behaviour of the three-loop splitting functions and coefficient functions is of special interest in connection with the soft-gluon (threshold) exponentiation [43, 44, 45] at next-to-next-leading logarithmic (NNL) accuracy. Here the coefficient function for $F_{2,\rm ns}$ can, up to terms which vanish for $N \to \infty$, be written as

$$
C_{2,\rm ns}(\alpha_{\rm s}, N) \;=\; (1 + a_{\rm s}\,g_{01} + a_{\rm s}^2\,g_{02} + \ldots)\,\exp[\,L\,g_1(a_{\rm s}L) + g_2(a_{\rm s}L) + a_{\rm s}\,g_3(a_{\rm s}L) + \ldots] \qquad (19)
$$

with $a_{\rm s} = \alpha_{\rm s}/(4\pi)$ and $L = \ln N$. The functions g_l depend on (universal) coefficients $A_{i \le l}$ and $B_{i \le l-1}$ and process-dependent parameters $D_{i \le l-1}^{\rm DIS}$ as described in ref. [46], where also the explicit expressions for the functions $g_{1,2,3}$ can be found. Hence the NNL function g_3 involves the new coefficients A_3, B_2 and $D_2^{\rm DIS}$. These coefficients can be fixed by expanding eq. (19) in powers of $\alpha_{\rm s}$ and comparing to the result of the full fixed-order calculations.

In the $\overline{\rm MS}$ scheme adopted in this article, the parameter A_3 is simply the coefficient of $\ln N$ in $\gamma_{\rm ns}^{(2)}(N)$ or, equivalently, of $1/(1-x)_+$ in $P_{\rm ns}^{(2)}(x)$. Its fermionic part is thus known from eq. (12),

$$A_3\Big|_{n_f} = C_A C_F n_f \left[-\frac{836}{27}+\frac{160}{9}\zeta_2-\frac{112}{3}\zeta_3\right]+C_F^2 n_f\left[-\frac{110}{3}+32\,\zeta_3\right]+C_F n_f^2\left[-\frac{16}{27}\right]\,. \qquad (20)$$

The numerical value can be read off from eq. (16). Like for the whole of $P_{\rm ns}^{(2)}(x)$, as shown in fig. 4, this result is consistent with, but supersedes the estimate derived in ref. [19] from the first six even-integer moments. Parallel to our work $A_3|_{n_f}$ has also been calculated in ref. [69].

The combination $B_2+D_2^{\rm DIS}$ has been determined in ref. [46] by comparing the expansion of eq. (19) to the $\ln N$ term of the two-loop coefficient function $c_{2,\rm ns}^{(2)}$ of ref. [2]. As the $\ln^2 N$ (or $\mathcal{D}_1$) contribution to $c_{2,\rm ns}^{(3)}$ involves a different linear combination, $\beta_0(B_2+2D_2^{\rm DIS})$, of the very same coefficients, B_2 and $D_2^{\rm DIS}$ can be disentangled using the three-loop coefficient function. The analytic results for the two new $+$-distribution coefficients read

$$\begin{aligned} c_{2,\rm ns}^{(3)}\Big|_{\mathcal{D}_1 n_f} &= C_A C_F n_f\left[-\frac{15062}{81}+\frac{512}{9}\zeta_2+16\,\zeta_3\right]+C_F^2 n_f\left[\frac{83}{9}+168\,\zeta_2+\frac{112}{3}\zeta_3\right]\\ &\quad + C_F n_f^2\left[\frac{940}{81}-\frac{32}{9}\zeta_2\right]\,, \end{aligned} \qquad (21)$$

$$\begin{aligned} c_{2,\rm ns}^{(3)}\Big|_{\mathcal{D}_0 n_f} &= C_A C_F n_f\left[-\frac{160906}{729}-\frac{9920}{81}\zeta_2-\frac{776}{9}\zeta_3+\frac{208}{15}\zeta_2^2\right]+C_F^2 n_f\left[-\frac{2003}{108}\right.\\ &\quad \left.-\frac{4226}{27}\zeta_2-60\,\zeta_3+16\,\zeta_2^2\right]+C_F n_f^2\left[-\frac{8714}{729}+\frac{232}{27}\zeta_2-\frac{32}{27}\zeta_3\right] \end{aligned} \qquad (22)$$

(the coefficients of $\mathcal{D}_{2,\ldots,5}$ can be found in ref. [65]). In fact, due to the prefactor β_0, the complete results for B_2 and $D_2^{\rm DIS}$ can already be inferred from fermionic result (21), yielding

$$\begin{aligned} B_2 &= C_F^2\left[-\frac{3}{2}-24\,\zeta_3+12\,\zeta_2\right]+C_F C_A\left[-\frac{3155}{54}+40\,\zeta_3+\frac{44}{3}\zeta_2\right]\\ &\quad + C_F n_f\left[\frac{247}{27}-\frac{8}{3}\zeta_2\right]\,, \end{aligned} \qquad (23)$$

$$D_2^{\rm DIS} = 0\,. \qquad (24)$$

The vanishing of $D_1^{\rm DIS}$ and $D_2^{\rm DIS}$ — in contrast to the Drell-Yan process, where D_2 is different from zero [46] — calls for a deeper explanation, possibly offering an all-order generalization.

