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Two-loop amplitudes with nested sums: Fermionic contributions to $e^+e^- \to q\bar{q}g$

Sven Moch[a], Peter Uwer[a] and Stefan Weinzierl[b]

[a] *Institut für Theoretische Teilchenphysik, Universität Karlsruhe*
76128 Karlsruhe, Germany

[b] *Dipartimento di Fisica, Università di Parma,*
INFN Gruppo Collegato di Parma, 43100 Parma, Italy

Abstract

We present the calculation of the n_f-contributions to the two-loop amplitude for $e^+e^- \to q\bar{q}g$ and give results for the full one-loop amplitude to order ε^2 in the dimensional regularization parameter. Our results agree with those recently obtained by Garland et al.. The calculation makes extensive use of an efficient method based on nested sums to calculate two-loop integrals with arbitrary powers of the propagators. The use of nested sums leads in a natural way to multiple polylogarithms with simple arguments, which allow a straightforward analytic continuation.

1 Introduction

The last thirty years of experimental studies at colliders together with the theoretical investigations, have taught us that perturbative quantum chromodynamics (QCD) gives an excellent description of short-distance scattering of strongly interacting partons. Indeed, today the theory has reached a sufficient maturity that it is no longer the target of experimental studies, but rather a tool in the search for new physics beyond the standard model.

The search for new physics in particle physics, being pursued at present and upcoming collider experiments at the Tevatron and the LHC, rely on our ability to make precise predictions for QCD and QCD-associated processes. The accuracy reached already in present collider experiments demands next-to-next-to-leading (NNLO) theoretical predictions within the framework of perturbation theory. For example the strong coupling constant α_s, whose precise value affects many cross sections, can be measured by using the data for $e^+e^- \to 3$ jets. At present, the error on the extraction of α_s from this measurement is dominated by theoretical uncertainties [1], among the main sources there being the truncation of the perturbative expansion at a fixed order. Up to now, event shapes in 3-jet events have been calculated at next-to-leading order (NLO) for massless [2]-[6] and massive quarks [7]-[11]. To reduce the theoretical uncertainties, it is necesarry to extend the calculation for massless quarks to next-to-next-to-leading order. The calculation of $e^+e^- \to 3$ jets at NNLO requires the tree-level amplitudes for $e^+e^- \to 5$ partons [12, 13], the one-loop amplitudes for $e^+e^- \to 4$ partons [14]-[17] as well as the two-loop amplitude for $e^+e^- \to q\bar{q}g$ together with the one-loop amplitude $e^+e^- \to q\bar{q}g$ to order ε^2 in the dimensional regularization parameter.

While for inclusive quantities like the total hadronic cross section in e^+e^--annihilation even higher orders have been calculated in the past [18]-[20], the calculation of two-loop four-point scattering amplitudes has been the main obstacle for a long time. Due to tremendous activity in that field during the past three years [21], this problem can be considered to be solved – at least for the case of massless internal quarks and only one external massive leg.

The by now more or less standard approach to calculate two-loop four-point functions has been inspired by the techniques developed in the calculation of two-point functions. The starting point is a reduction of the tensor integrals through Schwinger parametrization [22]-[24]. This yields immediately scalar integrals with a higher dimension and raised powers of the propagators. The usual way to proceed then, is to apply repeatedly algebraic relations between these integrals, which follow from Poincare- and Lorentz-invariance [25]-[27]. Finally one ends up with a small set of so-called master integrals which must be solved analytically. While for simple topologies it is often straightforward to find the reduction scheme (i.e. 'triangle rule'), in general it is a non-trivial task to solve this problem.

In a recent publication [28, 29], we have therefore proposed a different method to attack this problem. The basic idea is the following. We solve the scalar integrals in higher dimensions and with raised powers of propagators directly in terms of nested sums instead of reducing all the integrals to a small set of master integrals. The aim of this paper is to illustrate this method in the calculation of the fermionic contributions to the two-loop amplitude $e^+e^- \to q\bar{q}g$, i.e. the contributions proportional to the number of quark flavours n_f. We present our results in terms of multiple polylogarithms [30]-[32], which arise naturally from the use of nested sums. In addition, we show that these multiple polylogarithms can easily be continued analytically. As a consequence the amplitudes for $e^+e^- \to q\bar{q}g$ presented here can also be used for $(2+1)$-jet production in deep-inelastic scattering and the production of a vector boson (W, Z or Drell-Yan pair) in hadron-hadron collisions. The respective amplitudes can be obtained by the crossing symmetry and simple coupling constant modifications.

The outline of the paper is as follows. In section 2 we present a few properties of the two-loop amplitude. In particular, we discuss the kinematics, the ultraviolet (UV) renormalization, and the structure of the soft and collinear singularities. In section 3 we outline the calculation. In the following section 4 we present our results and compare them with those recently obtained by Garland et al. [33]. We give our conclusions in section 5. Appendix A contains the results for the one-loop amplitude to order ε^2 and appendix B summarizes properties of multiple polylogarithms under analytic continuation.

2 Preliminaries

2.1 Kinematics

In the following we study the reaction

$$e^+ + e^- \to q + g + \bar{q}\,. \tag{1}$$

We treat all quarks as massless, that means we work in QCD with n_f massless quark flavours. To be consistent with earlier work [14, 15] we calculate the amplitude for the reaction with all particles in the final state

$$0 \to q(p_1) + g(p_2) + \bar{q}(p_3) + e^-(p_4) + e^+(p_5). \tag{2}$$

The kinematical invariants are denoted by

$$s_{ij} = \left(p_i + p_j\right)^2, \quad s_{ijk} = \left(p_i + p_j + p_k\right)^2, \quad s = s_{123}\,, \tag{3}$$

and it is convenient to introduce the dimensionless quantities

$$x_1 = \frac{s_{12}}{s_{123}}, \quad x_2 = \frac{s_{23}}{s_{123}}. \tag{4}$$

For pure photon exchange, the complete amplitude $\mathcal{A}_\gamma$ for $e^+e^- \to q\bar{q}g$ can be written as the product of a leptonic current L_μ with the hadronic current H_μ:

$$\mathcal{A}_\gamma = -\frac{i}{s} e^2 Q^q T^a_{\bar{i}i} L_\mu H^\mu \equiv -e^2 Q^q T^a_{\bar{i}i} A_\gamma, \tag{5}$$

with Q^q denoting the electric charge of the outgoing quarks in units of the elementary charge $e = \sqrt{4\pi\alpha}$. The generator of the SU(N) gauge group is given by T^a. The indices $\bar{i}, i$ and a describe the color of the outgoing quarks and gluon. The normalization of the color matrices is taken to be

$$\mathrm{Tr}\left(T^a T^b\right) = \frac{1}{2}\delta^{ab}. \tag{6}$$

The leptonic current is given by

$$L_\mu = \bar{u}(p_4)\gamma_\mu v(p_5), \tag{7}$$

with u, v denoting the spinors of the outgoing leptons. As we will show later, it is sufficient to consider pure photon exchange: Working in a helicity basis, the pure photon exchange amplitude $\mathcal{A}_\gamma$ allows the reconstruction of the full amplitude including Z-boson exchange by adjusting the couplings. Using the anti-commutation relations for the γ-matrices, one can always achieve the following form of the hadronic current:

$$\begin{aligned}
H_\mu &= c_1 \frac{1}{s}\langle p_1|\not{p}_2|p_3\rangle \varepsilon_{g\mu} + c_2 \frac{1}{s^2}\langle p_1|\not{p}_2|p_3\rangle(\varepsilon_g\cdot p_1)p_{3\mu} \\
&+ c_3 \frac{1}{s^2}\langle p_1|\not{p}_2|p_3\rangle(\varepsilon_g\cdot p_3)p_{1\mu} + c_4 \frac{1}{s^2}\langle p_1|\not{p}_2|p_3\rangle(\varepsilon_g\cdot p_1)p_{1\mu} \\
&+ c_5 \frac{1}{s^2}\langle p_1|\not{p}_2|p_3\rangle(\varepsilon_g\cdot p_3)p_{3\mu} + c_6 \frac{1}{s}\langle p_1|\gamma_\mu|p_3\rangle(\varepsilon_g\cdot p_1) \\
&+ c_7 \frac{1}{s}\langle p_1|\gamma_\mu|p_3\rangle(\varepsilon_g\cdot p_3) + c_8 \frac{1}{s}\langle p_1|\not{\varepsilon}_g|p_3\rangle p_{1\mu} \\
&+ c_9 \frac{1}{s}\langle p_1|\not{\varepsilon}_g|p_3\rangle p_{3\mu} + c_{10} \frac{1}{s}\langle p_1|\not{\varepsilon}_g\not{p}_2\gamma_\mu|p_3\rangle \\
&+ c_{11} \frac{1}{s}\langle p_1|\not{\varepsilon}_g|p_3\rangle p_{2\mu} + c_{12} \frac{1}{s^2}\langle p_1|\not{p}_2|p_3\rangle(\varepsilon_g\cdot p_1)p_{2\mu} \\
&+ c_{13} \frac{1}{s^2}\langle p_1|\not{p}_2|p_3\rangle(\varepsilon_g\cdot p_3)p_{2\mu},
\end{aligned} \tag{8}$$

where we have used $\langle p_1|$ respective $|p_3\rangle$ as a short-hand notation for the spinors $\bar{u}(p_1)$ and $v(p_3)$ of the outgoing quark and anti-quark. The dimensionless functions c_i depend only on the ratios x_i, the spacetime dimension $d = 4-2\varepsilon$ and the renormalization scale μ,

$$c_i = c_i\left(x_1, x_2, \varepsilon, \frac{\mu}{s}\right). \tag{9}$$

Due to various constraints, for example current conservation,

$$(p_{1\mu} + p_{2\mu} + p_{3\mu})H^\mu = 0, \tag{10}$$

the functions c_i are not all independent of each other. It can be easily chown that c_2, c_4, c_6, c_{12} are sufficient to reconstruct all remaining functions. A similar conclusion has been drawn in ref. [33]. The relations between c_2, c_4, c_6, c_{12} and the remaing functions are:

$$\begin{aligned}
c_3(x_1,x_2) &= -c_2(x_2,x_1), \\
c_5(x_1,x_2) &= -c_4(x_2,x_1), \\
c_{13}(x_1,x_2) &= -c_{12}(x_2,x_1),
\end{aligned} \tag{11}$$

and

$$
\begin{aligned}
c_1 &= -\frac{1}{2}(1-x_2)c_3 - \frac{1}{2}(1-x_1)c_5 + \frac{x_1}{x_2}c_6 - \frac{1}{2}(x_1+x_2)c_{13}, \\
c_7 &= -\frac{x_1}{x_2}c_6, \\
c_8 &= -\frac{1}{2}x_2c_3 - \frac{1}{2}x_1c_4, \\
c_9 &= -\frac{1}{2}x_1c_2 - \frac{1}{2}x_2c_5, \\
c_{10} &= +\frac{1}{4}(1-x_1)(c_2-c_5) - \frac{1}{4}(1-x_2)(c_3-c_4) + \frac{1}{2}\frac{(x_1+x_2)}{x_2}c_6 \\
&\quad +\frac{1}{4}(x_1+x_2)(c_{12}-c_{13}), \\
c_{11} &= \frac{1}{2}(1-x_2)c_3 + \frac{1}{2}(1-x_1)c_5 - \frac{x_1}{x_2}c_6 + \frac{1}{2}x_1c_{13} - \frac{1}{2}x_1c_{12}.
\end{aligned}
\tag{12}
$$

In the actual calculation we have not used these constraints. Instead we calculated all c_1–c_{13} and used eqs. (11) and (12) as a cross-check on our calculation.

Beyond the leading-order, one encounters UV as well as soft and collinear singularities. We use dimensional regularization [25, 34] to regulate both types of singularities. There are several variants of dimensional regularization which are used in loop calculations in QCD: Conventional dimensional regularization (CDR) [35] continues all momenta and all polarization vectors to d dimensions. The 't Hooft-Veltman (HV) scheme [25] takes the momenta and the helicities of the unobserved particles in d dimensions, whereas the momenta and the helicities of the observed particles are four-dimensional. The CDR scheme is often employed within the interference method, but is not suited for the calculation of amplitudes. Enforcing the CDR scheme in the calculation of amplitudes requires the introduction of external states with "ε"-helicities [36]. Furthermore, there are several versions of four-dimensional schemes on the market. Despite the name "four-dimensional schemes", they are variants of dimensional regularization. The name refers to how these schemes treat unobserved internal particles and the Dirac algebra. The four-dimensional helicity scheme (FDH) [37] introduces an additional parameter d_s for unobserved internal states, which is set to 4 at the end of the calculation. It has the advantage that it respects supersymmetric Ward identities up to two loops. The four-dimensional scheme defined in [38] keeps the Dirac algebra in four dimensions and allows the use of four-dimensional Fierz- and Schouten identities, at the expense of having to restore Ward identities. These schemes can lead to considerable simplifications, in particular if many external particles are involved. For the process $e^+e^- \to 3$ jets, the number of external particles is relatively small and we do not consider these schemes further.

In this paper we keep our calculation rather general and we decompose the amplitude into spinor strings, which we can compute without any reference to the dimensionality of the external polarization vectors. Internal particles and the Dirac algebra are treated in d dimensions. From these spinor strings we can easily deduce the results in the CDR scheme and the HV scheme. The result in the CDR scheme is obtained by interfering the two-loop amplitude with the Born amplitude. On the other hand, by contracting with explicit representations of polarization vectors, we obtain helicity amplitudes in the 't Hooft-Veltman scheme.

Including the Z-boson exchange, the situation becomes slightly more complicated. Due to the presence of γ_5, a specific scheme has to be chosen. Schemes which allow for a consistent treatment of γ_5 are for example the HV scheme or the scheme defined in [38]. In both schemes, the regularization procedure violates certain Ward identities, which have to be restored by finite renormalizations. However, since the amplitudes considered in this paper do not contain closed fermion loops with axial-vector couplings, the results for Z-boson exchange can be obtained from the ones for pure photon exchange by a simple adjustment of the electro-weak couplings.

It should be noted that in general the finite part of the amplitudes are also scheme-dependent. As long as soft and collinear singularities are considered, this is not really an issue. Using the same scheme in the calculation of the divergent contributions from the real corrections any scheme dependence cancels out at the end. For the UV divergences however, one has to keep in mind that in general the coupling constant is scheme dependent. The values of the coupling constant in two different schemes are related by a finite renormalization. The coupling constants in the CDR scheme and the HV scheme are identical, since the internal states in these two schemes are treated in the same way. The CDR or HV scheme, together with the modified minimal subtraction prescription ($\overline{\mathrm{MS}}$), defines the usual coupling $\alpha_s^{\overline{\mathrm{MS}}}$ and it is most useful to quote our results in these schemes.

