Special case of sunset: reduction and ε-expansion.

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Abstract

We consider two loop sunset diagrams with two mass scales m and M at the threshold and pseudotreshold that cannot be treated by earlier published formula. The complete reduction to master integrals is given. The master integrals are evaluated as series in ratio m/M and in ε with the help of differential equation method. The rules of asymptotic expansion in the case when q^2 is at the (pseudo)threshold are given.

1 Introduction

The sunset diagram plays a key role in the 2-loop calculations with masses. Despite the fact that a lot of investigation has been devoted to the sunset diagram there still remain drawbacks. In Ref. [1] general reduction procedure is given in the case, when external momentum q and internal masses are arbitrary. But in the case when q^2 is equal to the threshold or one of the pseudothresholds values the formula of [1] become unapplicable. Therefore we turn to paper [2] where the reduction was given specifically for the (pseudo)threshold kinematics. However in two cases shown in Fig.1 the reduction of [2] fails.

In present paper we consider calculation of these two special cases. These integrals naturally arise in the threshold problems with given mass hierarchy. An immediate typical example, where it is the case, is a problem of matching vector and axial QCD currents to NRQCD ones with two heavy quark mass scales $m \ll M$. Another example is the calculation of masses of the heavy gauge bosons in 2-loop approximation [4], when q^2 is equal to m_Z^2 or m_W^2.

For the calculation of master integrals there are mainly three methods: 1) direct evaluation using α or Feynman parameter representation, 2) solving master differential equation in external Mandelstam variables, which can be written for the master integrals of any Feynman graph, 3) applying various asymptotic expansions [5, 6, 7]. Here, we will demonstrate the strong and weak features of the each mentioned method on the example of our particular problem and will advocate that a certain mixing of these methods can give us a desired answer in the most easiest way.

Now let us introduce notation, which will be used later in this paper and define master integrals we are going to calculate. For the two-loop sunset with arbitrary masses and propagator indices we have

$$J_{\nu_1\nu_2\nu_3}(q^2) = \frac{1}{\pi^d} \iint \frac{d^dk\ d^dl}{[k^2-m_1^2]^{\nu_1}[(k-l)^2-m_2^2]^{\nu_2}[(l-q)^2-m_3^2]^{\nu_3}}\,, \tag{1}$$

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where $d = 4 - 2\varepsilon$ is the dimension of space-time.

We will discuss only two special cases of the sunset integrals which are shown in Fig.1:

(a) $m_1 = M$, $m_2 = 0$, $m_3 = m$, $q^2 = (M+m)^2$ (threshold case),

(b) $m_1 = m$, $m_2 = m$, $m_3 = M$, $q^2 = M^2$ (pseudothreshold case).

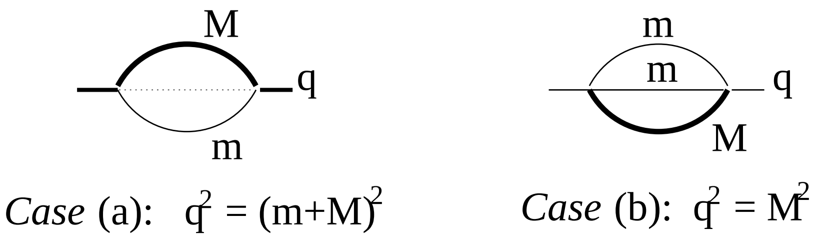

Figure 1: Sunset diagrams

With the help of recurrence relations [8], which will be given in next section, any integrals of the first type with arbitrary propagator indices can be reduced to two master integrals: $J_{111}((m+M)^2)$ and $J_{112}((m+M)^2)$, whereas in the second case to two master integrals: $J_{111}(M^2)$ and $J_{211}(M^2)$.

Another problem is the evaluation of the master integrals themselves. The general representation for the sunset diagram was obtained in Ref. [9] in terms of hypergeometric Lauricella function. For the practical purposes however one needs the ϵ-expansion of these formula which is not a trivial task.

The result for the threshold and pseudothreshold values of the sunset integrals with three arbitrary masses has been ontained in Ref. [10] up to $O(1)$ in ε-expansion. From these results we can easily obtain the finite parts of our integrals. However we established that in the reduction procedure for the matching of QCD to NRQCD currents one has to know also $O(\varepsilon)$ part for $J^{(a)}$ integrals and $O(\varepsilon^2)$ part for $J^{(b)}$ integrals. It is not easy to push forward the approach of [10] in order to evaluate these needed parts. Instead we use the differential equation method [11]-[13] (see also [14]). We solve differential equations as series in m/M and the desired number of coefficients can be always obtained. In order to find the boundary conditions for the solutions one can use two methods: the representation of Ref. [9] can be expanded in the limit $m/M \to 0$ and the asymptotic expansion procedure can be applied at the threshold and pseudothreshold.

2 Recurrence relations at the threshold and pseudothreshold

Here we give the recurrence relations [8] for the sunsets of two types introduced earlier. While the derivation of these recurrence relations is straightforward, they were not considered in the literature in detail till now. On the other hand they represent missing pieces to complete the generalized recurrence relations of Tarasov [1] and threshold relations of Davydychev and Smirnov [2] for the sunset diagrams in the case of threshold with one zero mass and pseudothreshold with two equal masses. Most of the formula below can be derived just by combining and reexpressing appropriate recurrence relations of [1] and [2].