Finally we note that, once the non-fermionic contributions to the 3-loop non-singlet splitting functions and coefficient functions are completed, the NNL threshold resummation facilitates a prediction of the first six towers of logarithms, i.e., the coefficients of $\alpha_{\rm s}^n \ln^{2n-i} N$, $i=0,\ldots,5$, of $C_{2,\rm ns}$ at all orders $n>3$. We will return to this issue in a later publication.

6 Summary

We have computed the fermionic (n_f-enhanced) third-order contributions to the structure functions F_2 and F_L in electromagnetic deep-inelastic scattering. The calculation has been carried out for all even-integer Mellin moments N, by solving the three-loop integrals by means of recursion relations (difference equations) in N. This progress with respect to previous computations restricted to some fixed moments N is especially due to an improved understanding of the mathematics of harmonic sums and difference equations, and the implementation of corresponding tools in the symbolic manipulation program FORM which we employed to handle the huge intermediate expressions. We are confident that our approach will enable us to compute all three-loop corrections in DIS.

We have thus been able to derive the complete expressions for the corresponding n_f-parts of the NNLO anomalous dimensions and splitting functions and the N^{3}LO coefficient functions for F_2 and F_L. The results have been presented in both Mellin-N and Bjorken-x space, in the latter case we have also provided easy-to-use accurate parametrizations. Our results agree with all partial and approximate results available in the literature for these quantities, in particular we reproduce the even-integer moments $N = 2, \ldots, 12$ computed before.

The present results for the three-loop splitting function represent a step towards completing the ingredients required for NNLO calculations of hard-scattering processes involving initial-state hadrons in perturbative QCD. The three-loop coefficient functions for the most important structure function F_2 form the dominant part of the N^{3}LO corrections at large x, thus facilitating extractions of α_s with a distinctly reduced theoretical uncertainty. Already the n_f-part computed in this article leads to a complete determination of the threshold-resummation parameters B_2 and D_2^{DIS} — including the non-fermionic contributions — of which only the sum was known so far, thus practically completing the information required for the next-to-next-to-leading logarithmic resummation.

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A Appendix

All results for the non-singlet anomalous dimensions and coefficient functions presented in this article can be obtained as a FORM file from the preprint server `http://arXiv.org` by downloading the source file. Furthermore they are available from the authors upon request.

The fermionic parts, i.e., all terms proportional to $C_A C_F n_f$, $C_F^2 n_f$ and $C_F n_f^2$ of the three-loop coefficient function for the electromagnetic structure function F_2 are given by