To obtain the results in the 't Hooft-Veltman scheme we work in the helicity basis. For fermions, spinors of definite helicity are given by:

$$
\begin{aligned}
u_\pm(p) &= \frac{1}{2}(1 \pm \gamma_5)\, u(p), \\
v_\pm(p) &= \frac{1}{2}(1 \mp \gamma_5)\, v(p).
\end{aligned} \tag{13}
$$

We introduce the following short-hand notation for spinors with definite helicity:

$$
\begin{aligned}
|i\pm\rangle &= |p_i\pm\rangle &&= u_\pm(p_i) = v_\mp(p_i), \\
\langle i\pm| &= \langle p_i\pm| &&= \bar{u}_\pm(p_i) = \bar{v}_\mp(p_i),
\end{aligned} \tag{14}
$$

and the spinor products are then defined as

$$
\begin{aligned}
\langle pq\rangle &= \langle p-|q+\rangle, \\
[pq] &= \langle p+|q-\rangle.
\end{aligned} \tag{15}
$$

Gluon polarization vectors are expressed in terms of Weyl spinors as [39]-[44]

$$
\varepsilon_\mu^+(k,q) = \frac{\langle q-|\gamma_\mu|k-\rangle}{\sqrt{2}\langle qk\rangle}, \qquad \varepsilon_\mu^-(k,q) = \frac{\langle q+|\gamma_\mu|k+\rangle}{\sqrt{2}[kq]}, \tag{16}
$$

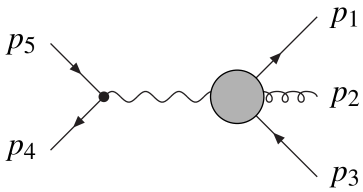

Figure 1: Labelling of the external legs for the amplitude. The lepton pair is denoted by p_4 and p_5.

where k is the gluon momentum and q is an arbitrary null reference momentum. The dependence on the reference momentum drops out in final gauge-invariant amplitudes. An appropriate choice of q can lead to a significant reduction in the number of diagrams which need to be evaluated.

Helicity conservation for massless quarks and leptons ensures that there are only $2^3 = 8$ possible helicity configurations for $e^+e^- \to q\bar{q}g$, namely two choices for each fermion line together with two possible gluon polarizations. Because the electron line couples through the current $\langle 4\pm|\gamma^\mu|5\pm\rangle = \langle 5\mp|\gamma^\mu|4\mp\rangle$, it is trivial to reverse its helicity simply by exchanging $p_4 \leftrightarrow p_5$ and by adjusting the weak couplings. Parity and charge conjugation can be used to further reduce the number of helicity amplitudes which need to be calculated. Parity reverses all helicities simultaneously and is implemented by complex conjugating all spinor products (e.g. $\langle ij\rangle \leftrightarrow [ji]$). Charge conjugation reverses the arrows of each fermion line. In addition there is a factor (-1) for each external gauge boson. Thus we are left with just one independent helicity amplitude, which we take to be $A_\gamma(1^+,2^+,3^-,4^+,5^-)$.

Keeping the reference momentum q in the gluon polarization vector arbitrary we obtain:

$$\begin{aligned}
A_\gamma(1^+,2^+,3^-,4^+,5^-) \;=\;& \frac{i}{\sqrt{2}}\frac{1}{\langle q2\rangle}\frac{1}{s^2}\Big\{\, 2c_1\langle q5\rangle[42][12]\langle 23\rangle + c_2\frac{1}{s}\langle q1\rangle[12][43]\langle 35\rangle[12]\langle 23\rangle \\
&+c_3\frac{1}{s}\langle q3\rangle[32][41]\langle 15\rangle[12]\langle 23\rangle + c_4\frac{1}{s}\langle q1\rangle[12][41]\langle 15\rangle[12]\langle 23\rangle \\
&+c_5\frac{1}{s}\langle q3\rangle[32][43]\langle 35\rangle[12]\langle 23\rangle + 2c_6\langle q1\rangle[12][41]\langle 35\rangle \\
&+2c_7\langle q3\rangle[32][41]\langle 35\rangle + 2c_8\langle q3\rangle[12][41]\langle 15\rangle \\
&+2c_9\langle q3\rangle[12][43]\langle 35\rangle + 4c_{10}\langle q2\rangle[12][42]\langle 35\rangle \\
&+2c_{11}\langle q3\rangle[12][42]\langle 25\rangle + c_{12}\frac{1}{s}\langle q1\rangle[12][42]\langle 25\rangle[12]\langle 23\rangle \\
&+c_{13}\frac{1}{s}\langle q3\rangle[32][42]\langle 25\rangle[12]\langle 23\rangle\Big\}. \qquad (17)
\end{aligned}$$

Using $q = p_3$ and the constraints of eq. (12) this can be simplified to:

$$A_\gamma(1^+,2^+,3^-,4^+,5^-) =$$

$$
\frac{i}{\sqrt{2}}\frac{[12]}{s^3}\left\{s\langle 35\rangle[42]\left[(1-x_1)\left(c_2+\frac{2}{x_2}c_6-c_{12}\right)+(1-x_2)(c_4-c_{12})+2c_{12}\right]\right.
$$
$$
\left.-\langle 31\rangle[12]\left[[43]\langle 35\rangle\left(c_2+\frac{2}{x_2}c_6-c_{12}\right)+[41]\langle 15\rangle(c_4-c_{12})\right]\right\}. \tag{18}
$$

We see, that for this choice of helicities and q the amplitude A_γ is described by three linear combinations of c_2, c_4, c_6, c_{12}.

The perturbative expansion of the functions c_i and A_γ is defined through

$$
\begin{aligned}
c_i &= \sqrt{4\pi\alpha_s}\left(c_i^{(0)}+\left(\frac{\alpha_s}{2\pi}\right)c_i^{(1)}+\left(\frac{\alpha_s}{2\pi}\right)^2 c_i^{(2)}+O(\alpha_s{}^3)\right),\\
A_\gamma &= \sqrt{4\pi\alpha_s}\left(A_\gamma^{(0)}+\left(\frac{\alpha_s}{2\pi}\right)A_\gamma^{(1)}+\left(\frac{\alpha_s}{2\pi}\right)^2 A_\gamma^{(2)}+O(\alpha_s{}^3)\right).
\end{aligned} \tag{19}
$$

Inserting the leading-order result for the c_i

$$
c_2^{(0)}=c_4^{(0)}=c_{12}^{(0)}=0, \quad c_6^{(0)}=\frac{2}{x_1}, \tag{20}
$$

one gets the following result for the tree amplitude

$$
A_\gamma^{(0)}(1^+,2^+,3^-,4^+,5^-) = 2\sqrt{2}i\frac{\langle 35\rangle^2}{\langle 12\rangle\langle 23\rangle\langle 45\rangle}. \tag{21}
$$

In the helicity basis one can easily account for the Z-boson exchange by adjusting the couplings:

$$
\mathcal{A}_{\gamma Z}(1^{\lambda_1},2^{\lambda_2},3^{\lambda_3},4^{\lambda_4},5^{\lambda_5}) = e^2(-Q^q+v^e_{\lambda_4}v^q_{\lambda_1}\mathcal{P}_Z(s))A_\gamma(1^{\lambda_1},2^{\lambda_2},3^{\lambda_3},4^{\lambda_4},5^{\lambda_5}), \tag{22}
$$

with

$$
\mathcal{P}_Z(s)=\frac{s}{s-m_Z^2+im_Z\Gamma_Z}. \tag{23}
$$

Here, m_Z and Γ_Z are the mass and the width of the Z-boson. The left- and right handed couplings of fermions to the Z-boson are

$$
v_-^f=\frac{I_3^f-Q_f\sin^2\theta_W}{\sin\theta_W\cos\theta_W}, \qquad v_+^f=\frac{-Q_f\sin\theta_W}{\cos\theta_W}, \tag{24}
$$

where Q_f and I_3^f are the charge and the third component of the weak isospin of the fermion f and θ_W is the Weinberg angle.

2.2 Ultraviolet renormalization

The amplitudes we present are the renormalized ones, i.e. the ultraviolet subtraction has been performed. To obtain the renormalized amplitudes in the $\overline{\text{MS}}$ scheme, one replaces the bare coupling α_0 with the renormalized coupling $\alpha_s(\mu^2)$ evaluated at the renormalization scale μ^2:

$$\alpha_0 = \alpha_s S_\varepsilon^{-1}\left[1 - \frac{\beta_0}{\varepsilon}\left(\frac{\alpha_s}{2\pi}\right) + \left(\frac{\beta_0^2}{\varepsilon^2} - \frac{\beta_1}{2\varepsilon}\right)\left(\frac{\alpha_s}{2\pi}\right)^2 + O(\alpha_s^3)\right], \tag{25}$$

where

$$S_\varepsilon = (4\pi)^\varepsilon e^{-\varepsilon\gamma_E}, \tag{26}$$

is the typical phase-space volume factor in $d = 4-2\varepsilon$ dimensions, γ_E is Euler's constant, and β_0 and β_1 are the first two coeffcients of the QCD β-function:

$$\beta_0 = \frac{11}{6}C_A - \frac{2}{3}T_R n_f, \qquad \beta_1 = \frac{17}{6}C_A^2 - \frac{5}{3}C_A T_R n_f - C_F T_R n_f, \tag{27}$$

with the color factors

$$C_A = N, \quad C_F = \frac{N^2-1}{2N}, \quad T_R = \frac{1}{2}. \tag{28}$$

It is convenient here and for the subsequent discussion of the soft and collinear singularities to introduce a different notation [45]. In an orthogonal basis of unit vectors $|\bar{i}, i, a\rangle$ in the three parton color space we define an abstract vector $|\mathcal{M}\rangle$ through

$$\mathcal{A} \equiv \langle \bar{i}, i, a | \mathcal{M}\rangle, \tag{29}$$

with the expansion in the coupling α_s defined by

$$|\mathcal{M}\rangle = 4\pi\alpha\sqrt{4\pi\alpha_s}\left[\left|\mathcal{M}^{(0)}\right\rangle + \left(\frac{\alpha_s}{2\pi}\right)\left|\mathcal{M}^{(1)}\right\rangle + \left(\frac{\alpha_s}{2\pi}\right)^2\left|\mathcal{M}^{(2)}\right\rangle + O(\alpha_s^3)\right]. \tag{30}$$

Then, the renormalized two-loop amplitude can be expessed as

$$\left|\mathcal{M}_{\text{ren}}\right\rangle = 4\pi\alpha\sqrt{4\pi\alpha_s}S_\varepsilon^{-1/2}\left[\left|\mathcal{M}_{\text{ren}}^{(0)}\right\rangle + \left(\frac{\alpha_s}{2\pi}\right)\left|\mathcal{M}_{\text{ren}}^{(1)}\right\rangle + \left(\frac{\alpha_s}{2\pi}\right)^2\left|\mathcal{M}_{\text{ren}}^{(2)}\right\rangle + O(\alpha_s^3)\right], \tag{31}$$

At two loops the relation between the renormalized and the bare amplitudes is given by

$$\begin{aligned}
\left|\mathcal{M}_{\text{ren}}^{(0)}\right\rangle &= \left|\mathcal{M}_{\text{bare}}^{(0)}\right\rangle, \\
\left|\mathcal{M}_{\text{ren}}^{(1)}\right\rangle &= S_\varepsilon^{-1}\left|\mathcal{M}_{\text{bare}}^{(1)}\right\rangle - \frac{\beta_0}{2\varepsilon}\left|\mathcal{M}_{\text{bare}}^{(0)}\right\rangle, \\
\left|\mathcal{M}_{\text{ren}}^{(2)}\right\rangle &= S_\varepsilon^{-2}\left|\mathcal{M}_{\text{bare}}^{(2)}\right\rangle - \frac{3\beta_0}{2\varepsilon}S_\varepsilon^{-1}\left|\mathcal{M}_{\text{bare}}^{(1)}\right\rangle + \left(\frac{3\beta_0^2}{8\varepsilon^2} - \frac{\beta_1}{4\varepsilon}\right)\left|\mathcal{M}_{\text{bare}}^{(0)}\right\rangle.
\end{aligned} \tag{32}$$

Thus, we obtain for the renormalized functions $c_{i,\text{ren}}$

$$\begin{aligned}
c_i^{(1),\text{ren}} &= S_\varepsilon^{-1}c_i^{(1),\text{bare}}, \\
c_i^{(2),\text{ren}} &= S_\varepsilon^{-2}c_i^{(2),\text{bare}} - \frac{3\beta_0}{2\varepsilon}S_\varepsilon^{-1}c_i^{(1),\text{bare}},
\end{aligned} \tag{33}$$

for $i = \{2, 4, 12\}$ and

$$
\begin{aligned}
c_6^{(1),\text{ren}} &= S_\varepsilon^{-1} c_6^{(1),\text{bare}} - \frac{\beta_0}{2\varepsilon} c_6^{(0)}, \\
c_6^{(2),\text{ren}} &= S_\varepsilon^{-2} c_6^{(2),\text{bare}} - \frac{3\beta_0}{2\varepsilon} S_\varepsilon^{-1} c_6^{(1),\text{bare}} + \left(\frac{3\beta_0^2}{8\varepsilon^2} - \frac{\beta_1}{4\varepsilon}\right) c_6^{(2),\text{bare}}.
\end{aligned}
\tag{34}
$$

In this paper we set the renormalization scale $\mu^2 = s$. The complete scale dependence is easily recovered by expanding the prefactor

$$
\left(\frac{\mu^2}{s}\right)^{l\varepsilon}, \tag{35}
$$

accompanying an l-loop amplitude to the appropriate order in ε.