2.1 JM0m

In this case we can not directly apply recurrence relations for general masses obtained in [2]. The point is as one of mass becomes zero the relations of [2] are degenerate. Instead we derive:

$$(d-2\nu_2-2)\,\nu_2\mathbf{2}^+ = -2m^2\nu_3(\nu_3+1)\,\mathbf{3}^{++} + (d-2\nu_3-2)\,\nu_3\mathbf{3}^+\,, \tag{2}$$

$$2M^2\nu_1(\nu_1+1)\,\mathbf{1}^{++} = 2m^2\nu_3(\nu_3+1)\,\mathbf{3}^{++} - (d-2\nu_3-2)\,\nu_3\mathbf{3}^+ + (d-2\nu_1-2)\,\nu_1\mathbf{1}^+\,, \tag{3}$$

$$\begin{aligned}
&2m^2(M+m)(3d-2\nu_1-2\nu_3-7)\nu_3(\nu_3+1)\mathbf{3}^{++} = \\
&\qquad \Big[m\Big((d-\nu_3-3)(2d-\nu_1-2\nu_3-4)+(d-\nu_1-\nu_3-2)(3d-2\nu_2-2\nu_3-6)\Big) \\
&\quad +M(2d-\nu_1-2\nu_3-5)(2d-\nu_1-2\nu_3-4)\Big]\,\nu_3\mathbf{3}^+ \\
&\quad +(d-\nu_3-2)\Big(m(d-\nu_3-3)+M(2d-\nu_1-2\nu_3-5)\Big)\,\nu_1\mathbf{1}^+ \\
&\quad +2m^2M\nu_1\nu_3(\nu_3+1)\mathbf{1}^+\mathbf{2}^-\mathbf{3}^{++} \\
&\quad +\Big[-m(d-\nu_3-3)-M(2d-\nu_1-2\nu_3-5)\Big]\,\nu_1\nu_3\mathbf{1}^+\mathbf{2}^-\mathbf{3}^+\,,
\end{aligned} \tag{4}$$

$$\begin{aligned}
&2mM(m+M)(3d-11)\,J_{212} = \\
&\quad -(d-3)\Big(m(d-3)+M(2d-7)\Big)\,J_{112} - (d-3)\Big(m(2d-7)+M(d-3)\Big)\,J_{211} \\
&\quad +\frac{m+M}{4m^2M^2}(d-2)^2(2d-7)\,J_{101}\,.
\end{aligned} \tag{5}$$

Here as usual $\mathbf{j}^+$ (or $\mathbf{j}^-$) means the operator raising (or diminishing) index on the j-th line.

With the help of (2)-(5) we reduce all integrals to three sunset diagrams J_{111}, J_{112} and J_{211} plus products of 1-loop tadpoles. In addition there is a relation between these three sunset integrals and one of them can be eliminated

$$M(m+2M)\,J_{211} + m(M+2m)\,J_{112} = \frac{3d-8}{2}\,J_{111} + \frac{(d-2)^2}{4mM(d-3)}\,J_{101}\,. \tag{6}$$

This finishes the reduction procedure.

2.2 JmmM

Here we are faced with pseudothreshold problem and again in this case general recurrence relations by Tarasov [1] and those for the threshold problems by Smirnov and Davydichev [2] become degenerate. The reduction procedure for this type of integrals has been studied in [16], where the topology with three and four lines has been considered. These integrals were also considered in [3], where asymptotic expansion has been used. For the completeness we give alternative reduction formula below.

First, using relations

$$2m^2\nu_1(\nu_1+1)\,\mathbf{1}^{++} = (d-2\nu_1-2)\,\nu_1\mathbf{1}^+ - (d-2\nu_3-2)\,\nu_3\mathbf{3}^+ + 2M^2\nu_3(\nu_3+1)\,\mathbf{3}^{++}\,, \tag{7}$$

$$2m^2\nu_2(\nu_2+1)\,\mathbf{2}^{++} = (d-2\nu_2-2)\,\nu_2\mathbf{2}^+ - (d-2\nu_3-2)\,\nu_3\mathbf{3}^+ + 2M^2\nu_3(\nu_3+1)\,\mathbf{3}^{++}\,, \tag{8}$$

we reduce indices of lines 1 and 2 to one or two. Thus there remain only integrals $J_{11\nu_3}$, $J_{12\nu_3}$ and $J_{22\nu_3}$.

For $J_{12\nu_3}$ we have

$$4(M^2 - m^2)\,\nu_3\mathbf{3}^+ = \Big[\mathbf{1}^- - (d-3)\,\mathbf{2}^-\Big]\,\nu_3\mathbf{3}^+ - \mathbf{1}^+\mathbf{3}^- + d - 3\nu_3\,. \tag{9}$$

For $J_{22\nu_3}$ we have

$$\begin{aligned}
4m^2(M^2 - m^2)(d-\nu_3-3) = \\
& m^2(d-\nu_3-3)\,\mathbf{3}^- + \Big[m^2\Big(-3d^2 + d(13+7\nu_3) \\
& -12 - 17\nu_3 - 4\nu_3^2\Big) + M^2(2d-\nu_3-6)(d-\nu_3-2)\Big]\,\mathbf{1}^- \\
& +\Big[-m^2 + M^2(d-\nu_3-2)\Big](d-3)\,\nu_3\mathbf{3}^+\mathbf{1}^-\mathbf{2}^- \\
& +\Big[m^2(d-3) - M^2(2d-\nu_3-6)\Big]\,\nu_3\mathbf{3}^+\mathbf{1}^{--}\,.
\end{aligned} \tag{10}$$