$$
\begin{aligned}
c^{(3)}_{2,\mathrm{ns}}(N) \;=\; & 16 C_A C_F n_f \Big(\delta(N-2)\Big[\frac{5}{3}\zeta_5 - \frac{119}{300}\zeta_3\Big] + \frac{142883}{7776} - \frac{1051}{72}\zeta_3 + \frac{3}{4}\zeta_4 + \frac{83}{54}S_4 - 2S_{1,4} \\
& -\frac{4}{9}S_{-4,1} + \frac{191}{81}S_{-3} - \frac{16}{3}S_{-3,-2} + \frac{20}{27}S_{-3,1} - \frac{4}{9}S_{-3,1,1} - \frac{29}{18}S_{-2} - \frac{14}{3}S_{-2}\zeta_3 - \frac{16}{3}S_{-2,-3} \\
& +\frac{13}{3}S_{-2,-2} + \frac{8}{3}S_{-2,-2,1} + \frac{23}{9}S_{-5} + \frac{199819}{8100}S_1 - \frac{101}{18}S_{1,2} + \frac{181}{108}S_{3,1} - \frac{83}{54}S_{-4} + \frac{4132}{135}S_{1,-2} \\
& +\frac{8}{3}S_{1,-2,-2} + \frac{56}{9}S_{1,-2,1} + \frac{21463}{1080}S_{1,1} - \frac{10}{3}S_{1,1}\zeta_3 - 4S_{1,1,-3} + \frac{58}{9}S_{1,1,1} - \frac{23}{18}S_{1,2,1} + 10S_{1,-4} \\
& +\frac{8}{3}S_{1,1,1,-2} + \frac{1}{3}S_{1,1,1,2} - 4S_{2,-2,1} - \frac{1}{3}S_{1,1,2,1} - \frac{8}{3}S_{1,2,-2} + S_{1,2,2} + \frac{35}{9}S_{1,3} + \frac{5}{3}S_{1,3,1} - \frac{1}{12}S_{2,2} \\
& -\frac{36719}{16200}S_2 + \frac{32}{3}S_2\zeta_3 + 10S_{2,-3} - \frac{218}{9}S_{2,-2} + \frac{23}{18}S_{1,1,2} - \frac{263}{60}S_{2,1} - \frac{8}{3}S_{2,1,-2} - 25S_1\zeta_3 \\
& -\frac{2}{3}S_{2,3} - \frac{4537}{1620}S_3 + \frac{28}{3}S_{3,-2} - \frac{208}{9}S_{1,-3} - \frac{112}{9}\mathbf{N}_- S_{1,1,-2} + \mathbf{N}_+\Big[\frac{4}{9}S_{3,1,1} - \frac{5}{3}S_{2,1,1} + \frac{4}{9}S_{4,1} \\
& -\frac{23}{9}S_5\Big] + (\mathbf{N}_{-3} - \mathbf{N}_{-2})\Big[\frac{2}{5}S_1\zeta_3 + \frac{2}{5}S_{1,-3} - \frac{119}{450}S_{1,-2} - \frac{2}{15}S_{1,-2,1} - \frac{2}{3}S_{1,1}\zeta_3 - \frac{2}{15}S_{1,1,-2} \\
& +\frac{1}{3}S_{1,1,3} - \frac{1}{3}S_{1,3,1} + \frac{4}{15}S_{2,-2}\Big] + (\mathbf{N}_{-2} - \mathbf{N}_-)\Big[\frac{2}{5}S_3 - \frac{179}{450}S_1 + \frac{2}{3}S_1\zeta_3 + \frac{1}{5}S_{1,1} - \frac{59}{450}S_2 \\
& +\frac{1}{5}S_{2,1} - \frac{1}{3}S_{1,3} + \frac{2}{15}S_{1,-2} + \frac{1}{3}S_{3,1}\Big] + (\mathbf{N}_- - 1)\Big[4S_{1,-4} + 2S_{1,-2,-2} + \frac{2}{3}S_{1,2,2}\Big] \\
& +(1-\mathbf{N}_+)\Big[\frac{11057}{324}S_1 - \frac{21}{2}S_1\zeta_3 + \frac{4}{9}S_{1,-3} - \frac{7}{3}S_{1,-2} - \frac{14}{3}S_{1,-2,-2} - \frac{8}{3}S_{1,-2,1} + \frac{3559}{216}S_{1,1} \\
& -8S_{1,1}\zeta_3 - 8S_{1,1,-3} + \frac{176}{9}S_{1,1,-2} + \frac{217}{36}S_{1,1,1} + \frac{16}{3}S_{1,1,1,-2} + \frac{2}{3}S_{1,1,1,2} - \frac{7}{36}S_{1,1,2} \\
& -\frac{2}{3}S_{1,1,2,1} + \frac{2}{3}S_{1,1,3} - \frac{55}{9}S_{1,2} - \frac{16}{3}S_{1,2,-2} + \frac{7}{36}S_{1,2,1} - \frac{4}{3}S_{1,2,2} + \frac{38}{9}S_{1,3} + \frac{8}{3}S_{1,3,1} \\
& -4S_{1,4} - \frac{231037}{5400}S_2 + 3S_2\zeta_3 - 6S_{2,-3} + \frac{118}{9}S_{2,-2} + \frac{20}{3}S_{2,-2,1} - \frac{793}{90}S_{2,1} - \frac{16}{3}S_{2,1,-2} \\
& +\frac{1}{6}S_{2,1,2} - \frac{4}{9}S_{2,2} - \frac{1}{6}S_{2,2,1} - 2S_{2,3} + \frac{49717}{1620}S_3 - \frac{22}{3}S_{3,-2} - \frac{47}{54}S_{3,1} - \frac{1}{6}S_{3,2} - \frac{166}{27}S_4 \\
& +\frac{5}{6}S_{4,1}\Big] + (\mathbf{N}_+ - \mathbf{N}_{+2})\Big[10S_1\zeta_3 - \frac{219}{50}S_1 + \frac{6}{5}S_{1,-2} + 4S_{1,-2,1} + \frac{24}{5}S_{1,1} + 4S_{1,1}\zeta_3 \\
& -4S_{1,1,-2} - 2S_{1,1,3} + 3S_{1,2} - 3S_{1,3} + 2S_{1,3,1} - \frac{21}{50}S_2 - 4S_2\zeta_3 + 4S_{2,-2} - \frac{24}{5}S_{2,1} + 2S_{2,3} \\
& -\frac{3}{5}S_3 - S_{3,1} - 2S_{4,1}\Big] + (\mathbf{N}_{+2} - \mathbf{N}_{+3})\Big[3S_{2,2} - 3S_{1,1,2} - \frac{72}{5}S_1\zeta_3 + \frac{18}{5}S_{1,-3} - \frac{159}{50}S_{1,-2} \\
& -\frac{6}{5}S_{1,-2,1} - 6S_{1,1}\zeta_3 - \frac{6}{5}S_{1,1,-2} + 3S_{1,1,3} + 3S_{1,2,1} + 6S_2\zeta_3 - 3S_{1,3,1} + \frac{6}{5}S_{2,-2} + 3S_{4,1} \\
& -3S_{2,3} - \frac{159}{50}S_3 - \frac{21}{5}S_{3,1} + \frac{18}{5}S_4\Big] + (\mathbf{N}_- + \mathbf{N}_+)\Big[\frac{1711}{108}S_1\zeta_3 - \frac{5608067}{291600}S_1 - \frac{1}{2}S_1\zeta_4 \\
& -\frac{79}{9}S_{1,-4} + 4S_{1,1,-2} + \frac{392}{27}S_{1,-3} + \frac{2}{9}S_{1,-3,1} - \frac{6613}{405}S_{1,-2} - \frac{104}{27}S_{1,-2,1} + \frac{4}{9}S_{1,-2,1,1}
\end{aligned}
$$