2.3 Infrared structure

Based on universal properties of soft and collinear limits, the infrared pole structure of two-loop amplitudes has been predicted by Catani [45]. Here, we briefly review how to organize these infrared poles for $e^+e^- \to q\bar{q}g$. We start with the one-loop amplitude, which can be written as

$$
\left|\mathcal{M}^{(1)}\right\rangle = \mathbf{I}^{(1)}(\varepsilon)\left|\mathcal{M}^{(0)}\right\rangle + \left|\mathcal{F}^{(1)}\right\rangle. \tag{36}
$$

Here $\mathbf{I}^{(1)}(\varepsilon)$ contains all infrared double and single poles in $1/\varepsilon$ and $|\mathcal{F}^{(1)}\rangle$ is a finite remainder. At two-loops, the corresponding formula reads:

$$
\left|\mathcal{M}^{(2)}\right\rangle = \mathbf{I}^{(1)}(\varepsilon)\left|\mathcal{M}^{(1)}\right\rangle + \mathbf{I}^{(2)}(\varepsilon)\left|\mathcal{M}^{(0)}\right\rangle + \left|\mathcal{F}^{(2)}\right\rangle. \tag{37}
$$

The one-loop insertion operator $\mathbf{I}^{(1)}$ is given by

$$
\mathbf{I}^{(1)}(\varepsilon) = \frac{1}{2}\frac{1}{\Gamma(1-\varepsilon)} e^{\varepsilon\gamma_E} \sum_i \frac{1}{\mathbf{T}_i^2} \mathcal{V}_i(\varepsilon) \sum_{j\neq i} \mathbf{T}_i \mathbf{T}_j \left(\frac{\mu^2}{-s_{ij}}\right)^{\varepsilon}, \tag{38}
$$

where

$$
\mathcal{V}_i(\varepsilon) = \mathbf{T}_i^2 \frac{1}{\varepsilon^2} + \gamma_i \frac{1}{\varepsilon}, \tag{39}
$$

and the coefficients $\mathbf{T}_i^2$ and γ_i are

$$
\begin{aligned}
&\mathbf{T}_q^2 = \mathbf{T}_{\bar{q}}^2 = C_F, \qquad \mathbf{T}_g^2 = C_A, \\
&\gamma_q = \gamma_{\bar{q}} = \frac{3}{2} C_F, \qquad \gamma_g = \beta_0.
\end{aligned}
\tag{40}
$$

In general, the color operators $\mathbf{T}_i\mathbf{T}_j$ give rise to color correlations. However, the color structure for the amplitude $e^+e^- \to q\bar{q}g$ is rather trivial and the color operators are proportional to the identity matrix in color space:

$$\mathbf{T}_q\mathbf{T}_{\bar{q}} = T_R\frac{1}{N},$$
$$\mathbf{T}_q\mathbf{T}_g = \mathbf{T}_g\mathbf{T}_{\bar{q}} = -T_R N. \tag{41}$$

Explicitly, the one-loop insertion operator reads for $e^+e^- \to q\bar{q}g$:

$$\begin{aligned}\mathbf{I}^{(1)}(\varepsilon) = {} & \frac{1}{2}\frac{1}{\Gamma(1-\varepsilon)}e^{\varepsilon\gamma_E}\left(\frac{\mu^2}{-s}\right)^{\varepsilon} T_R \\ & \times\left[\frac{2}{N}\left(1-x_1-x_2\right)^{-\varepsilon}\left(\frac{1}{\varepsilon^2}+\frac{3}{2}\frac{1}{\varepsilon}\right) - N\left(x_1^{-\varepsilon}+x_2^{-\varepsilon}\right)\left(\frac{2}{\varepsilon^2}+\left(\frac{\beta_0}{C_A}+\frac{3}{2}\right)\frac{1}{\varepsilon}\right)\right].\end{aligned} \tag{42}$$

The two-loop insertion operator has the form

$$\begin{aligned}& \mathbf{I}^{(2)}(\varepsilon) = \\ & \quad -\frac{1}{2}\mathbf{I}^{(1)}(\varepsilon)\left(\mathbf{I}^{(1)}(\varepsilon)+2\beta_0\frac{1}{\varepsilon}\right) + e^{-\varepsilon\gamma_E}\frac{\Gamma(1-2\varepsilon)}{\Gamma(1-\varepsilon)}\left(\beta_0\frac{1}{\varepsilon}+K\right)\mathbf{I}^{(1)}(2\varepsilon)+\mathbf{H}^{(2)},\end{aligned} \tag{43}$$

where

$$K = \left(\frac{67}{18}-\frac{\pi^2}{6}\right)C_A - \frac{10}{9}T_R n_f. \tag{44}$$

The function $\mathbf{H}^{(2)}$ is process- and scheme-dependent and for $e^+e^- \to q\bar{q}g$, it is given by [46, 47]

$$\mathbf{H}^{(2)} = \frac{1}{4}\frac{1}{\Gamma(1-\varepsilon)}e^{\varepsilon\gamma_E}\frac{1}{\varepsilon}\left(\mathbf{H}_q^{(2)}+\mathbf{H}_g^{(2)}+\mathbf{H}_{\bar{q}}^{(2)}\right), \tag{45}$$

where $\mathbf{H}_q^{(2)} = \mathbf{H}_{\bar{q}}^{(2)}$ and

$$\begin{aligned}\mathbf{H}_q^{(2)} &= \left(\frac{7}{4}\zeta_3-\frac{11}{96}\pi^2+\frac{409}{864}\right)N^2+\left(-\frac{1}{4}\zeta_3-\frac{\pi^2}{96}-\frac{41}{108}\right)+\left(-\frac{3}{2}\zeta_3+\frac{\pi^2}{8}-\frac{3}{32}\right)\frac{1}{N^2} \\ &\quad +\left(\frac{\pi^2}{48}-\frac{25}{216}\right)\frac{N^2-1}{N}n_f, \\ \mathbf{H}_g^{(2)} &= \left(\frac{\zeta_3}{2}+\frac{11}{144}\pi^2+\frac{5}{12}\right)N^2+\left(-\frac{\pi^2}{72}-\frac{89}{108}\right)Nn_f-\frac{n_f}{4N}+\frac{5}{27}n_f^2.\end{aligned} \tag{46}$$

Using the above results we define the finite functions $c_i^{(j),\mathrm{fin}}$:

$$\begin{aligned}c_i^{(1),\mathrm{fin}} &= c_i^{(1),\mathrm{ren}}, \\ c_i^{(2),\mathrm{fin}} &= c_i^{(1),\mathrm{ren}} - \mathbf{I}^{(1)}(\varepsilon)c_i^{(1),\mathrm{ren}}\end{aligned} \tag{47}$$

for $i = \{2,4,12\}$ and

$$\begin{aligned} c_6^{(1),\text{fin}} &= c_6^{(1),\text{ren}} - \mathbf{I}^{(1)}(\varepsilon) c_6^{(0)}, \\ c_6^{(2),\text{fin}} &= c_6^{(1),\text{ren}} - \mathbf{I}^{(1)}(\varepsilon) c_6^{(1),\text{ren}} - \mathbf{I}^{(2)}(\varepsilon) c_6^{(0)}. \end{aligned} \tag{48}$$

Explicit results for the functions $c_i^{(2),\text{fin}}$ are given in section 4 and for $c_i^{(1),\text{fin}}$ in the appendix A.

3 Method of calculation

In this section, we will discuss the method to calculate the virtual amplitudes for $e^+e^- \to q\bar{q}g$. We have used QGRAF [48] for the generation of all Feynman diagrams, which contribute to the process $e^+e^- \to q\bar{q}g$ up to two loops.

The evaluation of the diagrams leads to tensor integrals, which multiply the various spinor structures of eq. (8). Note, that there is no need here to consider projectors for the various spinor coefficients.

The tensor integrals are mapped to combinations of scalar integrals with higher powers of propagators and different values of d [22]-[24]. For this purpose, one introduces Schwinger parameters for the propagators

$$\frac{1}{(-k^2)^\nu} = \frac{1}{\Gamma(\nu)} \int\limits_0^\infty dx\, x^{\nu-1} \exp(xk^2). \tag{49}$$

Combining the exponentials arising from different propagators one obtains a quadratic form in the loop momenta. For instance, for a given two-loop integral with loop momenta k_1 and k_2, one has then

$$\begin{aligned} I(d,\nu_1,\dots,\nu_k) &= \int \frac{d^dk_1}{i\pi^{d/2}} \int \frac{d^dk_2}{i\pi^{d/2}} \frac{1}{(-k_1^2)^{\nu_1} \dots (-k_n^2)^{\nu_n}} \\ &= \int \frac{d^dk_1}{i\pi^{d/2}} \int \frac{d^dk_2}{i\pi^{d/2}} \left(\prod_{i=1}^n \frac{1}{\Gamma(\nu_i)} \int_0^\infty dx_i\, x_i^{\nu_i - 1} \right) \exp\left(\sum_{i=1}^n x_i k_i^2 \right), \end{aligned} \tag{50}$$

and

$$\sum_{i=1}^n x_i k_i^2 = a\, k_1^2 + b\, k_2^2 + 2\, c\, k_1 \cdot k_2 + 2\, d \cdot k_1 + 2\, e \cdot k_2 + f\ . \tag{51}$$

The momenta $k_3,\dots,k_k$ are linear combinations of the loop momenta k_1,k_2 and the external momenta. The coefficients a, b, c, d^μ, e^μ and f are directly readable from the actual graph: $a(b) = \sum x_i$, where the sum runs over the legs in the k_1 (k_2) loop, and $c = \sum x_i$ with the sum running over the legs common to both loops. With a suitable change of variables for the loop

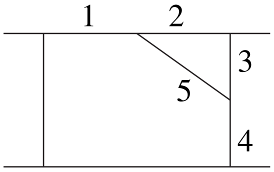

Figure 2: The PentaBox represents a typical diagram where the triangle relation can be applied. In the present calculation, the lower left leg is massive.

momenta k_1, k_2, one can diagonalize the quadratic form and the momentum integration can be performed as Gaussian integrals over the shifted loop momenta according to

$$\int \frac{d^d k}{i\pi^{d/2}} \exp\left(\mathcal{P}k^2\right) \quad = \quad \frac{1}{\mathcal{P}^{d/2}}. \tag{52}$$

Lorentz invariance allows immediately to relate the following symmetric tensor integrals to scalar integrals:

$$\begin{aligned}
\int \frac{d^d k}{i\pi^{d/2}} k^\mu k^\nu f(k^2) &= \frac{1}{d} g^{\mu\nu} \int \frac{d^d k}{i\pi^{d/2}} k^2 f(k^2), \\
\int \frac{d^d k}{i\pi^{d/2}} k^\mu k^\nu k^\rho k^\sigma f(k^2) &= \frac{1}{d(d+2)} \left(g^{\mu\nu} g^{\rho\sigma} + g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho}\right) \int \frac{d^d k}{i\pi^{d/2}} k^2 f(k^2),
\end{aligned} \tag{53}$$

and the generalization to arbitrary higher tensor structures is obvious.

In the remaining Schwinger parameter integrals, the tensor integrals introduce additional factors of the parameters x_i and of $1/\mathcal{P}$. These additional factors can be absorbed into scalar integrals with higher powers of propagators and shifted dimensions, by introducing operators $\mathbf{i}^+$, which raise the power of propagator i by one, or an operator $\mathbf{d}^+$ that increases the dimension by two,

$$\begin{aligned}
\nu_i \mathbf{i}^+ \frac{1}{\left(-k_i^2\right)^{\nu_i}} &= \nu_i \frac{1}{\left(-k_i^2\right)^{\nu_i+1}} = \frac{1}{\Gamma(\nu_i)} \int\limits_0^\infty dx_i \, x_i^{\nu_i - 1} x_i \exp(x_i k^2), \\
\mathbf{d}^+ \int \frac{d^d k}{i\pi^{d/2}} \exp\left(\mathcal{P}k^2\right) &= \int \frac{d^{(d+2)} k}{i\pi^{(d+2)/2}} \exp\left(\mathcal{P}k^2\right) = \frac{1}{\mathcal{P}} \frac{1}{\mathcal{P}^{d/2}}.
\end{aligned} \tag{54}$$

At this stage, we are left with sets of scalar integrals of a given topology in $4+2m-2\varepsilon$-dimensions, and with raised powers of propagators, where m is a non-negative integer. Then, for some topologies like the PentaBox in fig. (2), immediate simplifications are possible. By means of integration-by-parts [25, 26] these topologies reduce to simpler ones. For the PentaBox in fig.

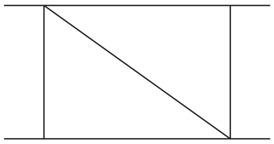

Figure 3: The generic CBox. For the CBox_2, the upper right external leg is massive.

(2), for instance, partial integration provides the following triangle relations,

$$\begin{aligned}\left[(d-2\nu_2-\nu_3-\nu_5)-\nu_3\mathbf{3}^+\mathbf{2}^- -\nu_5\mathbf{5}^+\left(\mathbf{2}^- -\mathbf{1}^-\right)\right]\mathrm{PentaBox}(m,\nu_1,\ldots,\nu_7) &= 0,\\ \left[(d-\nu_2-2\nu_3-\nu_5)-\nu_2\mathbf{2}^+\mathbf{3}^- -\nu_5\mathbf{5}^+\left(\mathbf{3}^- -\mathbf{4}^-\right)\right]\mathrm{PentaBox}(m,\nu_1,\ldots,\nu_7) &= 0. \quad (55)\end{aligned}$$

These relations can be used to eliminate the propagators 1 and 4, such that the PentaBox becomes reducible to the CBox_2 shown in fig. (3).

After having performed obvious simplifications based on triangle relations, we are then able, to calculate directly all necessary scalar integrals with the method of nested sums [28]. For each topology, this requires analytical solutions valid in any dimension and for any (not necessarily integer) power of the propagators in terms of nested sums. For the fermionic contributions to $e^+e^- \to q\bar{q}g$ up to two loops, it suffices to have these analytical expressions for the CBox_2 [28], the one-loop box with one external mass [49] and the one-loop triangle with two external masses [50].