For $J_{11\nu_3}$ we have

$$\begin{aligned}
4M^2(M^2 - m^2)(d-\nu_3-3)\nu_3(\nu_3+1)\,\mathbf{3}^{++} = \\
& \Big[-m^2(d-2\nu_3-3)(2d-2\nu_3-5) \\
& +M^2\Big(3d^2 - d(17+7\nu_3) + 24 + 19\nu_3 + 4\nu_3^2\Big)\Big]\,\nu_3\mathbf{3}^+ \\
& +\Big[-m^2\Big(d^2 - d(3+5\nu_3) + \nu_3(4\nu_3+11)\Big) - M^2(d-\nu_3-2)\nu_3\Big]\,\mathbf{2}^+ \\
& +\Big[-m^2(d-3) + M^2\nu_3\Big]\,\nu_3\mathbf{3}^+\mathbf{2}^+\mathbf{1}^- + m^2(d-\nu_3-3)\,\mathbf{3}^-\mathbf{1}^+\mathbf{2}^+\,.
\end{aligned} \tag{11}$$

Finaly there is relation between three integrals

$$M^2 J_{112} + m^2 J_{211} = \frac{3d-8}{4}J_{111} - \frac{(d-2)^2}{8m^2(d-3)}J_{110}\,. \tag{12}$$

This finishes the reduction.

3 Master differential equation

It is known, that a general two-loop sunset topology has four master integrals [1]: one with unit powers of propagators and three with a dot placed on one of the lines. For master integrals considered in this paper this number is however smaller, which is due to a symmetry of the mass distribution on the lines. In addition in case of $JM0m$ the index of massless line can be always reduced to 1 (see [1]). It can be shown, that these master integrals satisfy a sytem of linear nonhomogeneous differential equations in q^2 (q being external momentum) [13] or in masses [11]. However, in our case we want to write differential equations in $r = m/M$, where m and M are internal masses and q^2 lays at threshold or pseudothreshold. Differential equations of this type were considered earlier in [17].

3.1 JM0m

In this case we have two master integrals (J_{111} and J_{112}) and the system of two 1st order equations. It is convinient to rescale integrals introducing $\tilde{J}_{111}$ and $\tilde{J}_{112}$ by

$$J_{111} = M^{2d-6}\Gamma^2(3-d/2)\,\tilde{J}_{111} \quad \text{and} \quad J_{112} = M^{2d-8}\Gamma^2(3-d/2)\,\tilde{J}_{112}. \tag{13}$$

Then the differential equations read

$$\begin{cases} (r+1)(r+2)\dfrac{\partial}{\partial r}\tilde{J}_{111} - 6r(r+1)\tilde{J}_{112} - \Big(d + 2(d-3)r - 4\Big)\tilde{J}_{111} - \dfrac{8r^{d-3}}{(d-4)^2(d-3)} = 0\,, \\ (r+1)(r+2)\dfrac{\partial}{\partial r}\tilde{J}_{112} + \dfrac{1}{r}(2r^2 - 5dr + 20r - 4d + 14)\tilde{J}_{112} \\ \qquad\qquad - \dfrac{(d-3)(3d-8)}{2r}\tilde{J}_{111} - \dfrac{8r^{d-5}}{(d-4)^2} = 0\,. \end{cases} \tag{14}$$

Using this system one can write the second order differential equation for $\tilde{J}_{111}$:

$$(r+1)^2(r+2)^2\frac{\partial^2}{\partial r^2}\tilde{J}_{111} - \frac{2}{r}(r+1)(r+2)^2\Big(d + (d-4)r - 3\Big)\frac{\partial}{\partial r}\tilde{J}_{111}$$
$$+\frac{1}{r}(d-3)^2(r+2)^2(d-2r-4)\tilde{J}_{111} - \frac{8}{(d-4)^2}(r+2)^2 r^{d-4} = 0. \tag{15}$$

As usual we search the solution of (15) as a linear combination of two solutions of the homogeneous equation and the solution of the nonhomogeneous equation. We will find the solution as a series in r, namely, in the following form

$$\tilde{J}_{111} = \sum_i r^{\alpha_i}\left(\sum_{n=0}^{\infty} a_n^{(i)}(d)\, r^n\right). \tag{16}$$

By substituting (16) in Eq. (15), we find for leading exponents α_i three allowed values $\alpha_1 = 0$, $\alpha_2 = 2d-5$, corresponding to the two independent solutions of the associated homogeneous equation and $\alpha_3 = d-2$, as required by the nonhomogeneous part of Eq. (15). So the only thing we need yet to do is to find the coeffients in front of two independent solutions of homogeneous part of Eq. (15) using boundary conditions. One equation for these coefficients can be obtained from the value of J_{111} master integral at $r=0$, which can be written in terms of Γ-functions. One can *not* use J_{112} at $r=0$ as the second boundary condition. The reason is that the latter integral becomes infrared divergent and this divergency is regularized by ϵ ($d = 4-2\epsilon$) and not by r as in the limit $r \to 0$. In order to obtain a second boundary condition we need explicit expansion of function J_{111} in r up to the third order, which can be obtained by analysing representation of this integral in terms of Appel function F_4 or performing asymptotic expansion in small mass ratio r. Expansion of the Appel function will be considered in Appendix A, while asymptotic expansion technique is discussed in Section 4.1.