$$+\frac{8}{3}S_{1,1}\zeta_3-\frac{25511}{1620}S_{1,1}+\frac{52}{9}S_{1,1,-3}-\frac{184}{27}S_{1,1,1}-\frac{32}{9}S_{1,1,1,-2}-\frac{11}{9}S_{1,1,1,1}-\frac{17}{18}S_{1,1,1,2}$$
$$+\frac{1}{36}S_{1,1,2}+\frac{4}{9}S_{1,1,2,1}+\frac{13}{18}S_{1,1,3}+\frac{661}{108}S_{1,2}+\frac{34}{9}S_{1,2,-2}+\frac{29}{12}S_{1,2,1}+\frac{1}{2}S_{1,2,1,1}-4S_{1,3}+S_{4,1}$$
$$-\frac{8}{3}S_{1,3,1}+\frac{17}{9}S_{1,4}+\frac{44537}{2700}S_2-\frac{23}{3}S_2\zeta_3-\frac{79}{9}S_{2,-3}+\frac{377}{27}S_{2,-2}+\frac{20}{9}S_{2,-2,1}+\frac{5731}{540}S_{2,1}$$
$$+\frac{34}{9}S_{2,1,-2}+\frac{59}{18}S_{2,1,1}+\frac{5}{9}S_{2,1,2}-\frac{53}{18}S_{2,2}-\frac{5}{9}S_{2,2,1}+\frac{13}{18}S_{2,3}-\frac{4511}{405}S_3-\frac{67}{9}S_{3,-2}-\frac{83}{18}S_{3,1}$$
$$-\frac{5}{6}S_{3,1,1}+\frac{1}{2}S_{3,2}+\frac{253}{54}S_4\Big]\Big)+16C_Fn_f^2\Big(\frac{1}{18}S_{1,1,1}-\frac{9517}{7776}-\frac{1}{18}\zeta_3-\frac{757}{648}S_1-\frac{29}{108}S_{1,1}$$
$$-\frac{1}{18}S_{1,2}-\frac{43}{324}S_2-\frac{1}{6}S_{2,1}+\frac{19}{54}S_3+(1-\mathbf{N}_+)\Big[\frac{13}{18}S_{2,1}-\frac{19}{18}S_3+\frac{265}{108}S_2+\frac{7}{18}S_{1,2}-\frac{7}{18}S_{1,1,1}$$
$$-\frac{133}{108}S_{1,1}-\frac{1421}{648}S_1\Big]+(\mathbf{N}_-+\mathbf{N}_+)\Big[\frac{5585}{5832}S_1+\frac{1}{27}S_1\zeta_3+\frac{161}{324}S_{1,1}+\frac{13}{54}S_{1,1,1}+\frac{1}{9}S_{1,1,1,1}$$
$$-\frac{1}{9}S_{1,1,2}-\frac{13}{54}S_{1,2}-\frac{1}{9}S_{1,2,1}+\frac{1}{9}S_{1,3}-\frac{301}{324}S_2-\frac{29}{54}S_{2,1}-\frac{2}{9}S_{2,1,1}+\frac{2}{9}S_{2,2}+\frac{62}{81}S_3+\frac{1}{3}S_{3,1}$$
$$-\frac{23}{54}S_4\Big]\Big)+16C_F^2n_f\Big(\frac{119}{150}\zeta_3\delta(N-2)-\frac{341}{576}+\frac{139}{6}\zeta_3-\frac{3}{4}\zeta_4-\frac{46}{9}S_{-5}+\frac{8}{9}S_{-4,1}-\frac{382}{81}S_{-3}$$
$$+\frac{83}{27}S_{-4}+\frac{32}{3}S_{-3,-2}-\frac{40}{27}S_{-3,1}+\frac{8}{9}S_{-3,1,1}+\frac{29}{9}S_{-2}+\frac{28}{3}S_{-2}\zeta_3-\frac{16}{3}S_{-2,-2,1}+\frac{66367}{129600}S_1$$
$$+\frac{32}{3}S_{-2,-3}-\frac{26}{3}S_{-2,-2}+\frac{301}{6}S_1\zeta_3-4S_{1,-4}+\frac{416}{9}S_{1,-3}-\frac{8264}{135}S_{1,-2}+\frac{8}{3}S_{1,-2,-2}$$
$$-\frac{112}{9}S_{1,-2,1}+\frac{83}{144}S_{1,1}+\frac{20}{3}S_{1,1}\zeta_3+8S_{1,1,-3}-\frac{224}{9}S_{1,1,-2}-\frac{557}{72}S_{1,1,1}-\frac{16}{3}S_{1,1,1,-2}$$
$$-\frac{2}{3}S_{1,1,1,2}-\frac{31}{18}S_{1,1,2}+\frac{2171}{216}S_{1,2}+\frac{16}{3}S_{1,2,-2}+\frac{7}{3}S_{1,2,1}+\frac{2}{3}S_{1,2,2}-\frac{235}{36}S_{1,3}+4S_{1,4}$$
$$-\frac{10}{3}S_{1,3,1}-\frac{140237}{16200}S_2-22S_2\zeta_3-20S_{2,-3}+\frac{436}{9}S_{2,-2}+8S_{2,-2,1}+\frac{7133}{1080}S_{2,1}+\frac{16}{3}S_{2,1,-2}$$
$$+\frac{50}{9}S_{2,1,1}-4S_{2,2}+\frac{7}{3}S_{2,3}-\frac{7627}{3240}S_3+\mathbf{N}_+\Big[S_{2,1,1,1}-S_{2,1,2}+\frac{2}{3}S_{1,1,2,1}-S_{2,2,1}-\frac{56}{3}S_{3,-2}$$
$$-\frac{329}{27}S_{3,1}-\frac{29}{9}S_{3,1,1}+\frac{7}{3}S_{3,2}+\frac{401}{54}S_4+\frac{53}{18}S_{4,1}+\frac{1}{18}S_5\Big]+(\mathbf{N}_{-3}-\mathbf{N}_{-2})\Big[\frac{119}{225}S_{1,-2}$$
$$-\frac{4}{5}S_1\zeta_3+\frac{4}{15}S_{1,-2,1}-\frac{4}{5}S_{1,-3}+\frac{4}{15}S_{1,1,-2}-\frac{8}{15}S_{2,-2}\Big]+(\mathbf{N}_{-2}-\mathbf{N}_-)\Big[\frac{179}{225}S_1-\frac{4}{15}S_{1,-2}$$
$$+\frac{4}{15}S_{1,1}+\frac{59}{225}S_2+\frac{4}{15}S_{2,1}-\frac{4}{5}S_3\Big]+\frac{15439}{1440}(\mathbf{N}_--1)S_{1,1}+(\mathbf{N}_+-\mathbf{N}_{+2})\Big[\frac{219}{25}S_1-\frac{12}{5}S_{1,-2}$$
$$+4S_1\zeta_3-8S_{1,-2,1}-\frac{13}{5}S_{1,1}+8S_{1,1,-2}+2S_{1,1,2}-5S_{1,2}-2S_{1,2,1}-\frac{154}{25}S_2+\frac{36}{5}S_3+10S_{3,1}$$
$$+\frac{13}{5}S_{2,1}-2S_{2,2}-8S_{2,-2}\Big]+(\mathbf{N}_{+2}-\mathbf{N}_{+3})\Big[5S_{1,1,2}-\frac{12}{5}S_{2,-2}-5S_{2,2}+\frac{37}{5}S_{3,1}+\frac{159}{25}S_{1,-2}$$
$$+\frac{159}{25}S_3-5S_{1,2,1}+\frac{12}{5}S_{1,1,-2}+\frac{12}{5}S_{1,-2,1}+\frac{114}{5}S_1\zeta_3-\frac{36}{5}S_{1,-3}-\frac{36}{5}S_4\Big]+(1-\mathbf{N}_+)\Big[4S_4$$
$$-\frac{10337}{576}S_1-8S_{1,-4}-\frac{8}{9}S_{1,-3}+\frac{14}{3}S_{1,-2}+\frac{16}{3}S_{1,-2,-2}+\frac{16}{3}S_{1,-2,1}+\frac{40}{3}S_{1,1}\zeta_3+16S_{1,1,-3}$$
$$-\frac{76157}{4320}S_{1,1}-\frac{128}{9}S_{1,1,-2}+\frac{179}{72}S_{1,1,1}-\frac{32}{3}S_{1,1,1,-2}+\frac{14}{3}S_{1,1,1,1}-\frac{4}{3}S_{1,1,1,2}+\frac{565897}{16200}S_2$$