We give the explicit representations as nested sums for the basic integrals. As a short-hand notation, we use $\nu_{ij} = \nu_i + \nu_j$ for sums of powers of propagators in the following. The one-loop triangle with two external masses is defined by

$$\mathrm{Tri}_2(m,\nu_1,\nu_2,\nu_3;x_1) = (-s_{123})^{-m+\varepsilon+\nu_{123}} \int \frac{d^dk_1}{i\pi^{d/2}} \frac{1}{\left(-k_1^2\right)^{\nu_1}} \frac{1}{\left(-k_2^2\right)^{\nu_2}} \frac{1}{\left(-k_3^2\right)^{\nu_3}}, \quad (56)$$

where $k_2 = k_1 - p_1 - p_2$, $k_3 = k_2 - p_3$. It can be written as a combination of hypergeometric functions ${}_2F_1$. The series representation for this integral is given by

$$\begin{aligned}\mathrm{Tri}_2(m,\nu_1,\nu_2,\nu_3;x_1) &= \frac{\Gamma(\varepsilon-m+\nu_{23})\Gamma(1-\varepsilon+m-\nu_{23})\Gamma(m-\varepsilon-\nu_{13})}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu_3)\Gamma(2m-2\varepsilon-\nu_{123})} \quad (57)\\ &\times \sum_{i=0}^{\infty} \frac{x_1^i}{i!}\left[x_1^{m-\varepsilon-\nu_{23}} \frac{\Gamma(i_1+\nu_1)\Gamma(i_1-\varepsilon+m-\nu_2)}{\Gamma(i_1+1+m-\varepsilon-\nu_{23})} - \frac{\Gamma(i_1+\nu_3)\Gamma(i_1-m+\varepsilon+\nu_{123})}{\Gamma(i_1+1-m+\varepsilon+\nu_{23})}\right].\end{aligned}$$

The one-loop box integral is defined by

$$\begin{aligned}&\mathrm{Box}(m,\nu_1,\nu_2,\nu_3,\nu_4;x_1,x_2) = \quad (58)\\ &\quad (-s_{123})^{-m+\varepsilon+\nu_{1234}} \int \frac{d^dk_1}{i\pi^{d/2}} \frac{1}{\left(-k_1^2\right)^{\nu_1}} \frac{1}{\left(-k_2^2\right)^{\nu_2}} \frac{1}{\left(-k_3^2\right)^{\nu_3}} \frac{1}{\left(-k_4^2\right)^{\nu_4}},\end{aligned}$$

where $k_2 = k_1 - p_1$, $k_3 = k_2 - p_2$ and $k_4 = k_3 - p_3$. This integral can be expressed in terms of a combination of Appell functions of the second kind and the series representation is given by:

$$
\begin{aligned}
&\mathrm{Box}(m,\nu_1,\nu_2,\nu_3,\nu_4;x_1,x_2) = \\
&\quad \frac{\Gamma(m-\varepsilon-\nu_{123})\Gamma(m-\varepsilon-\nu_{234})\Gamma(1+\nu_{123}-m+\varepsilon)\Gamma(1+\nu_{234}-m+\varepsilon)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu_3)\Gamma(\nu_4)\Gamma(2m-2\varepsilon-\nu_{1234})} \\
&\quad \times \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\frac{x_1^{i_1}}{i_1!}\frac{x_2^{i_2}}{i_2!}\left[\frac{\Gamma(i_1+\nu_3)\Gamma(i_2+\nu_2)\Gamma(i_1+i_2-m+\varepsilon+\nu_{1234})}{\Gamma(i_1+1-m+\varepsilon+\nu_{123})\Gamma(i_2+1-m+\varepsilon+\nu_{234})}\right. \\
&\quad - x_1^{m-\varepsilon-\nu_{123}}\frac{\Gamma(i_1+m-\varepsilon-\nu_{12})\Gamma(i_2+\nu_2)\Gamma(i_1+i_2+\nu_4)}{\Gamma(i_1+1+m-\varepsilon-\nu_{123})\Gamma(i_2+1-m+\varepsilon+\nu_{234})} \\
&\quad - x_2^{m-\varepsilon-\nu_{234}}\frac{\Gamma(i_1+\nu_3)\Gamma(i_2+m-\varepsilon-\nu_{34})\Gamma(i_1+i_2+\nu_1)}{\Gamma(i_1+1-m+\varepsilon+\nu_{123})\Gamma(i_2+1+m-\varepsilon-\nu_{234})} \\
&\quad \left. + x_1^{m-\varepsilon-\nu_{123}}x_2^{m-\varepsilon-\nu_{234}}\frac{\Gamma(i_1+m-\varepsilon-\nu_{12})\Gamma(i_2+m-\varepsilon-\nu_{34})\Gamma(i_1+i_2+m-\varepsilon-\nu_{23})}{\Gamma(i_1+1+m-\varepsilon-\nu_{123})\Gamma(i_2+1+m-\varepsilon-\nu_{234})}\right].
\end{aligned}
\tag{59}
$$

Finally, the two-loop CBox_2 is defined by

$$
\begin{aligned}
&\mathrm{CBox}_2(m,\nu_1,\nu_2,\nu_3,\nu_4,\nu_5;x_1,x_2) = \\
&\quad (-s_{123})^{-2m+2\varepsilon+\nu_{12345}}\int\frac{d^dk_1}{i\pi^{d/2}}\int\frac{d^dl_5}{i\pi^{d/2}}\frac{1}{(-k_1^2)^{\nu_1}}\frac{1}{(-l_2^2)^{\nu_2}}\frac{1}{(-l_3^2)^{\nu_3}}\frac{1}{(-k_4^2)^{\nu_4}}\frac{1}{(-l_5^2)^{\nu_5}},
\end{aligned}
\tag{60}
$$

with $l_2 = k_1 + l_5 - p_1$, $l_3 = l_2 - p_2$, $k_4 = k_1 - p_{123}$. The formula for this integral is given by:

$$
\begin{aligned}
&\mathrm{CBox}_2(m,\nu_1,\nu_2,\nu_3,\nu_4,\nu_5;x_1,x_2) = \\
&\quad \frac{\Gamma(2m-2\varepsilon-\nu_{1235})\Gamma(1+\nu_{1235}-2m+2\varepsilon)\Gamma(2m-2\varepsilon-\nu_{2345})\Gamma(1+\nu_{2345}-2m+2\varepsilon)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu_3)\Gamma(\nu_4)\Gamma(\nu_5)\Gamma(3m-3\varepsilon-\nu_{12345})} \\
&\quad \times \frac{\Gamma(m-\varepsilon-\nu_5)\Gamma(m-\varepsilon-\nu_{23})}{\Gamma(2m-2\varepsilon-\nu_{235})}\sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\frac{x_1^{i_1}}{i_1!}\frac{x_2^{i_2}}{i_2!} \\
&\quad \times \left[\frac{\Gamma(i_1+\nu_3)\Gamma(i_2+\nu_2)\Gamma(i_1+i_2-2m+2\varepsilon+\nu_{12345})\Gamma(i_1+i_2-m+\varepsilon+\nu_{235})}{\Gamma(i_1+1-2m+2\varepsilon+\nu_{1235})\Gamma(i_2+1-2m+2\varepsilon+\nu_{2345})\Gamma(i_1+i_2+\nu_{23})}\right. \\
&\quad - x_1^{2m-2\varepsilon-\nu_{1235}} \\
&\quad \times \frac{\Gamma(i_1+2m-2\varepsilon-\nu_{125})\Gamma(i_2+\nu_2)\Gamma(i_1+i_2+\nu_4)\Gamma(i_1+i_2+m-\varepsilon-\nu_1)}{\Gamma(i_1+1+2m-2\varepsilon-\nu_{1235})\Gamma(i_2+1-2m+2\varepsilon+\nu_{2345})\Gamma(i_1+i_2+2m-2\varepsilon-\nu_{15})} \\
&\quad - x_2^{2m-2\varepsilon-\nu_{2345}} \\
&\quad \times \frac{\Gamma(i_1+\nu_3)\Gamma(i_2+2m-2\varepsilon-\nu_{345})\Gamma(i_1+i_2+\nu_1)\Gamma(i_1+i_2+m-\varepsilon-\nu_4)}{\Gamma(i_1+1-2m+2\varepsilon+\nu_{1235})\Gamma(i_2+1+2m-2\varepsilon-\nu_{2345})\Gamma(i_1+i_2+2m-2\varepsilon-\nu_{45})} \\
&\quad + x_1^{2m-2\varepsilon-\nu_{1235}}x_2^{2m-2\varepsilon-\nu_{2345}}\frac{\Gamma(i_1+2m-2\varepsilon-\nu_{125})\Gamma(i_2+2m-2\varepsilon-\nu_{345})}{\Gamma(i_1+1+2m-2\varepsilon-\nu_{1235})\Gamma(i_2+1+2m-2\varepsilon-\nu_{2345})} \\
&\quad \left. \times \frac{\Gamma(i_1+i_2+2m-2\varepsilon-\nu_{235})\Gamma(i_1+i_2+3m-3\varepsilon-\nu_{12345})}{\Gamma(i_1+i_2+4m-4\varepsilon-\nu_{12345}-\nu_5)}\right].
\end{aligned}
\tag{61}
$$

Then, the evaluation of the multiple nested sums proceeds systematically with the help of the algorithms of [28]. These algorithms rely on the algebraic properties of the so called Z-sums and allow to solve by means of recursion the nested sums in terms of a given basis in Z-sums to any order in the expansion parameter ε. Z-sums are defined by

$$Z(n;m_1,...,m_k;x_1,...,x_k) \quad = \sum_{n\geq i_1>i_2>...>i_k>0} \frac{x_1^{i_1}}{i_1{}^{m_1}} \cdots \frac{x_k^{i_k}}{i_k{}^{m_k}}. \tag{62}$$

They are generalizations of Euler-Zagier sums [51, 52]

$$Z_{m_1,...,m_k}(n) \quad = \quad Z(n;m_1,...,m_k;1,...,1), \tag{63}$$

or of harmonic sums [53] - [56] involving multiple ratios of scales. The latter are well known from calculations of Mellin moments of deep-inelastic structure functions [53], [57] - [60]. An important subset of Z-sums are multiple polylogarithms [30]-[32], obtained by letting n go to infinity in eq. (62):

$$\mathrm{Li}_{m_k,...,m_1}(x_k,...,x_1) \quad = \quad Z(\infty;m_1,...,m_k;x_1,...,x_k). \tag{64}$$

The key feature of Z-sums is the fact, that they interpolate between Goncharov's multiple polylogarithms and Euler-Zagier sums, which occur in the expansion of Γ-functions,

$$\begin{aligned}\Gamma(n+\varepsilon) &= \Gamma(1+\varepsilon)\Gamma(n) \\ &\times \left(1+\varepsilon Z_1(n-1)+\varepsilon^2 Z_{11}(n-1)+\varepsilon^3 Z_{111}(n-1)+...+\varepsilon^{n-1}Z_{11...1}(n-1)\right).\end{aligned} \tag{65}$$

In summary, the evaluation of the multiple nested sums proceeds by expanding the Γ-functions to the desired order, and by using then the algebraic properties of the Z-sums. In this way, we could calculate all loop integrals contributing to the one- and two-loop virtual amplitudes very efficiently in terms of multiple polylogarithms. All algorithms for this procedure have been implemented on a computer in symbolic manipulation programs. We have chosen to work with FORM [61] and in the GiNaC framework [62, 63].

4 Results

Let us now discuss our results for the finite coefficients $c_i^{(1),\mathrm{fin}}$ through order ε^2 and the n_f-contributions to $c_i^{(2),\mathrm{fin}}$.

We have made the following checks on our result. First of all, after UV renormalization, the infrared poles agree with the structure predicted by Catani [45]. This provides a strong check of the complete pole structure of our result. Secondly, having calculated all coefficients c_1–c_{13}, we could use the various relations eqs. (11) and (12) between c_1–c_{13} as a cross-check. Note that for

c_6 there is an additional symmetry: one can show that the combination $x_1 c_6$ is symmetric under the exchange $(x_1 \leftrightarrow x_2)$,

$$x_1 c_6(x_1,x_2) \;=\; x_2 c_6(x_2,x_1)\,. \tag{66}$$

Finally, we could compare with recent work of Garland et al.. They have obtained by an independent method the result for the squared matrix elements [46], i.e. the interference of the two-loop amplitude with the Born amplitude, and the interference of the one-loop amplitude with itself, as well as the result for the one- and two-loop amplitude [33]. Their results are given in terms of one- and two-dimensional harmonic polylogarithms, which form a subset of the multiple polylogarithms [30]-[32]. Thus, we have performed the comparision analytically. We agree with both of their results.

Here, we present explicitly the function $c_i^{(2),\mathrm{fin}}$, $i = 2,4,6,12$. All formulae for the one-loop amplitude are deferred to appendix A.

$$\begin{aligned}
c_2^{(2),\mathrm{fin}}(x_1,x_2) &= n_f N\Big(-2\frac{x_2}{(x_1+x_2)^3}\mathrm{R}_1(x_1,x_2) + \frac{\mathrm{R}_2(x_1,x_2)}{(1-x_1-x_2)^2} - \frac{3}{4}\frac{\ln(x_1)^2}{(1-x_1)(1-x_1-x_2)} \\
&+\frac{1}{6}\frac{(-6+19x_2+25x_1)}{(x_1+x_2)(1-x_1-x_2)(1-x_1)}\ln(x_1) + \frac{1}{12}\frac{(-2x_1+x_1x_2+x_1{}^2-2+2x_2)}{x_1(1-x_1)(1-x_2)(1-x_1-x_2)}\zeta_2 \\
&+\frac{1}{4}\frac{(\ln(x_1)+\ln(x_2))\zeta_2}{(1-x_1-x_2)^2} - \frac{1}{4}\frac{(2-2x_2+x_1)}{x_1(1-x_1-x_2)(1-x_2)}\ln(x_2)^2 \\
&+\frac{1}{18}\frac{(44x_1-31x_1x_2+13x_1{}^2+26x_2-26x_2{}^2)}{x_1(x_1+x_2)(1-x_2)(1-x_1-x_2)}\ln(x_2) + \frac{1}{12}\Big[-8-\frac{(1+x_1)}{(1-x_2)} - \frac{24x_1}{(x_1+x_2)} \\
&+2\frac{(13+6x_1)}{(1-x_1-x_2)} - 10\frac{x_2}{x_1} - 3\frac{(1+x_2)}{(1-x_1)}\Big]\frac{\mathrm{R}(x_1,x_2)}{(x_1+x_2)(1-x_1-x_2)} + \frac{1}{12}\Big[-\frac{12}{x_1(x_1+x_2)} \\
&+\frac{24}{(x_1+x_2)^2} - \frac{1}{x_1(1-x_2)} + 3\frac{(1+4x_1)}{x_1(1-x_1)(1-x_1-x_2)}\Big][\mathrm{Li}_2(1-x_1)-\mathrm{Li}_2(1-x_2)] \\
&+\frac{1}{2}\frac{1}{x_1}\Big[-\frac{1}{(1-x_2)} + 3\frac{1}{(1-x_1)(1-x_1-x_2)}\Big]\mathrm{Li}_2(1-x_2)\Big) \\
&+\frac{n_f}{N}\Big(\frac{(x_1{}^2-x_2{}^2)}{3x_1{}^2x_2{}^2}\Big[\mathrm{Li}_3(x_1+x_2)+\mathrm{Li}_3(1-x_1-x_2)+\frac{1}{2}\ln(x_1+x_2)\ln(1-x_1-x_2)^2 \\
&-\frac{1}{4}\ln(x_1)\mathrm{R}(x_1,1-x_1-x_2) + \frac{1}{4}(\ln(x_1)^2-\ln(x_1)\ln(x_2)+\ln(x_2)^2)\ln(1-x_1-x_2) \\
&-\frac{1}{4}\ln(x_2)\mathrm{R}(x_2,1-x_1-x_2)\Big] + \frac{1}{6x_1x_2}\Big[\frac{1}{2}\mathrm{Li}_2(1-x_2) - \frac{5}{2}\mathrm{R}(x_1,1-x_1-x_2) - 2\zeta_2 \\
&+\frac{5}{2}\mathrm{Li}_2(1-x_1) - \mathrm{Li}_2(1-x_1-x_2) - \ln(1-x_1-x_2)^2 - \ln(x_1+x_2)\ln(1-x_1-x_2) \\
&-\frac{1}{2}\mathrm{R}(x_2,1-x_1-x_2)\Big] + \frac{1}{3x_2{}^2}\Big[\frac{1}{4}\ln(x_1)^2\ln(1-x_1-x_2) - \frac{1}{2}\ln(x_1)\ln(1-x_1-x_2)^2 \\
&-\frac{1}{2}\ln(x_1)^2\ln(1-x_1) - \frac{1}{4}\ln(x_2)(\mathrm{Li}_2(1-x_1)-\mathrm{Li}_2(1-x_2)) - \mathrm{Li}_3(x_1) - \mathrm{Li}_3(1-x_1)
\end{aligned}$$

