TWO-LOOP AMPLITUDES FOR $e^+e^- \to q\bar{q}g$: THE n_f-CONTRIBUTION

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We discuss the calculation of the n_f-contributions to the two-loop amplitude for $e^+e^- \to qg\bar{q}$. The calculation uses an efficient method based on nested sums. The result is presented in terms of multiple polylogarithms with simple arguments, which allow for analytic continuation in a straightforward manner.

1. Introduction

Searches for new physics in particle physics rely to a large extend on our ability to constrain the parameters of the standard model. For instance, the strong coupling constant α_s can be measured by using the data for $e^+e^- \to$ 3-jets. At present, the error on the extraction of α_s from this measurement is dominated by theoretical uncertainties [1], most prominently, by the truncation of the perturbative expansion at a fixed order.

The perturbative QCD calculation of $e^+e^- \to$ 3-jets at next-to-next-to-leading order (NNLO) requires the tree-level amplitudes for $e^+e^- \to$ 5 partons [2], the one-loop amplitudes for $e^+e^- \to$ 4 partons [3, 4] as well as the two-loop amplitude for $e^+e^- \to q\bar{q}g$ together with the one-loop amplitude $e^+e^- \to q\bar{q}g$ to order ε^2 in the parameter of dimensional regularization.

The helicity averaged squared matrix elements at the two-loop level for $e^+e^- \to q\bar{q}g$ have recently been given [5]. In contrast, having the two-loop amplitude available, one keeps the full correlation between the incoming e^+e^- and the outgoing parton's spins and momenta. Thus, one can study oriented event-shape observables. In addition, one has the option to investigate event-shape observables in polarized e^+e^--annihillation at a future linear e^+e^--collider TESLA.

2. Calculation

We are interested in the following reaction

$$e^+ + e^- \to q + g + \bar{q}\,, \tag{1}$$

which we consider in the form, $0 \to q(p_1) + g(p_2) + \bar{q}(p_3) + e^-(p_4) + e^+(p_5)$, with all particles in the final state, to be consistent with earlier work [3]. The kinematical invariants for this reaction are denoted by

$$s_{ij} = (p_i + p_j)^2\,, \quad s_{ijk} = (p_i + p_j + p_k)^2\,, \quad s = s_{123}\,, \tag{2}$$

and it is convenient to introduce the dimensionless quantities

$$x_1 = \frac{s_{12}}{s_{123}}, \quad x_2 = \frac{s_{23}}{s_{123}}. \tag{3}$$

Working in a helicity basis, it suffices to consider the pure photon exchange amplitude $\mathcal{A}_\gamma$ as it allows the reconstruction of the full amplitude with Z-boson exchange by adjusting the couplings. Furthermore, the complete information about $\mathcal{A}_\gamma$ is given by just one independent helicity amplitude, which we take to be $A_\gamma(1^+, 2^+, 3^-, 4^+, 5^-)$. All other helicity configurations can be obtained from parity and charge conjugation.

We can write $A_\gamma(1^+, 2^+, 3^-, 4^+, 5^-)$ in terms of coefficients c_2, c_4, c_6 and c_{12} for the various independent spinor structure as

$$\begin{aligned} A_\gamma(1^+, 2^+, 3^-, 4^+, 5^-) \;=\; & \frac{i}{\sqrt{2}}\frac{[12]}{s^3} \\ \times \Bigg\{ s\langle 35\rangle[42] \Bigg[(1-x_1)\left(c_2 + \frac{2}{x_2}c_6 - c_{12}\right) + (1-x_2)\,(c_4 - c_{12}) + 2c_{12} \Bigg] & \\ -\langle 31\rangle[12] \Bigg[[43]\langle 35\rangle \left(c_2 + \frac{2}{x_2}c_6 - c_{12}\right) + [41]\langle 15\rangle\,(c_4 - c_{12}) \Bigg] \Bigg\}\,, & \end{aligned} \tag{4}$$

where we have introduced the short-hand notation for spinors of definite helicity, $|i\pm\rangle = |p_i\pm\rangle = u_\pm(p_i) = v_\mp(p_i)$, $\langle i \pm | = \langle p_i \pm | = \bar{u}_\pm(p_i) = \bar{v}_\mp(p_i)$, and for the spinor products $\langle pq\rangle = \langle p - |q+\rangle$ and $[pq] = \langle p + |q-\rangle$.