Having obtained expansion in r up to order $O(r^3)$ using one of the mentioned methods, one can easily reconstruct all other expansion coefficients from differential equation. Returning back to functions J_{111} and J_{112} instead of $\tilde{J}_{111}$ and $\tilde{J}_{112}$ we give the first seven coefficients of expansion:

$$e^{2\epsilon\gamma_E}M^{-2+4\epsilon}J_{111} = -\frac{1+r^2}{2\epsilon^2} + \frac{(8L-5)r^2 + 2r - 5}{4\epsilon} - \frac{1}{8}(11 + 20\zeta_2) + \frac{5r}{4}$$
$$-\frac{1}{8}(16L^2 - 48L - 12\zeta_2 + 7)r^2 - \frac{2}{9}(9L^2 + 15L + 54\zeta_2 - 8)r^3 + \frac{1}{8}(24L^2 + 20L + 144\zeta_2 - 3)r^4$$
$$-\frac{2}{225}(450L^2 + 255L + 2700\zeta_2 - 16)r^5 + \frac{1}{72}(360L^2 + 156L + 2160\zeta_2 - 5)r^6$$
$$+\epsilon\left(\frac{55}{16} - \frac{25\zeta_2}{4} - \frac{11\zeta_3}{3} + \frac{1}{8}(20\zeta_2 + 11)r + \left(\frac{4L^3}{3} - 6L^2 + 2(\zeta_2 + 7)L\right.\right.$$

$$+\frac{1}{48}(276\zeta_2+208\zeta_3+321)\Big)r^2+\left(4L^3+\frac{2L^2}{3}+(40\zeta_2-\frac{238}{9})L-\frac{68\zeta_2}{3}-24\zeta_3+\frac{773}{54}\right)r^3$$
$$+\left(-6L^3-\frac{L^2}{2}+(\frac{463}{12}-60\zeta_2)L+17\zeta_2+36\zeta_3-\frac{4639}{144}\right)r^4$$
$$+\left(8L^3+\frac{23L^2}{15}+(80\zeta_2-\frac{22681}{450})L-\frac{134\zeta_2}{15}-48\zeta_3+\frac{1242497}{27000}\right)r^5$$
$$+\left(-10L^3-\frac{19L^2}{6}+(\frac{3751}{60}-100\zeta_2)L-\frac{5\zeta_2}{3}+60\zeta_3-\frac{648161}{10800}\right)r^6\Big)+O(r^7)+O(\varepsilon^2) \quad (17)$$

and

$$e^{2\epsilon\gamma_E}M^{4\epsilon}J_{112}=-\frac{1}{2\epsilon^2}+\frac{2L-\frac{1}{2}}{\epsilon}+\frac{1}{4}(6\zeta_2+2)+4L-2L^2+(-2L^2-4L-12\zeta_2+2)r$$
$$+(3L^2+3L+18\zeta_2-\frac{1}{2})r^2+(-4L^2-\frac{8L}{3}-24\zeta_2+\frac{2}{9})r^3+(5L^2+\frac{5L}{2}+30\zeta_2-\frac{1}{8})r^4$$
$$+(-6L^2-\frac{12L}{5}-36\zeta_2+\frac{2}{25})r^5+(7L^2+\frac{7L}{3}+\frac{1}{18}(756\zeta_2-1))r^6$$
$$+\epsilon\Bigg(\frac{1}{12}(66\zeta_2+52\zeta_3+66)+8L+2L\zeta_2-4L^2+\frac{4L^3}{3}+(4L^3+4L^2+40\zeta_2L-24L$$
$$-8\zeta_2-24\zeta_3+14)r+(-6L^3-5L^2-60\zeta_2L+37L-6\zeta_2+36\zeta_3-\frac{67}{2})r^2$$
$$+\left(8L^3+\frac{22L^2}{3}+80\zeta_2L-\frac{440L}{9}+\frac{68\zeta_2}{3}-48\zeta_3+\frac{1277}{27}\right)r^3$$
$$+\left(-10L^3-\frac{31L^2}{3}-100\zeta_2L+\frac{1097L}{18}-42\zeta_2+60\zeta_3-\frac{4403}{72}\right)r^4$$
$$+\left(12L^3+\frac{69L^2}{5}+120\zeta_2L-\frac{5489L}{75}+\frac{318\zeta_2}{5}-72\zeta_3+\frac{75179}{1000}\right)r^5$$
$$+\left(-14L^3-\frac{529L^2}{30}-140\zeta_2L+\frac{6418L}{75}-\frac{1307\zeta_2}{15}+84\zeta_3-\frac{4823273}{54000}\right)r^6\Bigg)+O(r^7)+O(\varepsilon^2), \quad (18)$$

where $L=\log r$. The $O(1)$ part of J_{111} is in agreement with [10].

3.2 JmmM

This diagram was recently studied in Ref. [15] by means of differential equation method in the regime when $m \gg M$, while we are interested in the case $m \ll M$. Here we also have two master integrals and the system of two differential equations. Again introducing rescaled functions according to (13) we have

$$\begin{cases} \frac{\partial}{\partial r}\tilde{J}_{111}-4r\tilde{J}_{211}=0\,, \\ (r^2-1)\frac{\partial}{\partial r}\tilde{J}_{211}+\frac{r^2(13-4d)+2d-7}{r}\tilde{J}_{211}+\frac{(d-3)(3d-8)}{4r}\tilde{J}_{111}-\frac{2r^{d-7}(r^d-2r^2)}{(d-4)^2}=0. \end{cases} \quad (19)$$