$$
\begin{aligned}
&-\ln(x_1)\mathrm{R}(x_1,1-x_1-x_2)+\frac{19}{6}\mathrm{R}(x_1,1-x_1-x_2)\Big]+\frac{1}{3{x_1}^2}\Big[\frac{1}{2}\ln(x_2)\ln(1-x_1-x_2)^2\\
&-\frac{1}{4}\ln(x_2)^2\ln(1-x_1-x_2)+\frac{1}{2}\ln(x_2)^2\ln(1-x_2)-\frac{1}{4}\ln(x_1)(\mathrm{Li}_2(1-x_1)-\mathrm{Li}_2(1-x_2))\\
&+\mathrm{Li}_3(x_2)+\mathrm{Li}_3(1-x_2)+\ln(x_2)\mathrm{R}(x_2,1-x_1-x_2)-\frac{19}{6}\mathrm{R}(x_2,1-x_1-x_2)\Big]\\
&+\frac{1}{18}\frac{(19x_1+13x_2)}{x_1x_2(x_1+x_2)}\ln(1-x_1-x_2)+\frac{1}{4}\frac{(-1+3x_2+x_1)}{x_2(1-x_1)(1-x_1-x_2)}\ln(x_1)^2\\
&-\frac{1}{18}\frac{(-19+19x_1+51x_2)}{x_2(1-x_1)(1-x_1-x_2)}\ln(x_1)+\frac{1}{3}\frac{(1+x_1+x_2)}{x_1(1-x_1-x_2)(x_1+x_2)}\mathrm{R}(x_2,1-x_1-x_2)\\
&+\frac{1}{4}\frac{(1-x_2+x_1)}{x_1(1-x_2)(1-x_1-x_2)}\ln(x_2)^2-\frac{1}{18}\frac{(19-19x_2+13x_1)}{x_1(1-x_2)(1-x_1-x_2)}\ln(x_2)+\frac{1}{3}\frac{\mathrm{Li}_2(1-x_2)}{x_1(1-x_2)}\\
&-\frac{1}{12}\Big[\frac{1}{x_1(1-x_2)}+\frac{1}{x_2(1-x_1)}-\frac{2}{x_1(1-x_1)(1-x_1-x_2)}\Big](\zeta_2+\mathrm{R}(x_1,x_2)\\
&-\mathrm{Li}_2(1-x_1)-\mathrm{Li}_2(1-x_2))+\frac{1}{3}\frac{(\mathrm{Li}_2(1-x_1)-\mathrm{Li}_2(1-x_2)-\mathrm{R}(x_1,1-x_1-x_2))}{x_1(x_1+x_2)}\\
&-\frac{\mathrm{Li}_2(1-x_1)}{(1-x_1)(1-x_1-x_2)}+\frac{1}{3}\frac{(\mathrm{Li}_2(1-x_1)+2\mathrm{R}(x_1,1-x_1-x_2))}{x_2(1-x_1-x_2)}-\frac{2}{3}\frac{\mathrm{Li}_2(1-x_2)}{x_1(1-x_1-x_2)}\Big)\\
&+i\pi n_f N\Big(\frac{3}{2}\frac{\ln(x_1)}{(1-x_1)(1-x_1-x_2)}+\frac{1}{2}\frac{(2-2x_2+x_1)}{x_1(1-x_2)(1-x_1-x_2)}\ln(x_2)+\frac{3}{2}\frac{\mathrm{R}(x_1,x_2)}{(1-x_1-x_2)^2}\Big)\\
&+\frac{i\pi n_f}{N}\Big(\frac{1}{2}\frac{1}{{x_2}^2}\mathrm{R}(x_1,1-x_1-x_2)-\frac{1}{2}\frac{1}{{x_1}^2}\mathrm{R}(x_2,1-x_1-x_2)+\frac{1}{2}\frac{\ln(1-x_1-x_2)}{x_1x_2}\\
&-\frac{1}{2}\frac{(-1+3x_2+x_1)}{x_2(1-x_1)(1-x_1-x_2)}\ln(x_1)-\frac{1}{2}\frac{(1-x_2+x_1)}{x_1(1-x_2)(1-x_1-x_2)}\ln(x_2)\Big),
\end{aligned}
\tag{67}
$$

$$
\begin{aligned}
&c_4^{(2),\mathrm{fin}}(x_1,x_2)=n_fN\Big(\frac{1}{36}\frac{\ln(x_2)}{(1-x_2)}\Big[36\frac{(1+x_1)}{(x_1+x_2)}-\frac{59}{x_1}-\frac{50}{x_1(1-x_2)}+12\frac{(1-3x_1)(1-x_1)}{x_1(1-x_1-x_2)}\Big]\\
&-\frac{19}{18}\frac{1}{x_1(1-x_2)}+\frac{(x_1-x_2)}{(x_1+x_2)^3}\mathrm{R}_1(x_1,x_2)-\Big[\frac{5}{12}\frac{(2-x_2)}{x_1(1-x_2)^2}+\frac{2}{(x_1+x_2)^2}\Big]\mathrm{Li}_2(1-x_2)\\
&+\frac{1}{4}\frac{(2-x_2)}{x_1(1-x_2)^2}\Big[\ln(x_2)^2+\frac{1}{3}\zeta_2\Big]-\frac{1}{12}\frac{(7x_1-5x_2-6x_1x_2+5{x_2}^2+{x_1}^2)}{(1-x_2)x_1(1-x_1-x_2)(x_1+x_2)}\ln(x_1)\\
&-\frac{\mathrm{R}(x_1,x_2)}{(1-x_1-x_2)(1-x_2)(x_1+x_2)}\Big[\frac{17}{6}x_2-\frac{35}{12}x_1-\frac{19}{12}+\frac{1}{12}\frac{(1+x_1)}{(1-x_2)}-\frac{1}{3}\frac{(1-3x_1)(1-x_1)}{(1-x_1-x_2)}\\
&+\frac{5{x_2}^2}{12x_1}-\frac{5}{6}\frac{x_2}{x_1}+2\frac{x_1(1+x_1)}{(x_1+x_2)}\Big]-\Big[\frac{1}{12}\frac{(2-x_2)}{x_1(1-x_2)^2}-\frac{2}{(x_1+x_2)^2}\Big]\mathrm{Li}_2(1-x_1)\Big)\\
&+\frac{n_f}{N}\Big(-\frac{1}{3}\frac{1}{{x_1}^2}\Big[\mathrm{Li}_2(1-x_1-x_2)+\ln(1-x_1-x_2)^2+\ln(x_1+x_2)\ln(1-x_1-x_2)\Big]\\
&+\frac{19}{18}\frac{1}{(1-x_2)x_1}+\frac{(x_1+x_2)}{3{x_1}^3}\Big[\frac{1}{2}\ln(x_1)^2\ln(1-x_1-x_2)+(\ln(x_1+x_2)-\ln(x_2))\ln(1-x_1-x_2)^2
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{2}\ln(x_1)\mathrm{Li}_2(1-x_1)-\frac{1}{2}\mathrm{Li}_2(1-x_2)\ln(x_1)-2\mathrm{Li}_3(1-x_2)+2\mathrm{Li}_3(x_1+x_2)\\
&+2\mathrm{Li}_3(1-x_1-x_2)-2\mathrm{Li}_3(x_2)-\frac{1}{2}\ln(x_2)\ln(x_1)\ln(1-x_1-x_2)+\ln(x_2)^2\ln(1-x_1-x_2)\\
&-\frac{1}{2}\ln(x_1)\mathrm{R}(x_1,1-x_1-x_2)-\frac{5}{2}\ln(x_2)\mathrm{R}(x_2,1-x_1-x_2)-\ln(x_2)^2\ln(1-x_2)\Big]\\
&+\frac{1}{36}\frac{(76x_2(1-x_2)+61x_1-11x_1x_2)}{x_1{}^2(1-x_2)^2}\ln(x_2)+\frac{(17x_1{}^2+61x_1x_2+38x_2{}^2)}{18x_1{}^3(x_1+x_2)}\mathrm{R}(x_2,1-x_1-x_2)\\
&+\frac{1}{4}\frac{(-2x_2+2x_2{}^2-2x_1+x_1x_2)}{x_1{}^2(1-x_2)^2}\Big[\frac{1}{3}\mathrm{R}(x_1,x_2)+\ln(x_2)^2\Big]+\frac{(x_1-x_2)}{6x_1{}^2(x_1+x_2)}\mathrm{R}(x_1,1-x_1-x_2)\\
&+\frac{(-8+14x_2-6x_2{}^2-2x_1+x_1x_2)}{12x_1{}^2(1-x_2)^2}\zeta_2-\frac{1}{12}\frac{\ln(x_1)}{x_1(1-x_2)}+\frac{1}{9}\frac{(16x_1+19x_2)}{x_1{}^2(x_1+x_2)}\ln(1-x_1-x_2)\\
&-\frac{1}{12}\frac{(-10x_1{}^2+5x_1{}^2x_2-14x_1+8x_1x_2+x_1x_2{}^2-10x_2+10x_2{}^2)}{(x_1+x_2)x_1{}^2(1-x_2)^2}\mathrm{Li}_2(1-x_2)\\
&-\frac{1}{12}\frac{(-2x_1{}^2+x_1{}^2x_2+2x_1-8x_1x_2+5x_1x_2{}^2-2x_2+2x_2{}^2)}{(x_1+x_2)x_1{}^2(1-x_2)^2}\mathrm{Li}_2(1-x_1)\Big)\\
&+\frac{i\pi n_f}{N}\Big(\frac{1}{x_1{}^2}\ln(1-x_1-x_2)-\frac{1}{2}\frac{(-2x_2+2x_2{}^2-2x_1+x_1x_2)}{x_1{}^2(1-x_2)^2}\ln(x_2)+\frac{1}{2}\frac{1}{x_1(1-x_2)}\\
&+\frac{(x_1+x_2)}{x_1{}^3}\mathrm{R}(x_2,1-x_1-x_2)\Big)-i\pi n_f N\Big(\frac{1}{2}\frac{(2-x_2)}{x_1(1-x_2)^2}\ln(x_2)+\frac{1}{2}\frac{1}{x_1(1-x_2)}\Big),
\end{aligned}\tag{68}
$$

$$
\begin{aligned}
x_1c_6^{(2),\mathrm{fin}}(x_1,x_2)&=\frac{n_f}{N}\Big(\frac{1}{2}\Big[\frac{1}{3}\ln(1-x_1-x_2)-\frac{91}{108}+\frac{1}{4}\zeta_2+\frac{1}{3}\ln(x_1+x_2)\Big]\ln(1-x_1-x_2)\\
&+\frac{1}{36}\frac{(52x_1{}^2x_2+47x_1x_2{}^2-45x_1x_2-14x_2{}^2+14x_2{}^3-19x_1{}^2+19x_1{}^3)}{(1-x_1-x_2)(x_1+x_2)x_2}\mathrm{R}(x_1,1-x_1-x_2)\\
&+\frac{(x_1+2x_2)}{6x_2}\Big[+\frac{1}{2}\ln(1-x_1)\ln(x_1)^2-\frac{1}{4}\ln(x_2)\mathrm{Li}_2(1-x_2)+\frac{1}{4}\mathrm{Li}_2(1-x_1)\ln(x_2)\\
&+\frac{1}{2}\ln(x_1)\ln(1-x_1-x_2)^2+\mathrm{Li}_3(x_1)+\mathrm{Li}_3(1-x_1)\Big]+\frac{1}{6}\mathrm{Li}_2(1-x_1-x_2)\\
&+\frac{(x_2{}^2+x_1{}^2+4x_1x_2)}{12x_1x_2}\Big[+\frac{1}{4}\ln(x_2)\ln(x_1)\ln(1-x_1-x_2)-\frac{1}{2}\ln(x_1+x_2)\ln(1-x_1-x_2)^2\\
&-\mathrm{Li}_3(1-x_1-x_2)-\mathrm{Li}_3(x_1+x_2)\Big]+\frac{1}{24}\frac{(x_2{}^2+5x_1{}^2+12x_1x_2)}{x_1x_2}\ln(x_1)\mathrm{R}(x_1,1-x_1-x_2)\\
&+\frac{1}{36}\frac{(-43x_1+31x_1{}^2+63x_1x_2+12-12x_2)}{(1-x_1)(1-x_1-x_2)}\ln(x_1)+\frac{1}{8}\frac{x_1(1-x_1-3x_2)}{(1-x_1)(1-x_1-x_2)}\ln(x_1)^2\\
&-\frac{1}{24}\frac{(2x_1{}^2+x_2{}^2+6x_1x_2)}{x_1x_2}\ln(x_1)^2\ln(1-x_1-x_2)-\frac{4085}{2592}+\frac{1}{72}\zeta_3\\
&-\frac{1}{24}\frac{(x_1+2x_1x_2-1+x_2)}{(1-x_2)(1-x_1-x_2)}\zeta_2-\frac{1}{24}\frac{x_1(-1+4x_2)}{(1-x_2)(1-x_1-x_2)}\mathrm{R}(x_1,x_2)
\end{aligned}
$$