$$
\begin{aligned}
&+2S_{1,1,2,1}-\frac{295}{72}S_{1,2}+\frac{32}{3}S_{1,2,-2}-\frac{17}{3}S_{1,2,1}+\frac{4}{3}S_{1,2,2}-\frac{115}{36}S_{1,3}+8S_{1,4}-\frac{29}{6}S_{1,1,2}-\frac{2}{3}S_2\zeta_3\\
&-\frac{20}{3}S_{1,3,1}+12S_{2,-3}-\frac{236}{9}S_{2,-2}-\frac{40}{3}S_{2,-2,1}-\frac{4663}{1080}S_{2,1}+\frac{32}{3}S_{2,1,-2}-\frac{37}{3}S_{2,1,1}-\frac{22817}{810}S_3\\
&+\frac{32}{3}S_{2,2}+\frac{5}{3}S_{2,3}-4S_{3,-2}+\frac{65}{18}S_{3,1}+\frac{85}{6}S_1\zeta_3\Big]+(\mathbf{N}_-+\mathbf{N}_+)\Big[\frac{86}{9}S_{1,-4}-\frac{91}{36}S_5-4S_{3,1,1}\\
&-\frac{8}{9}S_{1,-2,1,1}+\frac{13226}{405}S_{1,-2}-\frac{104}{9}S_{1,1,-3}+\frac{152}{9}S_{1,1,-2}-\frac{784}{27}S_{1,-3}-4S_{1,-2,-2}+\frac{41929}{129600}S_1\\
&-\frac{244}{9}S_1\zeta_3+\frac{1}{2}S_1\zeta_4+\frac{208}{27}S_{1,-2,1}-\frac{4}{9}S_{1,-3,1}-\frac{13}{3}S_{1,1}\zeta_3-\frac{7}{8}S_{1,1,1}+\frac{64}{9}S_{1,1,1,-2}-\frac{35}{6}S_{1,1,1,1}\\
&-\frac{10}{3}S_{1,1,1,1,1}+\frac{44}{9}S_{1,1,1,2}+\frac{25}{3}S_{1,1,2}+3S_{1,1,2,1}-\frac{91}{18}S_{1,1,3}+\frac{577}{648}S_{1,2}+\frac{40}{3}S_2\zeta_3+\frac{107}{18}S_{1,2,1}\\
&+\frac{34}{9}S_{1,2,1,1}-\frac{79151}{64800}S_2-\frac{68}{9}S_{1,2,-2}-\frac{239}{54}S_{2,2}-\frac{49}{9}S_{1,2,2}-\frac{275}{108}S_{1,3}+\frac{73}{18}S_{2,1,1,1}-\frac{31}{6}S_{2,1,2}\\
&+\frac{158}{9}S_{2,-3}-\frac{754}{27}S_{2,-2}-\frac{40}{9}S_{2,-2,1}+\frac{133}{36}S_{4,1}-\frac{403}{810}S_{2,1}-\frac{68}{9}S_{2,1,-2}+\frac{79}{18}S_{2,1,1}-\frac{67}{18}S_{2,2,1}\\
&+\frac{17}{18}S_{1,4}+\frac{5}{72}S_4-\frac{5}{18}S_{1,3,1}+\frac{25}{9}S_{2,3}+\frac{7871}{3240}S_3+\frac{134}{9}S_{3,-2}-\frac{241}{72}S_{3,1}+\frac{38}{9}S_{3,2}\Big]\Big)\ . \qquad \text{(A.1)}
\end{aligned}
$$