Using (19) the corresponding second order differential equation for J_{111} looks like

$$(r^2-1)\frac{\partial^2}{\partial r^2}J_{111}-\frac{2(d-3)(2r^2-1)}{r}\frac{\partial}{\partial r}J_{111}+(d-3)(3d-8)J_{111}-\frac{8r^{d-6}(r^d-2r^2)}{(d-4)^2}=0. \quad (20)$$

To find a solution of this equation we again use Ansatz (16) for the most general form of solution at $r\to 0$. For leading exponents α_i there are four allowed values $\alpha_1=0$, $\alpha_2=2d-5$, corresponding to the two independent solutions of the associated homogeneous equation and $\alpha_3=d-2$, $\alpha_4=2d-4$, as required by nonhomogeneous part of Eq. (20). All other steps in this case are in one to one correspondence with those considered in previous subsection. The first boundary condition is given by the value of master integral J_{111} at $r=0$. In order to find the second boundary condition we need explicit expression for r-expansion of J_{111} up to the third order, which can be obtained from threshold large mass assymptotic expansion of J_{111}. As a result of all these steps we have the following expressions for our master integrals

$$\begin{aligned}
e^{2\epsilon\gamma_E}M^{-2+4\epsilon}J_{111} &= -\frac{1+2r^2}{2\epsilon^2}+\frac{4(4L-3)r^2-5}{4\epsilon}-\frac{1}{8}(11+20\zeta_2)-(4L^2-12L-3\zeta_2+5)r^2\\
&-\frac{1}{4}(8L^2-12L+8\zeta_2+7)r^4-\frac{1}{18}(12L-11)r^6+\epsilon\left(\frac{55}{16}-\frac{11\zeta_3}{3}-\frac{25\zeta_2}{4}\right.\\
&+\left(\frac{8L^3}{3}-12L^2+4(\zeta_2+7)L+9\zeta_2+\frac{26\zeta_3}{3}-3\right)r^2-32\zeta_2r^3\\
&+\left.\left(4L^3-5L^2-\frac{7L}{2}+4\zeta_2-4\zeta_3+\frac{95}{8}\right)r^4+\frac{32\zeta_2r^5}{5}+\frac{2}{9}(8L-9\zeta_2-14)r^6\right)\\
&+\epsilon^2\left(\frac{949}{32}-\frac{55\zeta_2}{8}-\frac{55\zeta_3}{6}-\frac{1}{720}\pi^4(296r^4-578r^2+303)\right.\\
&+\left(-\frac{4L^4}{3}+8L^3-4(\zeta_2+7)L^2+\frac{4}{3}(9\zeta_2-2\zeta_3+45)L+31\zeta_2+26\zeta_3+19\right)r^2\\
&+\frac{64}{3}(6L+12\log 2-11)\zeta_2r^3+\left(-\frac{14L^4}{3}+6L^3+(\frac{3}{2}-2\zeta_2)L^2+(3\zeta_2+\frac{37}{4})L\right.\\
&-\left.\frac{71\zeta_2}{4}+8\zeta_3-\frac{885}{16}\right)r^4-\frac{32}{75}(60L+120\log 2-77)\zeta_2r^5\\
&+\left.\frac{1}{324}(288L^3-792L^2-108(2\zeta_2+11)L+1926\zeta_2-1296\zeta_3+5417)r^6\right)+O(r^7)+O(\varepsilon^3) \quad (21)
\end{aligned}$$

and

$$\begin{aligned}
e^{2\epsilon\gamma_E}M^{4\epsilon}J_{211} &= -\frac{1}{2\epsilon^2}+\frac{4L-1}{2\epsilon}+\frac{1}{2}+\frac{3\zeta_2}{2}+4L-2L^2-(2L^2-2L+2\zeta_2+1)r^2\\
&+(\frac{3}{4}-L)r^4+(\frac{5}{36}-\frac{L}{3})r^6+\epsilon\left(\frac{11}{2}+\frac{11\zeta_2}{2}+\frac{13\zeta_3}{3}+8L+2L\zeta_2-4L^2+\frac{4L^3}{3}-24\zeta_2r\right.\\
&+(4L^3-2L^2-6L+4\zeta_2-4\zeta_3+11)r^2+8\zeta_2r^3+\frac{1}{9}(24L-27\zeta_2-38)r^4+\frac{8\zeta_2r^5}{5}\\
&+\left.(\frac{L}{45}-\zeta_2-\frac{83}{300})r^6\right)+\epsilon^2\left(\frac{49}{2}+\frac{37\zeta_2}{2}+\frac{37\zeta_3}{3}+\frac{289\pi^4}{720}-\frac{4}{3}L(\zeta_3-12)+4L\zeta_2\right.\\
&-2L^2\zeta_2-8L^2+\frac{8L^3}{3}-\frac{2L^4}{3}+48\zeta_2(2L+4\log 2-3)r
\end{aligned}$$

$$
\begin{aligned}
&+\left(-\frac{14L^4}{3}+\frac{4L^3}{3}-2(\zeta_2-3)L^2+2(\zeta_2+5)L-17\zeta_2+8\zeta_3-\frac{37\pi^4}{90}-53\right)r^2\\
&-\frac{8}{3}(12L+24\log 2-13)\zeta_2 r^3+\left(\frac{4L^3}{3}-3L^2-(\zeta_2+\frac{121}{18})L+\frac{35\zeta_2}{4}-6\zeta_3+\frac{5219}{216}\right)r^4\\
&-\frac{4}{75}(120L+240\log 2-199)\zeta_2 r^5\\
&+\left(\frac{4L^3}{9}-\frac{5L^2}{9}-\frac{1}{450}(150\zeta_2+203)L+\frac{37\zeta_2}{180}-2\zeta_3+\frac{237511}{81000}\right)r^6\Bigg)+O(r^7)+O(\varepsilon^3)\,, \quad (22)
\end{aligned}
$$

where $L=\log r$. The $O(1)$ part of J_{111} is in agreement with [10].