$$+\frac{1}{24}\Big[-\frac{3}{(1-x_2)}-\frac{15}{(1-x_1)}-\frac{2(4x_2-5)}{(1-x_1-x_2)}-\frac{8x_2}{(x_1+x_2)}\Big]\mathrm{Li}_2(1-x_1)\Big)$$
$$+n_fN\Big(-\frac{1}{72}\frac{(-82x_1+41x_1{}^2+14x_1x_2+41-41x_2)}{(1-x_1)(1-x_1-x_2)}\ln(x_1)^2-\frac{7}{36}\zeta_3+\frac{4345}{2592}$$
$$+\frac{1}{36}\frac{(31-56x_1-31x_2+25x_1{}^2-20x_1x_2)}{(1-x_1)(1-x_1-x_2)}\ln(x_1)+\frac{1}{72}\Big[-33x_1{}^2+58x_1+58-\frac{58}{(1-x_1)}$$
$$+\frac{(9x_1{}^2+9x_1+58)}{(1-x_2)}-\frac{(4x_1{}^3-51x_1{}^2+58-69x_1)}{(1-x_1-x_2)}\Big]\frac{\mathrm{R}(x_1,x_2)}{(1-x_1-x_2)(x_1+x_2)}$$
$$-\frac{1}{12}\frac{(2x_1{}^2+x_1x_2+2x_2{}^2)}{(x_1+x_2)^2}\mathrm{R}_1(x_1,x_2)+\frac{1}{12}\frac{(2-4x_1-4x_2+2x_1{}^2+x_1x_2+2x_2{}^2)}{(1-x_1-x_2)^2}\mathrm{R}_2(x_1,x_2)$$
$$+\frac{1}{72}\Big[11+\frac{9}{(1-x_2)}+\frac{45}{(1-x_1)}-\frac{9(4x_1+1)}{(1-x_1-x_2)}-\frac{36x_1}{(x_1+x_2)}\Big]\mathrm{Li}_2(1-x_1)$$
$$-\frac{1}{8}\frac{x_1x_2\ln(x_1)}{(1-x_1-x_2)^2}\zeta_2-\frac{1}{144}\frac{(1340x_1-1023x_1x_2-670x_1{}^2+688x_1{}^2x_2-335)}{(1-x_1)(1-x_2)(1-x_1-x_2)}\zeta_2\Big)$$
$$+i\pi n_fN\Big(\frac{1}{24}\frac{(-31+13x_1)x_2+31(1-x_1)^2}{(1-x_1)(1-x_1-x_2)}\ln(x_1)+\frac{1}{8}\frac{x_1x_2-8x_1+2+4x_1{}^2}{(1-x_1-x_2)^2}\mathrm{R}(x_1,x_2)$$
$$+\frac{1}{12}\zeta_2-\frac{5}{72}\Big)+\frac{i\pi n_f}{N}\frac{1}{4}\Big(\frac{x_1(-1+x_1+3x_2)}{(1-x_1)(1-x_1-x_2)}\ln(x_1)-\frac{1}{x_1}(x_2+2x_1)\mathrm{R}(x_2,1-x_1-x_2)$$
$$-2\ln(1-x_1-x_2)-\frac{1}{2}\zeta_2-\frac{191}{54}\Big)+(x_1\leftrightarrow x_2), \tag{69}$$

$$c_{12}^{(2),\mathrm{fin}}(x_1,x_2)=n_fN\Big(3\frac{\ln(x_1)}{(x_1+x_2)^2}-\frac{1}{2}\frac{1}{x_1(1-x_2)}\Big[-\frac{1}{2}\ln(x_2)^2-\frac{1}{6}\zeta_2+\mathrm{Li}_2(1-x_2)\Big]$$
$$-\frac{1}{18}\frac{(13x_1{}^2+36x_1-10x_1x_2-18x_2+31x_2{}^2)}{(x_1+x_2)^2x_1(1-x_2)}\ln(x_2)+\frac{(-x_2{}^2-2x_1+4x_2+x_1{}^2)}{(x_1+x_2)^4}\mathrm{R}_1(x_1,x_2)$$
$$-\frac{1}{12}\frac{1}{x_1(x_1+x_2)^2}\Big[5x_2+42x_1+5-\frac{(1+x_1)^2}{(1-x_2)}-4\frac{(1-3x_1+3x_1{}^2)}{(1-x_1-x_2)}-72\frac{x_1{}^2}{(x_1+x_2)}\Big]\mathrm{R}(x_1,x_2)$$
$$+\Big[\frac{(1+2x_1)}{x_1(x_1+x_2)^2}-\frac{6}{(x_1+x_2)^3}-\frac{1}{12}\frac{1}{x_1(1-x_2)}\Big](\mathrm{Li}_2(1-x_1)-\mathrm{Li}_2(1-x_2))-\frac{1}{(x_1+x_2)x_1}\Big)$$
$$+\frac{n_f}{N}\Big(-\frac{19}{9}\frac{1}{x_1(x_1+x_2)}+\frac{1}{18}\frac{(13x_1{}^2-29x_1+42x_1x_2-38x_2+38x_2{}^2)}{x_1{}^2(x_1+x_2)(1-x_2)}\ln(x_2)$$
$$+\frac{1}{4}\frac{(2-2x_2-x_1)}{x_1{}^2(1-x_2)}\ln(x_2)^2+\frac{1}{9}\frac{(-32x_1-19x_2+16x_1{}^2+35x_1x_2+19x_2{}^2)}{x_1{}^2(x_1+x_2)^2}\ln(1-x_1-x_2)$$
$$-\frac{(1-x_1-x_2)}{3x_1{}^3}\Big[\Big\{\ln(x_2)^2-\frac{1}{2}\ln(x_1)\ln(x_2)+\frac{1}{2}\ln(x_1)^2+\ln(x_1+x_2)\ln(1-x_1-x_2)$$
$$-\ln(x_2)\ln(1-x_1-x_2)\Big\}\ln(1-x_1-x_2)-\ln(x_2)^2\ln(1-x_2)-\frac{1}{2}\mathrm{Li}_2(1-x_2)\ln(x_1)$$

$$
\begin{aligned}
&-2\mathrm{Li}_3(1-x_2)-2\mathrm{Li}_3(x_2)+2\mathrm{Li}_3(1-x_1-x_2)+2\mathrm{Li}_3(x_1+x_2)+\frac{1}{2}\ln(x_1)\mathrm{Li}_2(1-x_1)\\
&-\frac{1}{2}\ln(x_1)\mathrm{R}(x_1,1-x_1-x_2)-\frac{5}{2}\ln(x_2)\mathrm{R}(x_2,1-x_1-x_2)\Big]+\frac{1}{2}\frac{\ln(x_1)}{x_1(x_1+x_2)}\\
&-\frac{1}{3}\frac{(x_1{}^2+2x_1x_2+x_2{}^2-2x_1-x_2)}{x_1{}^2(x_1+x_2)^2}\Big[\ln(1-x_1-x_2)^2+\ln(x_1+x_2)\ln(1-x_1-x_2)\\
&+\mathrm{Li}_2(1-x_1-x_2)\Big]+\frac{(2-2x_2-x_1)}{12x_1{}^2(1-x_2)}\mathrm{R}(x_1,x_2)+\frac{(4x_1+x_2+x_1{}^2-x_2{}^2)}{6x_1{}^2(x_1+x_2)^2}\mathrm{R}(x_1,1-x_1-x_2)\\
&-\frac{\zeta_2}{12x_1}\Big[\frac{1}{(1-x_2)}+\frac{6}{x_1}-8\frac{(2x_1+x_2)}{x_1(x_1+x_2)^2}\Big]-\frac{\mathrm{Li}_2(1-x_1)}{12x_1(x_1+x_2)}\Big[5-\frac{(1+x_1)}{(1-x_2)}+\frac{2}{x_1}+\frac{6}{(x_1+x_2)}\Big]\\
&-\frac{\mathrm{Li}_2(1-x_2)}{12x_1(x_1+x_2)}\Big[1-5\frac{(1+x_1)}{(1-x_2)}+\frac{10}{x_1}+\frac{6}{(x_1+x_2)}\Big]\\
&+\frac{1}{18}\frac{\mathrm{R}(x_2,1-x_1-x_2)}{x_1(x_1+x_2)}\Big[17+61\frac{x_2}{x_1}-\frac{23}{x_1}-38\frac{x_2(1-x_2)}{x_1{}^2}+\frac{9}{(x_1+x_2)}\Big]\Big)\\
&+\frac{i\pi n_f}{N}\Big(\frac{(x_1{}^2+2x_1x_2+x_2{}^2-2x_1-x_2)}{x_1{}^2(x_1+x_2)^2}\ln(1-x_1-x_2)-\frac{1}{2}\frac{(2-2x_2-x_1)}{x_1{}^2(1-x_2)}\ln(x_2)\\
&-\frac{1}{(x_1+x_2)x_1}-\frac{(1-x_1-x_2)}{x_1{}^3}\mathrm{R}(x_2,1-x_1-x_2)\Big)-\frac{1}{2}i\pi n_f N\frac{\ln(x_2)}{x_1(1-x_2)}. \qquad (70)
\end{aligned}
$$

We have introduced the function $\mathrm{R}(x_1,x_2)$, which is well known from ref. [2],

$$
\mathrm{R}(x_1,x_2)=\left(\frac{1}{2}\ln(x_1)\ln(x_2)-\ln(x_1)\ln(1-x_1)+\frac{1}{2}\zeta_2-\mathrm{Li}_2(x_1)\right)+(x_1\leftrightarrow x_2)\,. \qquad (71)
$$

In addition, it is convenient, to define the symmetric functions $\mathrm{R}_1(x_1,x_2)$ and $\mathrm{R}_2(x_1,x_2)$ as

$$
\begin{aligned}
\mathrm{R}_1(x_1,x_2) &= \left(\ln(x_1)\mathrm{Li}_{1,1}\left(\frac{x_1}{x_1+x_2},x_1+x_2\right)-\frac{1}{2}\ln(1-x_1-x_2)\zeta_2-\ln(x_1)\mathrm{Li}_2(x_1+x_2)\right.\\
&\quad -\frac{1}{2}\ln(x_1)\ln(x_2)\ln(1-x_1-x_2)-\mathrm{Li}_{1,2}\left(\frac{x_1}{x_1+x_2},x_1+x_2\right)\\
&\quad \left.-\mathrm{Li}_{2,1}\left(\frac{x_1}{x_1+x_2},x_1+x_2\right)+\mathrm{Li}_3(x_1+x_2)\right)+(x_1\leftrightarrow x_2)\,.\\
\mathrm{R}_2(x_1,x_2) &= \left(\frac{1}{2}\ln(x_1)^2\ln(1-x_1)-\frac{3}{4}\ln(x_1)^2\ln(x_2)-\frac{1}{4}\ln(x_1)\mathrm{Li}_2(1-x_1)\right.\\
&\quad \left.-\frac{1}{4}\ln(x_2)\mathrm{Li}_2(1-x_1)-\mathrm{Li}_3(1-x_1)+\mathrm{Li}_3(x_1)\right)+(x_1\leftrightarrow x_2)\,. \qquad (72)
\end{aligned}
$$

We see in eqs. (67)–(70) that the multiple polylogarithms all have simple arguments of a particular structure, that can easily be continued analytically. Details of this procedure are discussed in appendix B.

5 Conclusions

Determinations of the strong coupling constant α_s from data for $e^+e^- \to 3$ jets demand next-to-next-to-leading (NNLO) theoretical predictions in perturbative QCD. In this paper we have calculated the fermionic contributions to the two-loop amplitude $e^+e^- \to q\bar{q}g$. Furthermore, we have obtained the full one-loop amplitude to order ε^2 in dimensional regularization, needed for the interference of the one-loop amplitude with itself.

We have used a systematic, fast and efficient method for the calculation of loop integrals. Our procedure allows for a direct evaluation of all scalar integrals in arbitrary dimensions and with arbitrary powers of propagators by means of nested sums. The approach relies on the ability to solve all nested sums in terms of multiple polylogarithms, which appear to be the natural class of functions for virtual corrections in perturbative QCD. As a consequence, the results allow for analytic continuation in a straightforward manner. Therefore, they apply also to $(2+1)$-jet production in deep-inelastic scattering or to the production of a massive vector boson in hadron-hadron collisions.

The results presented in this paper represent one contribution to the full next-to-next-to-leading order calculation of $e^+e^- \to 3$ jets. At the same time, it provides an important cross check on the results recently obtained by Garland et al. [33] with a completely independent method. Extending our approach to the calculation of the remaining contributions to the two-loop amplitude for $e^+e^- \to q\bar{q}g$, i.e. the terms which are not enhanced by n_f, can be done along the lines of section 3.