For the sake of completeness, we include the result for the complete first and second-order longitudinal coefficient functions $c^{(1)}_{L,\mathrm{ns}}$ and $c^{(2)}_{L,\mathrm{ns}}$ known from refs. [56, 70, 2, 6]

$$
\begin{aligned}
c^{(1)}_{L,\mathrm{ns}}(N) \;=\;& -4C_F(1-\mathbf{N}_+)S_1\ , \qquad \text{(A.2)}\\
c^{(2)}_{L,\mathrm{ns}}(N) \;=\;& 4C_AC_F\Big(\frac{12}{5}\zeta_3\delta(N{-}2)+\frac{12}{5}(\mathbf{N}_+-\mathbf{N}_{+2})[S_1-S_2]+\frac{12}{5}(\mathbf{N}_{+2}-\mathbf{N}_{+3})[S_{1,-2}+S_3]\\
&-\frac{98}{15}S_1+\frac{8}{5}S_2+\frac{8}{5}(\mathbf{N}_{-3}-\mathbf{N}_{-2})S_{1,-2}+8(\mathbf{N}_--1)S_{1,-2}+\frac{8}{5}(\mathbf{N}_{-2}-\mathbf{N}_-)[S_1+S_2]\\
&+(1-\mathbf{N}_+)\Big[12S_1\zeta_3-4S_{1,-2}-\frac{23}{3}S_{1,1}-8S_{1,1,-2}-4S_{1,3}-\frac{287}{18}S_1+4S_{1,-3}+\frac{176}{15}S_2-4S_3\Big]\\
&+(\mathbf{N}_-+\mathbf{N}_+)\Big[\frac{49}{15}S_1-\frac{4}{5}S_2\Big]\Big)+4C_Fn_f\Big((1-\mathbf{N}_+)\Big[\frac{2}{3}S_{1,1}-\frac{4}{3}S_2+\frac{25}{9}S_1\Big]-\frac{2}{3}(\mathbf{N}_--1)S_1\Big)\\
&+4{C_F}^2\Big(-\frac{24}{5}\zeta_3\delta(N{-}2)+\frac{22}{5}S_1+\frac{4}{5}S_2-\frac{16}{5}(\mathbf{N}_{-3}-\mathbf{N}_{-2})S_{1,-2}-\frac{16}{5}(\mathbf{N}_{-2}-\mathbf{N}_-)[S_1+S_2]\\
&+(\mathbf{N}_--1)[2S_{1,1}-16S_{1,-2}]-\frac{24}{5}(\mathbf{N}_+-\mathbf{N}_{+2})[S_1-S_2]-\frac{24}{5}(\mathbf{N}_{+2}-\mathbf{N}_{+3})[S_{1,-2}+S_3]\\
&+(1-\mathbf{N}_+)\Big[\frac{33}{2}S_1+8S_{1,3}-24S_1\zeta_3-8S_{1,-3}+8S_{1,-2}+7S_{1,1}+16S_{1,1,-2}-4S_{1,1,1}+4S_{1,2}\\
&-\frac{54}{5}S_2+6S_{2,1}+4S_3\Big]-(\mathbf{N}_-+\mathbf{N}_+)\Big[\frac{11}{5}S_1+\frac{2}{5}S_2\Big]\Big)\ . \qquad \text{(A.3)}
\end{aligned}
$$