4 Asymptotic large mass expansion at the threshold

In this section we consider asymtotic large mass expansion for master integrals introduced above. A type of expansion one needs to perform in order to obtain an analytic expression for the master integral of case (b) was already considered in Ref. [18] and [3] and one may just follow along the lines of procedure described there. However, in the case of master integral JM0m a somewhat different prescription for setting loop momenta is required[2] and thus we will consider this case in detail below.

In order to establish the expansion procedure we use "the strategy of regions" [19]. Let us remind you first a general prescription for large mass expansion at the threshold of Ref. [18]. We consider a general case of threshold Feynman integral F_Γ, corresponding to a graph Γ when the masses M_i and external momenta Q_i are considered large with respect to small masses m_i and external momenta q_i. We are interested in a case, when external momenta Q_i are on the following mass shell: $Q_i^2=(\sum_j a_{ij}M_j+\sum_k b_{ik}m_k)^2$. Here a_{ij} and b_{ik} are some numbers. It is just the generalization of the on-mass-shell condition of Ref. [18]. Then the asymptotic expansion in the limit $Q_i, M_i\to\infty$ takes the following explicit form [5]

$$
F_\Gamma(Q_i,M_i,q_i,m_i;\varepsilon)\overset{M_i\to\infty}{\sim}\sum_\gamma \mathcal{M}_\gamma F_\Gamma(Q_i,M_i,q_i,m_i;\varepsilon). \quad (23)
$$

Here the sum runs over subgraphs γ of Γ such that

(a) in γ there is a path between any pair of external vertices associated with the large external momenta Q_i;

(b) γ containes all the lines with the large masses;

(c) every connectivity component γ_j of the graph $\hat\gamma$ obtained from γ by collapsing all the external vertices with the large external momenta to apoint is 1PI with respect to the lines with small masses.

Operator $\mathcal{M}_\gamma$ in (23) can be written as a product $\Pi_i\mathcal{M}_{\gamma_i}$ over different connectivity components, where $\mathcal{M}_{\gamma_i}$ are operators of Taylor expansion in certain momenta and masses. In what follows we will distinguish the connectivity component γ_0, which is defined to contain all external vertices with large momenta. For connectivity components γ_i, different from γ_0,

[2] Asymptotic expansions in momentum space are not invariant under the redefinition of loop momenta contrary to the expansions, performed in α-representation for Feynman integrals. Therefore special care should be taken choosing a correct one set of momenta.

the corresponding operator $\mathcal{M}_{\gamma_i}$ performs Taylor expansion of the Feynman integral F_{γ_i} in its small masses and external momenta. To describe the action of $\mathcal{M}_{\gamma_0}$ one uses representation of γ_0-component in terms of a union of its 1PI components and cut heavy lines (that is subgraph becomes disconnected after a cut line is removed). Here we can again factorize $\mathcal{M}_{\gamma_0}$ and the Taylor expansion of the 1PI components of γ_0 is performed as in a case of other connectivity components γ_i. As for the action of operator $\mathcal{M}$ on the cut lines there are two different cases. Let $P+k$ be the momentum of a line with large mass M_i, where P is a linear combination of large external momenta and k is a linear combination of loop momenta and small external momenta. Then the mentioned two cases can be written as follows:

- $P^2 = M_i^2$ and $\mathcal{M}$ for this line is given by

$$\mathcal{M} = \mathcal{T}_x \frac{1}{xk^2 + 2Pk}\bigg|_{x=1}, \tag{24}$$

  Here $\mathcal{T}_x$ denotes the operator of Taylor expansion in x around $x = 0$.

- $P^2 \neq M_i^2$ and the operator $\mathcal{M}$ reduces to the ordinary Taylor expansion in small (with respect to this line) external momenta.

As for the optimal set of internal loop momenta we choose a rule when a large external momenta are divided between lines with masses M_i and m_k in order to satisfy the following conditions: $P_i^2 = M_i^2$ and $P_k^2 = m_k^2$. We do not know whether such separation is always possible or not, but in most cases of interest it certainly works. As an example in the next subsection we will consider an expansion for master integral $J_{111}((m+M)^2)$ from our case (a).

4.1 Second boundary condition for JM0m

For master integral J_{111}, according to our prescription for choosing internal momenta we have the following expression:

$$J_{111} = \frac{1}{\pi^d} \iint \frac{d^d k d^d l}{[k^2][(k+l)^2 + 2a(k+l,q)][l^2 + 2b\, lq]}, \tag{25}$$

Here q is an external momentum; $a = M/(M+m)$ and $b = -m/(M+m)$. From the general formula (23) in the asymptotic expansion of master integral J_{111} in the limit $m/M \to 0$ we have four subgraphs[3]:

1) graph Γ itself;

2) subgraph γ_1 consisting from lines with masses M and m;

3) subgraph γ_2 consisting from lines with mases M and zero;

4) subgraph γ_3 consisting from one heavy line.

For 1) we expand the integrand around $m = 0$ with the result

$$\sum_{n=0}^{\infty} \frac{(-1)^n (2b)^n}{\pi^d} \iint \frac{d^d k d^d l (l \cdot q)^n}{[k^2][(k+l)^2 + 2a(k+l,q)][l^2]^{n+1}}. \tag{26}$$

[3]Here Γ is the graph, corresponding to Feynman integral J_{111}.