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A One-loop amplitude

Here we list the result for the one-loop function $c_{12}^{(1),\text{fin}}$ to order ε^2. The remaining functions $c_i^{(1),\text{fin}}$, $i = 2,4,6$, are of a similar length, and in order to save a few trees, we do not list them explicitly here. All results for the one-loop functions $c_i^{(1),\text{fin}}$, $i = 2,4,6,12$, together with the fermionic contributions $c_i^{(2),\text{fin}}$, $i = 2,4,6,12$ to the two-loop amplitude can be obtained as a FORM file from the preprint server http://arXiv.org by downloading the source file of this article. Furthermore they are available upon request from the authors.

$$c_{12}^{(1),\text{fin}}(x_1,x_2) = N(1+i\pi\varepsilon)\frac{\ln(x_2)}{x_1(1-x_2)} + (\varepsilon + i\pi\varepsilon^2)N\Big(\frac{1}{2}\frac{(6\ln(x_2) - \ln(x_2)^2)}{(1-x_2)x_1} - \frac{\text{R}(x_1,x_2)}{(1-x_1-x_2)x_1}\Big)$$

$$
\begin{aligned}
&+\frac{(1+i\pi\varepsilon)}{N}\Big(\frac{2}{x_1x_2}-\frac{\ln(x_2)}{(1-x_2)x_1}-\frac{2}{(x_1+x_2)x_2}-2\frac{\ln(1-x_1-x_2)}{(x_1+x_2)^2x_2}+2\frac{\ln(1-x_1-x_2)}{x_1{}^2x_2}\\
&-2\frac{\ln(1-x_1-x_2)-\ln(x_2)}{x_1{}^2}+2\frac{(1-x_1-x_2)}{x_1{}^3}\mathrm{R}(x_2,1-x_1-x_2)\Big)\\
&+\frac{(\varepsilon+i\pi\varepsilon^2)}{N}\Big(\frac{(4-\ln(x_1)-\ln(1-x_1-x_2))}{x_1x_2}-\frac{(1-x_2)}{x_1{}^2x_2}(\ln(1-x_1-x_2)-3)\ln(1-x_1-x_2)\\
&+\frac{1}{x_1{}^2}(3-\ln(x_2))\ln(x_2)-\frac{(1-x_1+x_2)}{x_1x_2{}^2}\mathrm{R}(x_1,1-x_1-x_2)+\frac{(1-3x_1-x_2)}{x_1{}^3}\mathrm{R}(x_2,1-x_1-x_2)\\
&-\frac{1}{2}\frac{(6\ln(x_2)-\ln(x_2)^2)}{(1-x_2)x_1}-2\frac{(2-\ln(1-x_1-x_2))}{(x_1+x_2)x_2}-\frac{(4-\ln(1-x_1-x_2))}{(x_1+x_2)^2x_2}\ln(1-x_1-x_2)\\
&+\frac{(1-x_1-x_2)}{x_1{}^3}\Big[(3\ln(1-x_2)-\ln(1-x_1-x_2))\ln(x_2)^2-2\mathrm{R}_1(1-x_1-x_2,x_2)\\
&+(2\mathrm{Li}_2(1-x_2)-4\zeta_2-\ln(1-x_1-x_2)^2)\ln(x_2)+6\mathrm{Li}_3(1-x_1-x_2)+6\mathrm{Li}_3(x_2)\\
&+\ln(x_1+x_2)\ln(1-x_1-x_2)^2-2(\mathrm{Li}_2(1-x_1-x_2)+\zeta_2)\ln(1-x_1-x_2)\Big]\Big)\\
&+\varepsilon^2N\Big(\frac{1}{6}\frac{(-21\zeta_2+36-9\ln(x_2)+\ln(x_2)^2)}{x_1(1-x_2)}\ln(x_2)+\frac{1}{2}\frac{1}{x_1(1-x_1-x_2)}\Big[-6\mathrm{Li}_3(x_1)\\
&-6\mathrm{Li}_3(x_2)-3\ln(x_2)^2\ln(1-x_2)+(-3\ln(1-x_1)+\ln(x_2))\ln(x_1)^2+4\ln(x_2)\zeta_2\\
&-(2\mathrm{Li}_2(1-x_1)-\ln(x_2)^2-4\zeta_2)\ln(x_1)-6\mathrm{R}(x_1,x_2)+2\mathrm{R}_1(x_1,x_2)-2\mathrm{Li}_2(1-x_2)\ln(x_2)\Big]\Big)\\
&+\frac{\varepsilon^2}{N}\Big(\frac{1}{6}\frac{(21\zeta_2-36+9\ln(x_2)-\ln(x_2)^2)}{x_1(1-x_2)}\ln(x_2)-\frac{(1-x_1+3x_2)}{x_1x_2{}^2}\mathrm{R}(x_1,1-x_1-x_2)\\
&-\frac{(8-7\zeta_2-4\ln(1-x_1-x_2)+\ln(1-x_1-x_2)^2)}{(x_1+x_2)x_2}+\frac{1}{3}\frac{\ln(1-x_1-x_2)}{(x_1+x_2)^2x_2}(21\zeta_2-24\\
&+6\ln(1-x_1-x_2)-\ln(1-x_1-x_2)^2)+\frac{(1-x_1+x_2)}{x_1x_2{}^2}\Big[\mathrm{R}_1(x_1,1-x_1-x_2)-\mathrm{Li}_2(1-x_1)\ln(x_1)\\
&+\frac{1}{2}(\ln(x_1)-\ln(x_1+x_2))\ln(1-x_1-x_2)^2+2\ln(x_1)\zeta_2-\frac{3}{2}\ln(x_1)^2\ln(1-x_1)-3\mathrm{Li}_3(x_1)\\
&+\frac{1}{2}(2\zeta_2+2\mathrm{Li}_2(1-x_1-x_2)+\ln(x_1)^2)\ln(1-x_1-x_2)-3\mathrm{Li}_3(1-x_1-x_2)\Big]\\
&+\frac{(1-3x_1-x_2)}{x_1{}^3}\Big[\mathrm{Li}_2(1-x_2)\ln(x_2)-\mathrm{R}_1(1-x_1-x_2,x_2)+3\mathrm{Li}_3(1-x_1-x_2)+3\mathrm{Li}_3(x_2)\\
&+\frac{1}{2}(\ln(x_1+x_2)-\ln(x_2))\ln(1-x_1-x_2)^2-\mathrm{Li}_2(1-x_1-x_2)\ln(1-x_1-x_2)\\
&+3\mathrm{R}(x_2,1-x_1-x_2)-\frac{1}{2}\ln(x_2)^2(\ln(1-x_1-x_2)-3\ln(1-x_2))\Big]+\frac{\ln(1-x_1-x_2)}{x_1{}^2x_2}\Big[6-7\zeta_2\\
&+\frac{1}{3}\ln(1-x_1-x_2)^2-\frac{3}{2}\ln(1-x_1-x_2)\Big]+\frac{1}{x_1x_2}\Big[8-7\zeta_2-2\ln(1-x_1-x_2)+\frac{1}{2}\ln(x_1)^2\\
&+\frac{1}{2}\ln(1-x_1-x_2)^2-2\ln(x_1)\Big]+\frac{1}{x_1{}^2}\Big[\frac{3}{2}\ln(1-x_1-x_2)^2+9\ln(1-x_1-x_2)\zeta_2
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{3}\ln(1-x_1-x_2)^3+4\mathrm{R}(x_2,1-x_1-x_2)-6\ln(1-x_1-x_2)-3\ln(x_2)\zeta_2+6\ln(x_2)\\
&-\frac{3}{2}\ln(x_2)^2+\frac{1}{3}\ln(x_2)^3\Big]+\frac{(1-x_1-x_2)}{{x_1}^3}\Big[-2\mathrm{R}_1(1-x_1-x_2,x_2)\ln(1-x_1-x_2)-\frac{3}{5}{\zeta_2}^2\\
&-(2\ln(x_2)+\ln(1-x_1-x_2)+2\mathrm{R}(x_1,x_2)+4\mathrm{R}(x_1,1-x_1-x_2)+5\mathrm{R}(x_2,1-x_1-x_2))\zeta_2\\
&+12\mathrm{Li}_4(1-x_1)-14\mathrm{Li}_4(1-x_1-x_2)-14\mathrm{Li}_4(x_2)-3\mathrm{Li}_2(1-x_1)\ln(1-x_1-x_2)^2\\
&+(2\zeta_2-\ln(x_1)\ln(1-x_1-x_2)-\mathrm{Li}_2(1-x_1)+\frac{1}{2}\ln(1-x_1-x_2)^2-\mathrm{Li}_2(1-x_2))\ln(x_2)^2\\
&-\frac{1}{3}\ln(x_1+x_2)\ln(1-x_1-x_2)^3+\mathrm{Li}_2(1-x_1-x_2)\ln(1-x_1-x_2)^2+4\mathrm{Li}_{1,3}(1,1-x_1)\\
&+\frac{1}{3}(\ln(1-x_1-x_2)-4\ln(1-x_2))\ln(x_2)^3+\mathrm{Li}_{1,1}\left(\frac{x_2}{1-x_1},1-x_1\right)\ln(x_2)^2\\
&+4\mathrm{Li}_{2,2}(1,1-x_1)+4\mathrm{Li}_{3,1}(1,1-x_1)+2(\mathrm{Li}_3(1-x_1)+\mathrm{Li}_3(1-x_1-x_2))\ln(1-x_1-x_2)\\
&-2(\mathrm{Li}_{2,1}(1,1-x_1)+\mathrm{Li}_{1,2}(1,1-x_1))(\ln(1-x_1-x_2)+\ln(x_2))+\Big[2\mathrm{Li}_{2,1}\left(\frac{x_2}{1-x_1},1-x_1\right)\\
&-2\mathrm{Li}_3(1-x_1)+\ln(1-x_1-x_2)\ln(x_1)^2-2\ln(1-x_1-x_2)\mathrm{Li}_2(1-x_1)+2\mathrm{Li}_3(x_2)\\
&+2\mathrm{Li}_{1,1}\left(\frac{x_2}{1-x_1},1-x_1\right)\ln(1-x_1-x_2)+2\mathrm{Li}_{1,2}\left(\frac{x_2}{1-x_1},1-x_1\right)+\frac{1}{3}\ln(1-x_1-x_2)^3\\
&-3\ln(x_1)\ln(1-x_1-x_2)^2+2\mathrm{Li}_{1,1,1}\left(\frac{x_2}{1-x_1},1,1-x_1\right)\Big]\ln(x_2)-6\mathrm{Li}_{3,1}\left(\frac{x_2}{1-x_1},1-x_1\right)\\
&-2\mathrm{Li}_{2,1,1}\left(\frac{x_2}{1-x_1},1,1-x_1\right)-2\mathrm{Li}_{1,1,2}\left(\frac{1-x_1-x_2}{1-x_1},1,1-x_1\right)-6\mathrm{Li}_{2,2}\left(\frac{x_2}{1-x_1},1-x_1\right)\\
&-2\mathrm{Li}_{1,1,2}\left(\frac{x_2}{1-x_1},1,1-x_1\right)-2\mathrm{Li}_{1,2,1}\left(\frac{x_2}{1-x_1},1,1-x_1\right)-6\mathrm{Li}_{2,2}\left(\frac{1-x_1-x_2}{1-x_1},1-x_1\right)\\
&-6\mathrm{Li}_{1,3}\left(\frac{x_2}{1-x_1},1-x_1\right)-2\mathrm{Li}_{2,1,1}\left(\frac{1-x_1-x_2}{1-x_1},1,1-x_1\right)-6\mathrm{Li}_{1,3}\left(\frac{1-x_1-x_2}{1-x_1},1-x_1\right)\\
&-2\mathrm{Li}_{1,2,1}\left(\frac{1-x_1-x_2}{1-x_1},1,1-x_1\right)-6\mathrm{Li}_{3,1}\left(\frac{1-x_1-x_2}{1-x_1},1-x_1\right)+\Big[\ln(1-x_1-x_2)^2\\
&+\ln(x_1)^2-2\mathrm{Li}_2(1-x_1-x_2)-2\ln(x_1+x_2)\ln(1-x_1-x_2)+6\mathrm{Li}_2(1-x_1)\Big]\zeta_2\\
&+3\mathrm{Li}_{1,1}\left(\frac{1-x_1-x_2}{1-x_1},1-x_1\right)\ln(1-x_1-x_2)^2+2\Big[\mathrm{Li}_{1,1,1}\left(\frac{1-x_1-x_2}{1-x_1},1,1-x_1\right)\\
&-\mathrm{Li}_{2,1}\left(\frac{x_2}{1-x_1},1-x_1\right)-\mathrm{Li}_{1,2}\left(\frac{x_2}{1-x_1},1-x_1\right)\Big]\ln(1-x_1-x_2)\Big]\Big).
\end{aligned}
\tag{73}
$$

B Analytic continuation

The multiple polylogarithms $\mathrm{Li}_{m_k,...,m_1}(y_k,...,y_1)$ are analytic functions in k complex variables y_j, $j=1,...,k$. To discuss the branch cuts it is convenient to change the variables according to

$$z_j = 1 - y_1 y_2 ... y_j, \tag{74}$$

and one has

$$\mathrm{Li}_{m_k,...,m_1}(y_k,...,y_1) = \mathrm{Li}_{m_k,...,m_1}\left(\frac{1-z_k}{1-z_{k-1}},...,\frac{1-z_2}{1-z_1},1-z_1\right). \tag{75}$$

The multiple polylogarithms have a representation as nested sums. From this represenation one deduces that the multiple polylogarithms are real, if all z_j are in the intervall $0 \leq z_j \leq 2$ and $(m_1,z_1) \neq (1,0)$. If z_1, ..., z_{j-1}, z_{j+1}, ..., z_k are fixed in this interval, a given multiple polylogarithm is an analytic function in the remaining variable z_j with a branch cut along the negative real axis starting from 0. We denote by Re_{z_j} and Im_{z_j} the real and imaginary part with respect to the variable z_j. In the calculation presented here, the z_j's are ratios of two kinematical invariants:

$$z_j = \frac{-s_j}{-t_j} \tag{76}$$

In fact, the only ratios which occur are:

$$\begin{aligned} x_1 &= \frac{-s_{12}}{-s_{123}}, & 1-x_1 &= \frac{-s_{23}-s_{13}}{-s_{123}}, \\ x_2 &= \frac{-s_{23}}{-s_{123}}, & 1-x_2 &= \frac{-s_{12}-s_{13}}{-s_{123}}, \\ 1-x_1-x_2 &= \frac{-s_{13}}{-s_{123}}, & x_1+x_2 &= \frac{-s_{12}-s_{23}}{-s_{123}}. \end{aligned} \tag{77}$$

The real and imaginary part of the logarithm is given by

$$\begin{aligned} \mathrm{Re}_z\, \mathrm{Li}_1(1-z) &= \ln|z|, \\ \mathrm{Im}_z\, \mathrm{Li}_1(1-z) &= -\mathrm{Im}\, \ln\left(\frac{-s-i0}{-t-i0}\right) = \pi\left[\theta(s)-\theta(t)\right]. \end{aligned} \tag{78}$$

Eq. (78) defines the sign of the imaginary part for a ratio of two invariants. All imaginary parts of higher multiple polylogarithms can be related to the imaginary part of the logarithm. They are easily obtained from the iterated integral represantation

$$\mathrm{Li}_{m_k,...,m_1}(x_k,...,x_1) = \\ \int\limits_0^{x_1}\left(\frac{dt}{t}\circ\right)^{m_1-1}\frac{dt}{1-t}\circ\int\limits_0^{tx_2}\left(\frac{dt}{t}\circ\right)^{m_2-1}\frac{dt}{1-t}\circ...\circ\int\limits_0^{tx_k}\left(\frac{dt}{t}\circ\right)^{m_k-1}\frac{dt}{1-t}, \tag{79}$$

and the fact that

$$\mathrm{Im}_z \frac{1}{1-z\pm i0} \;=\; \mp\pi\frac{\partial}{\partial z}\Theta(z-1). \tag{80}$$