The contributions to the 3-loop longitudinal coefficient function corresponding to eq. (A.1) read

$$
\begin{aligned}
c^{(3)}_{L,\mathrm{ns}}(N) \;=\;& 16C_AC_Fn_f\Big(\delta(N{-}2)\Big[\frac{20}{3}\zeta_5-\frac{149}{75}\zeta_3\Big]+(\mathbf{N}_{-3}-\mathbf{N}_{-2})\Big[\frac{16}{15}S_{2,-2}-\frac{4}{3}S_{1,3,1}+\frac{4}{3}S_{1,1,3}\\
&-\frac{8}{3}S_{1,1}\zeta_3-\frac{8}{15}S_{1,1,-2}-\frac{8}{15}S_{1,-2,1}+\frac{8}{5}S_{1,-3}-\frac{298}{225}S_{1,-2}+\frac{8}{5}S_1\zeta_3\Big]+(\mathbf{N}_{-2}-\mathbf{N}_-)\Big[\frac{4}{3}S_{1,1}\zeta_3\\
&-\frac{418}{225}S_1+\frac{8}{5}S_3+\frac{8}{15}S_{1,-2}+\frac{4}{3}S_{1,-2,1}-\frac{4}{3}S_{1,1,-2}+\frac{4}{5}S_{1,1}-\frac{178}{225}S_2+\frac{4}{3}S_{3,1}+4S_1\zeta_3+\frac{4}{5}S_{2,1}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{2}{3}S_{1,1,3}-\frac{4}{3}S_{1,3}+\frac{2}{3}S_{1,3,1}\Big]+(\mathbf{N}_{-}-1)\Big[\frac{16}{3}S_{2,-2}-\frac{13033}{1350}S_1+8S_1\zeta_3-\frac{518}{45}S_{1,-2}-\frac{2}{5}S_{2,1}\\
&-\frac{8}{3}S_{1,-2,1}-\frac{254}{45}S_{1,1}-\frac{2}{15}S_3+8S_{1,-3}+\frac{76}{25}S_2-\frac{8}{3}S_{1,1,-2}\Big]+(\mathbf{N}_{+}-\mathbf{N}_{+2})\Big[\frac{8}{3}S_{1,-2,1}-\frac{8}{3}S_2\zeta_3\\
&-\frac{7}{25}S_2-\frac{8}{3}S_{1,1,-2}-2S_{1,3}+\frac{4}{3}S_{1,3,1}-\frac{4}{3}S_{1,1,3}+2S_{1,2}+\frac{16}{5}S_{1,1}+\frac{8}{3}S_{1,1}\zeta_3+\frac{4}{5}S_{1,-2}-\frac{16}{5}S_{2,1}\\
&+\frac{20}{3}S_1\zeta_3-\frac{73}{25}S_1-\frac{2}{5}S_3-\frac{2}{3}S_{3,1}-\frac{4}{3}S_{4,1}+\frac{4}{3}S_{2,3}+\frac{8}{3}S_{2,-2}\Big]+(\mathbf{N}_{+2}-\mathbf{N}_{+3})\Big[\frac{4}{5}S_{2,-2}+2S_{2,2}\\
&-\frac{4}{5}S_{1,-2,1}-2S_{2,3}-\frac{14}{5}S_{3,1}-\frac{53}{25}S_3+\frac{12}{5}S_4+2S_{4,1}-\frac{48}{5}S_1\zeta_3+\frac{12}{5}S_{1,-3}+2S_{1,1,3}-4S_{1,1}\zeta_3\\
&-\frac{4}{5}S_{1,1,-2}-\frac{53}{25}S_{1,-2}+2S_{1,2,1}-2S_{1,3,1}+4S_2\zeta_3-2S_{1,1,2}\Big]+(1-\mathbf{N}_{+})\Big[\frac{125599}{4050}S_1-\frac{40}{3}S_1\zeta_3\\
&+4S_{1,-4}-\frac{68}{9}S_{1,-3}+\frac{413}{45}S_{1,-2}-\frac{8}{3}S_{1,-2,-2}+\frac{4469}{270}S_{1,1}-8S_{1,1}\zeta_3-8S_{1,1,-3}+\frac{88}{9}S_{1,1,-2}\\
&+\frac{40}{9}S_{1,1,1}+\frac{16}{3}S_{1,1,1,-2}+\frac{2}{3}S_{1,1,1,2}+\frac{2}{3}S_{1,1,2}-\frac{2}{3}S_{1,1,2,1}+\frac{2}{3}S_{1,1,3}-\frac{40}{9}S_{1,2}-\frac{16}{3}S_{1,2,-2}\\
&-\frac{10}{3}S_4-\frac{2}{3}S_{1,2,1}-\frac{2}{3}S_{1,2,2}+\frac{2}{3}S_{3,1}+\frac{8}{3}S_{1,3,1}-4S_{1,4}-\frac{20953}{675}S_2+8S_2\zeta_3+\frac{4}{3}S_{2,-3}-\frac{2}{3}S_{2,-2}\\
&+\frac{8}{3}S_{2,-2,1}-\frac{238}{45}S_{2,1}-\frac{16}{3}S_{2,1,-2}-2S_{2,3}+\frac{824}{45}S_3+\frac{38}{9}S_{1,3}+\frac{2}{3}S_{4,1}\Big]\Big)\\