Each term in this expansion can be evaluated rewriting it via scalar integrals with shifted space-time dimension [1] and for first three of them we have:

$$\frac{1}{\pi^d}\iint \frac{d^dkd^dl(lq)}{[k^2][l^2]^2[(k+l)^2+2a(k+l,q)]} = -aq^2\mathrm{JV}^{d+2}_{222},$$
$$\frac{1}{\pi^d}\iint \frac{d^dkd^dl(lq)^2}{[k^2][l^2]^3[(k+l)^2+2a(k+l,q)]} = -\frac{q^2}{2}\mathrm{JV}^{d+2}_{231} - \frac{q^2}{2}\mathrm{JV}^{d+2}_{132} + 4a^2q^4\mathrm{JV}^{d+4}_{333},$$
$$\frac{1}{\pi^d}\iint \frac{d^dkd^dl(lq)^3}{[k^2][l^2]^4[(k+l)^2+2a(k+l,q)]} = -36a^3q^6\mathrm{JV}^{d+6}_{444} + 3aq^4\mathrm{JV}^{d+4}_{243} + 3aq^4\mathrm{JV}^{d+4}_{342},$$

where

$$\mathrm{JV}^d_{\nu_1\nu_2\nu_3} = \frac{1}{\pi^d}\iint \frac{d^dkd^dl}{[k^2]^{\nu_1}[l^2]^{\nu_2}[(k+l)^2+2(k+l,q)]^{\nu_3}} \quad \text{and} \quad q^2 = M^2\,.$$

The latter can be easily expressed in terms of Γ-functions for arbitrary values of ν_1, ν_2 and ν_3.

The Taylor expansion for subgraph γ_2 gives

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{\pi^d}\iint \frac{d^dkd^dl(k^2+2kl+2akq)^n}{[k^2][l^2+2alq]^{n+1}[l^2+2blq]} = 0. \tag{27}$$

In case of subgraph γ_3 we have the following expression

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{\pi^d}\iint \frac{d^dkd^dl(l^2+2kl+2alq)^n}{[k^2][k^2+2akq]^{n+1}[l^2+2blq]}. \tag{28}$$

Each term of this sum is just a product of two one-loop integrals and hence can be easily evaluated.

An expansion for subgraph γ_4 leads to the following result

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{\pi^d}\iint \frac{d^dkd^dl((k+l)^2)^n}{[k^2][l^2+2b\,lq][2a(k+l,q)]^{n+1}}. \tag{29}$$

Here we can see, that it is the most difficult from computation point of view type of contribution, as we are dealing in this case with eikonal integrals. As one can easily see, the evaluation of each term from this contribution can be reduced to the evaluation of integrals of the following type

$$\frac{1}{\pi^d}\iint \frac{d^dkd^dl(kl)^m}{[k^2][l^2+2blq][(k+l,q)]^n}. \tag{30}$$

In order to calculate these integrals it is naturally to express the products $(kl)^m$ in terms of traceless products $(kl)^{(m)} \equiv k^{(\alpha,m)}l_{(\alpha,m)}$ [4]. Then we notice, that integration over k of traceless products $k^{(\alpha,m)}$ times the part of integrand, which depends only on k results in expression propotional to the traceless product $q^{(\alpha,m)}$. Thus, we can replace the factor $(kl)^{(m)}$ by $(qk)^{(m)}(ql)^{(m)}/(qq)^{(m)}$ and finally replace the factors involved through ordinary products qk and lq. After performing these steps we reduce the evaluation of integrals (30) to the integrals:

$$\frac{1}{\pi^d}\iint \frac{d^dkd^dl(lq)^m}{[k^2][l^2+2b\,lq][(k+l,q)]^n}. \tag{31}$$

[4] Here α is a collective index representing m Lorenz indices $\alpha_1, \alpha_2, \ldots, \alpha_m$

These integrals with the use of integration by parts identities $(d-3-n)n^+ + (n+1)m^+n^{++} = 0$, where $n(m)^+$ are operators, which increase corresponding indices n or m) can be further reduced to integrals of the form

$$\frac{1}{\pi^d}\iint \frac{d^dkd^dl(lq)^m}{[k^2][l^2+2b\,lq][(k+l,q)]}. \tag{32}$$

To take these last integrals one can first perform k_0 and l_0 integrations using Cauchy theorem and remained angular integrations are trivial since there are no products kl in the integrand left.

However, as our calculations showed, it is far more simple just to calculate using asymtotic expansion this master integral up to the order $(m/M)^3$ and then use this result as the nessesary boundary conditions for the solution of master differential equation. It takes then much less efforts to construct the expansion of this master integral up to any order in $r = m/M$ using differential equation method.

5 Conclusion

We considered two loop sunset diagrams with two mass scales m and M at the threshold and pseudotreshold that cannot be threated by earlier published formula [1, 2]. The complete reduction to the master integrals is given. The master integrals are evaluated as series in ratio m/M and up to needed in applications order in ε with the help of differential equation method. The rules of asymptotic expansion in the case when q^2 is at the (pseudo)threshold are given.