The imaginary part in the variable z_j is given by

$$\begin{aligned}
& \mathrm{Im}_{z_j}\, \mathrm{Li}_{m_k,...,m_1}\left(\frac{1-z_k}{1-z_{k-1}},...,\frac{1-z_2}{1-z_1},1-z_1\right) = \\
& \quad \left[\mathrm{Im}\,\mathrm{Li}_1\left(1-z_j\right)\right] \times \int\limits_0^{1-z_1}\left(\frac{dt}{t}\circ\right)^{m_1-1}\frac{dt}{1-t}\circ ... \\
& \quad \int\limits_0^{\frac{1-z_j}{1-z_{j-1}}t}\left(\frac{dt}{t}\circ\right)^{m_j-1}\int\limits_0^{t} dt\left[\frac{\partial}{\partial t}\theta(t-1)\right]\circ...\circ\int\limits_0^{\frac{1-z_k}{1-z_{k-1}}t}\left(\frac{dt}{t}\circ\right)^{m_k-1}\frac{dt}{1-t}.
\end{aligned} \tag{81}$$

Using partial integration, the iterated integral is then related to multiple polylogarithms of reduced weight. For the dilogarithm and the trilogarithm one obtains the well-known formulae

$$\begin{aligned}
\mathrm{Im}\,\mathrm{Li}_2(1-z) &= \ln(1-z)\,\mathrm{Im}\,\mathrm{Li}_1(1-z), \\
\mathrm{Im}\,\mathrm{Li}_3(1-z) &= \frac{1}{2}\ln^2(1-z)\,\mathrm{Im}\,\mathrm{Li}_1(1-z).
\end{aligned} \tag{82}$$

In the two-loop amplitude there are also multiple polylogarithms depending on two variables in the form $\mathrm{Li}_{ab}((1-z_2)/(1-z_1),1-z_1)$, where $(a,b)=(1,1)$, $(1,2)$ or $(2,1)$. In general we have

$$\begin{aligned}
& \mathrm{Li}_{ab}\left(\frac{1-z_2}{1-z_1},1-z_1\right) = \\
& \quad \mathrm{Re}_{z_1}\,\mathrm{Re}_{z_2}\,\mathrm{Li}_{ab}\left(\frac{1-z_2}{1-z_1},1-z_1\right) + i\,\mathrm{Re}_{z_1}\,\mathrm{Im}_{z_2}\,\mathrm{Li}_{ab}\left(\frac{1-z_2}{1-z_1},1-z_1\right) \\
& \quad + i\,\mathrm{Im}_{z_1}\,\mathrm{Re}_{z_2}\,\mathrm{Li}_{ab}\left(\frac{1-z_2}{1-z_1},1-z_1\right) - \mathrm{Im}_{z_1}\,\mathrm{Im}_{z_2}\,\mathrm{Li}_{ab}\left(\frac{1-z_2}{1-z_1},1-z_1\right).
\end{aligned} \tag{83}$$

For the imaginary parts we find:

$$\begin{aligned}
& \mathrm{Im}_{z_1}\,\mathrm{Li}_{11}\left(\frac{1-z_2}{1-z_1},1-z_1\right) = \mathrm{Li}_1\left(\frac{1-z_2}{1-z_1}\right)\,\mathrm{Im}_{z_1}\,\mathrm{Li}_1\left(1-z_1\right), \\
& \mathrm{Im}_{z_1}\,\mathrm{Li}_{12}\left(\frac{1-z_2}{1-z_1},1-z_1\right) = -\mathrm{Li}_1(z_1)\mathrm{Li}_1\left(\frac{1-z_2}{1-z_1}\right)\,\mathrm{Im}_{z_1}\,\mathrm{Li}_1\left(1-z_1\right), \\
& \mathrm{Im}_{z_1}\,\mathrm{Li}_{21}\left(\frac{1-z_2}{1-z_1},1-z_1\right) = \mathrm{Li}_2\left(\frac{1-z_2}{1-z_1}\right)\,\mathrm{Im}_{z_1}\,\mathrm{Li}_1\left(1-z_1\right).
\end{aligned} \tag{84}$$

$$\mathrm{Im}_{z_2}\,\mathrm{Li}_{11}\left(\frac{1-z_2}{1-z_1},1-z_1\right) = \left[\mathrm{Li}_1\left(1-z_1\right)-\mathrm{Li}_1\left(\frac{1-z_1}{1-z_2}\right)\right]\,\mathrm{Im}_{z_2}\,\mathrm{Li}_1\left(1-z_2\right),$$

$$\begin{aligned}
&\mathrm{Im}_{z_2}\,\mathrm{Li}_{12}\left(\frac{1-z_2}{1-z_1},1-z_1\right)\\
&= \left[\mathrm{Li}_2\left(1-z_1\right)-\mathrm{Li}_2\left(\frac{1-z_1}{1-z_2}\right)+\mathrm{Li}_1(z_2)\mathrm{Li}_1\left(\frac{1-z_1}{1-z_2}\right)\right]\,\mathrm{Im}_{z_2}\,\mathrm{Li}_1\left(1-z_2\right),\\
&\mathrm{Im}_{z_2}\,\mathrm{Li}_{21}\left(\frac{1-z_2}{1-z_1},1-z_1\right)\\
&= -\left[\mathrm{Li}_2\left(1-z_1\right)-\mathrm{Li}_2\left(\frac{1-z_1}{1-z_2}\right)+\mathrm{Li}_1\left(z_2\right)\mathrm{Li}_1\left(1-z_1\right)\right]\,\mathrm{Im}_{z_2}\,\mathrm{Li}_1\left(1-z_2\right). \qquad (85)
\end{aligned}$$

$$\begin{aligned}
&\mathrm{Im}_{z_1}\,\mathrm{Im}_{z_2}\,\mathrm{Li}_{11}\left(\frac{1-z_2}{1-z_1},1-z_1\right)=\theta(z_1-z_2)\,\mathrm{Im}_{z_1}\,\mathrm{Li}_1\left(1-z_1\right)\,\mathrm{Im}_{z_2}\,\mathrm{Li}_1\left(1-z_2\right),\\
&\mathrm{Im}_{z_1}\,\mathrm{Im}_{z_2}\,\mathrm{Li}_{12}\left(\frac{1-z_2}{1-z_1},1-z_1\right)=-\theta(z_1-z_2)\mathrm{Li}_1(z_1)\,\mathrm{Im}_{z_1}\,\mathrm{Li}_1\left(1-z_1\right)\,\mathrm{Im}_{z_2}\,\mathrm{Li}_1\left(1-z_2\right),\\
&\mathrm{Im}_{z_1}\,\mathrm{Im}_{z_2}\,\mathrm{Li}_{21}\left(\frac{1-z_2}{1-z_1},1-z_1\right)\\
&\quad= \theta(z_1-z_2)\left[\mathrm{Li}_1(z_1)-\mathrm{Li}_1(z_2)\right]\,\mathrm{Im}_{z_1}\,\mathrm{Li}_1\left(1-z_1\right)\,\mathrm{Im}_{z_2}\,\mathrm{Li}_1\left(1-z_2\right). \qquad (86)
\end{aligned}$$

The imaginary parts of the harmonic polylogarithms $\mathrm{Li}_{ab}(1,1-z_1)$ can be obtained from the formulae above by setting $z_2=z_1$. Special care has to be taken for the double imaginary part. Here one uses

$$\theta(x-1)\frac{\partial}{\partial x}\theta(x-1) = \frac{1}{2}\frac{\partial}{\partial x}\left[\theta(x-1)\right]^2 \qquad (87)$$

and a factor $1/2$ appears in the final formulae:

$$\begin{aligned}
\lim_{z_2\to z_1}\mathrm{Im}_{z_1}\,\mathrm{Im}_{z_2}\,\mathrm{Li}_{11}\left(\frac{1-z_2}{1-z_1},1-z_1\right) &= \frac{1}{2}\left[\,\mathrm{Im}_{z_1}\,\mathrm{Li}_1\left(1-z_1\right)\right]^2,\\
\lim_{z_2\to z_1}\mathrm{Im}_{z_1}\,\mathrm{Im}_{z_2}\,\mathrm{Li}_{12}\left(\frac{1-z_2}{1-z_1},1-z_1\right) &= -\frac{1}{2}\mathrm{Li}_1(z_1)\left[\,\mathrm{Im}_{z_1}\,\mathrm{Li}_1\left(1-z_1\right)\right]^2,\\
\lim_{z_2\to z_1}\mathrm{Im}_{z_1}\,\mathrm{Im}_{z_2}\,\mathrm{Li}_{21}\left(\frac{1-z_2}{1-z_1},1-z_1\right) &= 0. \qquad (88)
\end{aligned}$$

In the one-loop amplitude we encounter additional multiple polylogarithms. We discuss here as an example the imaginary parts of $\mathrm{Li}_{111}\left((1-z_3)/(1-z_2),(1-z_2)/(1-z_1),1-z_1\right)$:

$$\begin{aligned}
&\mathrm{Im}_{z_1}\,\mathrm{Li}_{111}\left(\frac{1-z_3}{1-z_2},\frac{1-z_2}{1-z_1},1-z_1\right)=\mathrm{Li}_{11}\left(\frac{1-z_3}{1-z_2},\frac{1-z_2}{1-z_1}\right)\,\mathrm{Im}_{z_1}\,\mathrm{Li}_1\left(1-z_1\right),\\
&\mathrm{Im}_{z_2}\,\mathrm{Li}_{111}\left(\frac{1-z_3}{1-z_2},\frac{1-z_2}{1-z_1},1-z_1\right)=\mathrm{Li}_1\left(\frac{1-z_3}{1-z_2}\right)\left[\mathrm{Li}_1\left(1-z_1\right)-\mathrm{Li}_1\left(\frac{1-z_1}{1-z_2}\right)\right]\\
&\qquad\times\,\mathrm{Im}_{z_2}\,\mathrm{Li}_1\left(1-z_2\right),\\
&\mathrm{Im}_{z_3}\,\mathrm{Li}_{111}\left(\frac{1-z_3}{1-z_2},\frac{1-z_2}{1-z_1},1-z_1\right)=\left\{\mathrm{Li}_{11}\left(\frac{1-z_2}{1-z_1},1-z_1\right)-\mathrm{Li}_{11}\left(\frac{1-z_2}{1-z_1},\frac{1-z_1}{1-z_3}\right)\right.
\end{aligned}$$

$$
\begin{aligned}
& -\mathrm{Li}_1\left(\frac{1-z_2}{1-z_3}\right)\left[\mathrm{Li}_1\left(1-z_1\right)-\mathrm{Li}_1\left(\frac{1-z_1}{1-z_3}\right)\right]\Bigg\} \ \mathrm{Im}_{z_3}\,\mathrm{Li}_1\left(1-z_3\right), \\
\mathrm{Im}_{z_1}\,\mathrm{Im}_{z_2}\,\mathrm{Li}_{111}\left(\frac{1-z_3}{1-z_2},\frac{1-z_2}{1-z_1},1-z_1\right) &= \theta(z_1-z_2)\mathrm{Li}_1\left(\frac{1-z_3}{1-z_2}\right)\ \mathrm{Im}_{z_1}\,\mathrm{Li}_1\left(1-z_1\right) \\
& \times \mathrm{Im}_{z_2}\,\mathrm{Li}_1\left(1-z_2\right), \\
\mathrm{Im}_{z_1}\,\mathrm{Im}_{z_3}\,\mathrm{Li}_{111}\left(\frac{1-z_3}{1-z_2},\frac{1-z_2}{1-z_1},1-z_1\right) &= \theta(z_1-z_3)\left[\mathrm{Li}_1\left(\frac{1-z_2}{1-z_1}\right)-\mathrm{Li}_1\left(\frac{1-z_2}{1-z_3}\right)\right] \\
& \times \mathrm{Im}_{z_1}\,\mathrm{Li}_1\left(1-z_1\right)\ \mathrm{Im}_{z_3}\,\mathrm{Li}_1\left(1-z_3\right), \\
\mathrm{Im}_{z_2}\,\mathrm{Im}_{z_3}\,\mathrm{Li}_{111}\left(\frac{1-z_3}{1-z_2},\frac{1-z_2}{1-z_1},1-z_1\right) &= \theta(z_2-z_3)\left[\mathrm{Li}_1\left(1-z_1\right)-\mathrm{Li}_1\left(\frac{1-z_1}{1-z_2}\right)\right] \\
& \times \mathrm{Im}_{z_2}\,\mathrm{Li}_1\left(1-z_2\right)\ \mathrm{Im}_{z_3}\,\mathrm{Li}_1\left(1-z_3\right), \\
\mathrm{Im}_{z_1}\,\mathrm{Im}_{z_2}\,\mathrm{Im}_{z_3}\,\mathrm{Li}_{111}\left(\frac{1-z_3}{1-z_2},\frac{1-z_2}{1-z_1},1-z_1\right) &= \theta(z_1-z_2)\theta(z_2-z_3)\ \mathrm{Im}_{z_1}\,\mathrm{Li}_1\left(1-z_1\right) \\
& \times \mathrm{Im}_{z_2}\,\mathrm{Li}_1\left(1-z_2\right)\ \mathrm{Im}_{z_3}\,\mathrm{Li}_1\left(1-z_3\right). \qquad (89)
\end{aligned}
$$

From these formulae the imaginary parts of $\mathrm{Li}_{111}\left((1-z_3)/(1-z_1),1,1-z_1\right)$ can be obtained by setting $z_2=z_1$. Double imaginary parts in z_1 and z_2 are given by

$$
\begin{aligned}
& \lim_{z_2\to z_1}\mathrm{Im}_{z_1}\,\mathrm{Im}_{z_2}\,\mathrm{Li}_{111}\left(\frac{1-z_3}{1-z_2},\frac{1-z_2}{1-z_1},1-z_1\right) = \frac{1}{2}\mathrm{Li}_1\left(\frac{1-z_3}{1-z_1}\right)\left[\,\mathrm{Im}_{z_1}\,\mathrm{Li}_1\left(1-z_1\right)\right]^2, \\
& \lim_{z_2\to z_1}\mathrm{Im}_{z_1}\,\mathrm{Im}_{z_2}\,\mathrm{Im}_{z_3}\,\mathrm{Li}_{111}\left(\frac{1-z_3}{1-z_2},\frac{1-z_2}{1-z_1},1-z_1\right) \\
& \qquad = \frac{1}{2}\theta(z_1-z_3)\left[\,\mathrm{Im}_{z_1}\,\mathrm{Li}_1\left(1-z_1\right)\right]^2\ \mathrm{Im}_{z_3}\,\mathrm{Li}_1\left(1-z_3\right). \qquad (90)
\end{aligned}
$$

At weight four there are in addition the multiple polylogarithms with indices Li_{22}, Li_{13}, Li_{31}, as well as Li_{211}, Li_{121} and Li_{112}. The imaginary parts of those are obtained in complete analogy.

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