&+16C_Fn_f^{\,2}\Big((\mathbf{N}_{-}-1)\Big[\frac{19}{27}S_1+\frac{2}{9}S_{1,1}-\frac{2}{9}S_2\Big]+(1-\mathbf{N}_{+})\Big[\frac{2}{9}S_{1,2}-\frac{317}{162}S_1-\frac{25}{27}S_{1,1}-\frac{2}{9}S_{1,1,1}\\
&+\frac{50}{27}S_2+\frac{4}{9}S_{2,1}-\frac{2}{3}S_3\Big]\Big)+16C_F^{\,2}n_f\Big(\frac{298}{75}\zeta_3\delta(N-2)+(\mathbf{N}_{-3}-\mathbf{N}_{-2})\Big[\frac{16}{15}S_{1,1,-2}-\frac{16}{5}S_{1,-3}\\
&+\frac{16}{15}S_{1,-2,1}+\frac{596}{225}S_{1,-2}-\frac{32}{15}S_{2,-2}-\frac{16}{5}S_1\zeta_3\Big]+(\mathbf{N}_{-2}-\mathbf{N}_{-})\Big[\frac{16}{15}S_{2,1}-\frac{16}{5}S_3+\frac{356}{225}S_2\\
&+\frac{836}{225}S_1-\frac{8}{3}S_1\zeta_3+\frac{16}{15}S_{1,1}-\frac{8}{3}S_{1,-2,1}+\frac{8}{3}S_{1,1,-2}-\frac{16}{15}S_{1,-2}\Big]+(\mathbf{N}_{-}-1)\Big[\frac{2141}{1350}S_1-16S_1\zeta_3\\
&-16S_{1,-3}+\frac{1036}{45}S_{1,-2}+\frac{16}{3}S_{1,-2,1}+\frac{73}{45}S_{1,1}+\frac{16}{3}S_{1,1,-2}-\frac{5}{3}S_{1,1,1}+\frac{5}{3}S_{1,2}+\frac{232}{225}S_2+\frac{1}{10}S_3\\
&-\frac{32}{3}S_{2,-2}+\frac{14}{5}S_{2,1}\Big]+(\mathbf{N}_{+}-\mathbf{N}_{+2})\Big[\frac{146}{25}S_1-\frac{10}{3}S_{1,2}-\frac{308}{75}S_2+\frac{20}{3}S_{3,1}-\frac{4}{3}S_{2,2}+\frac{8}{3}S_1\zeta_3\\
&-\frac{16}{3}S_{2,-2}-\frac{4}{3}S_{1,2,1}+\frac{16}{3}S_{1,1,-2}+\frac{4}{3}S_{1,1,2}-\frac{8}{5}S_{1,-2}-\frac{16}{3}S_{1,-2,1}-\frac{26}{15}S_{1,1}+\frac{24}{5}S_3+\frac{26}{15}S_{2,1}\Big]\\
&+(\mathbf{N}_{+2}-\mathbf{N}_{+3})\Big[\frac{76}{5}S_1\zeta_3+\frac{106}{25}S_{1,-2}+\frac{8}{5}S_{1,-2,1}-\frac{24}{5}S_{1,-3}+\frac{10}{3}S_{1,1,2}+\frac{8}{5}S_{1,1,-2}-\frac{10}{3}S_{1,2,1}\\
&-\frac{8}{5}S_{2,-2}+\frac{74}{15}S_{3,1}-\frac{24}{5}S_4+\frac{106}{25}S_3-\frac{10}{3}S_{2,2}\Big]+(1-\mathbf{N}_{+})\Big[\frac{74}{3}S_1\zeta_3-\frac{173653}{10800}S_1-8S_{1,-4}\\
&+\frac{136}{9}S_{1,-3}-\frac{826}{45}S_{1,-2}+\frac{16}{3}S_{1,-2,-2}-\frac{1609}{135}S_{1,1}+\frac{40}{3}S_{1,1}\zeta_3+16S_{1,1,-3}-\frac{176}{9}S_{1,1,-2}\\
&+\frac{7}{18}S_{1,1,1}-\frac{32}{3}S_{1,1,1,-2}+\frac{8}{3}S_{1,1,1,1}-\frac{4}{3}S_{1,1,1,2}+8S_{1,4}-4S_{1,1,2}+\frac{4}{3}S_{1,1,2,1}-\frac{23}{18}S_{1,2}+\frac{17}{6}S_4\\
&+\frac{32}{3}S_{1,2,-2}-2S_{1,2,1}+\frac{4}{3}S_{1,2,2}-\frac{49}{9}S_{1,3}-\frac{20}{3}S_{1,3,1}+\frac{30737}{1350}S_2-\frac{40}{3}S_2\zeta_3-\frac{8}{3}S_{2,-3}+\frac{4}{3}S_{2,-2}\\
&-\frac{16}{3}S_{2,-2,1}-\frac{17}{10}S_{2,1}+\frac{32}{3}S_{2,1,-2}-\frac{13}{3}S_{2,1,1}+\frac{13}{3}S_{2,2}+\frac{8}{3}S_{2,3}-\frac{913}{45}S_3+\frac{13}{3}S_{3,1}\Big]\Big)\,. \qquad \text{(A.4)}
\end{aligned}
$$

Recall that in all our formulae the expansion parameter is normalized as in eq. (2). The operators $\mathbf{N}_{\pm}$ and $\mathbf{N}_{\pm \mathbf{i}}$ have been defined in eq. (4).

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