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A Expansion of F_4 Appel function: second boundary condition for JM0m

In order to obtain the second boundary integral in case of JM0m we can use the representation of the sunset diagram in term of Lauricella function which can be obtained from [9]. However since one mass is zero, the Lauricella function simplifies to Appel function:

$$\begin{aligned} e^{2\gamma\varepsilon}M^{-2+4\varepsilon}J_{111}(m,M,0;\,q^2) = -\left(\frac{m^2}{M^2}\right)^{1-\varepsilon}\Gamma^2(-1+\varepsilon)\,F_4\left(\begin{matrix}1,\,\varepsilon\\ 2-\varepsilon,\,2-\varepsilon\end{matrix};\,\frac{m^2}{M^2},\,\frac{q^2}{M^2}\right)\\ -\Gamma(1-\varepsilon)\Gamma(-1+\varepsilon)\Gamma(-1+2\varepsilon)\,F_4\left(\begin{matrix}-1+2\varepsilon,\,\varepsilon\\ \varepsilon,\,2-\varepsilon\end{matrix};\,\frac{m^2}{M^2},\,\frac{q^2}{M^2}\right) \end{aligned} \tag{33}$$

and in our case $q^2 = (m+M)^2$. We need the expansion of this integal up to the order $O(r^3)$ and $O(\varepsilon)$. Then the rest coefficients can be found from the differential equation.

In order to expand (33) in series over m/M we use the following representation for F_4

$$F_4\left(\begin{matrix}a,\,b\\ c,\,c'\end{matrix};\,x,\,y\right) = \sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k}\frac{x^k}{k!}\,{}_2F_1\left(\begin{matrix}a+k,\,b+k\\ c'\end{matrix};\,y\right). \tag{34}$$

In (34) function ${}_2F_1$ has to be transformed to argument $1-y$. Then we have rot the r.h.s. $(r=m/M)$

$$-r^{2-2\varepsilon}\Gamma^2(-1+\varepsilon)\sum_{k=0}^{\infty}\frac{(1)_k(\varepsilon)_k}{(2-\varepsilon)_k}\frac{r^{2k}}{k!}\Bigg\{$$
$$\Gamma\left(\begin{smallmatrix}2-\varepsilon,\,1-2\varepsilon-2k\\ 1-\varepsilon-k,\,2-2\varepsilon-k\end{smallmatrix}\right)\,{}_2F_1\left(\begin{smallmatrix}1+k,\,\varepsilon+k\\ 2\varepsilon+2k\end{smallmatrix};-r(2+r)\right)$$
$$+[-r(2+r)]^{1-2\varepsilon-2k}\,\Gamma\left(\begin{smallmatrix}2-\varepsilon,\,2\varepsilon+2k\\ 1+k,\,\varepsilon+k\end{smallmatrix}\right)\,{}_2F_1\left(\begin{smallmatrix}1-\varepsilon-k,\,2-2\varepsilon-k\\ 2-2\varepsilon-2k\end{smallmatrix};-r(2+r)\right)\Bigg\}$$
$$-\Gamma(-1+\varepsilon)\Gamma(1-\varepsilon)\Gamma(-1+2\varepsilon)\sum_{k=0}^{\infty}(-1+2\varepsilon)_k\frac{r^{2k}}{k!}\Bigg\{$$
$$\Gamma\left(\begin{smallmatrix}2-\varepsilon,\,3-4\varepsilon-2k\\ 3-3\varepsilon-k,\,2-2\varepsilon-k\end{smallmatrix}\right)\,{}_2F_1\left(\begin{smallmatrix}-1+2\varepsilon+k,\,\varepsilon+k\\ 2\varepsilon+2k\end{smallmatrix};-r(2+r)\right)$$
$$+[-r(2+r)]^{3-4\varepsilon-2k}\,\Gamma\left(\begin{smallmatrix}2-\varepsilon,\,-3+4\varepsilon+2k\\ -1+2\varepsilon+k,\,\varepsilon+k\end{smallmatrix}\right)\,{}_2F_1\left(\begin{smallmatrix}3-3\varepsilon-k,\,2-2\varepsilon-k\\ 4-4\varepsilon-2k\end{smallmatrix};-r(2+r)\right)\Bigg\}. \quad (35)$$

The first and third terms in (35) can be easily expanded since the series in k and ${}_2F_1$ series can be truncated at the given order. In the second and fourth terms we still can truncate ${}_2F_1$ series, however, we cannot truncate the sum over k because of factor $[-r(2+r)]^{-2k}$. Thus we have to resum the whole k-sum. Thus in order $O(r^3)$ we have for the second term

$$\frac{(-r)^{-2\varepsilon}(+r)^{-2\varepsilon}}{2\sqrt{\pi}}\Gamma(2-\varepsilon)\Gamma^2(-1+\varepsilon)\Gamma(-1/2+\varepsilon)\,{}_2F_1\left(\begin{smallmatrix}3-3\varepsilon-k,\,2-2\varepsilon-k\\ 4-4\varepsilon-2k\end{smallmatrix};1\right)$$

and the fourth term

$$\frac{(-r)^{-4\varepsilon}}{2\sqrt{\pi}\Gamma(\varepsilon)}\Gamma(2-\varepsilon)\Gamma^2(1-\varepsilon)\Gamma(-1+\varepsilon)\Gamma(-3/2+2\varepsilon)\Gamma(-1+2\varepsilon)\,{}_2F_1\left(\begin{smallmatrix}3-3\varepsilon-k,\,2-2\varepsilon-k\\ 4-4\varepsilon-2k\end{smallmatrix};1\right).$$

Each of this two terms has an imaginary part but it cancels in the sum of the two. Adding contributions from the first and third terms of (35) and expanding in ε we get $O(r^3)$ term of the formula (17).

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