The coefficients c_i depend on the x_1 and x_2 of eq.(3) and can be calculated in conventional dimensional regularization. To that end, we proceed as follows [6]-[8]. In a first step, with the help of Schwinger parameters [9], we map the tensor integrals to combinations of scalar integrals in various dimensions and with various powers ν_i of the propagators. For every basic topology, these scalar integrals can be written as nested sums involving Γ-functions. The evaluation of the nested sums proceeds systematically with the help of the algorithms of [6], which rely on the algebraic properties of the so called Z-sums,

$$Z(n;m_1,...,m_k;x_1,...,x_k) \;=\; \sum_{n\geq i_1>i_2>\ldots>i_k>0} \frac{x_1^{i_1}}{{i_1}^{m_1}} \ldots \frac{x_k^{i_k}}{{i_k}^{m_k}} \,. \tag{5}$$

By means of recursion the algorithms allow to solve the nested sums in terms of a given basis in Z-sums to any order in ε. Z-sums can be viewed as generalizations of harmonic sums [10] and an important subset of Z-sums are multiple polylogarithms [11],

$$\mathrm{Li}_{m_k,...,m_1}(x_k,...,x_1) \;=\; Z(\infty;m_1,...,m_k;x_1,...,x_k). \tag{6}$$

All algorithms for this procedure have been implemented in FORM [12] and in the GiNaC framework [13, 14]. In this way, we could calculate all loop integrals contributing to the one- and two-loop virtual amplitudes very efficiently in terms of multiple polylogarithms.

The perturbative expansion in α_s of the functions c_i is defined through

$$c_i \;=\; \sqrt{4\pi\alpha_s}\left(c_i^{(0)} + \left(\frac{\alpha_s}{2\pi}\right)c_i^{(1)} + \left(\frac{\alpha_s}{2\pi}\right)^2 c_i^{(2)} + O({\alpha_s}^3)\right) \,. \tag{7}$$

Then, after ultraviolet renormalization, the infrared pole structure of the renormalized coefficients c_i^{ren} agrees with the prediction made by Catani [15] using an infrared factorization formula. We use this formula to organize the finite part into terms arising from the expansion of the pole coefficients and a finite remainder,

$$c_i^{(2),\mathrm{fin}} \;=\; c_i^{(1),\mathrm{ren}} - \mathbf{I}^{(1)}(\varepsilon)c_i^{(1),\mathrm{ren}} - \mathbf{I}^{(2)}(\varepsilon)c_i^{(0)} \,, \tag{8}$$

for $i = \{2,4,6,12\}$, and with the one- and two-loop insertion operators $\mathbf{I}^{(1)}(\varepsilon)$ and $\mathbf{I}^{(2)}(\varepsilon)$ given in [15].

As an example, we present our result for $n_f N$-contribution to the finite part $c_{12}^{(2),\mathrm{fin}}$ at two loops,

$$\begin{aligned} c_{12}^{(2),\mathrm{fin}}(x_1,x_2) &= n_f N\Bigg(3\frac{\ln(x_1)}{(x_1{+}x_2)^2} + \frac{1}{4}\frac{\ln(x_2)^2 - 2\mathrm{Li}_2(1{-}x_2)}{x_1(1{-}x_2)} \\ &+\frac{1}{12}\frac{\zeta(2)}{(1{-}x_2)x_1} - \frac{1}{18}\frac{13{x_1}^2{+}36x_1{-}10x_1x_2{-}18x_2{+}31{x_2}^2}{(x_1{+}x_2)^2x_1(1{-}x_2)}\ln(x_2) \\ &+\frac{{x_1}^2{-}{x_2}^2{-}2x_1{+}4x_2}{(x_1{+}x_2)^4}\mathrm{R}_1(x_1,x_2) - \frac{1}{12}\frac{\mathrm{R}(x_1,x_2)}{x_1(x_1{+}x_2)^2}\Big[5x_2 + 42x_1 + 5 \end{aligned} \tag{9}$$

$$
-\frac{(1+x_1)^2}{1-x_2} - 4\frac{1-3x_1+3{x_1}^2}{1-x_1-x_2} - 72\frac{{x_1}^2}{x_1+x_2}\Bigg] + \Bigg[\frac{1}{12}\frac{1}{x_1(1-x_2)} + \frac{6}{(x_1+x_2)^3}
$$
$$
-\frac{1+2x_1}{x_1(x_1+x_2)^2}\Bigg](\mathrm{Li}_2(1-x_2) - \mathrm{Li}_2(1-x_1)) - \frac{1}{(x_1+x_2)x_1}\Bigg)
$$
$$
-\frac{1}{2}I\pi n_f N\frac{\ln(x_2)}{x_1(1-x_2)}\,.
$$

We have introduced the function $\mathrm{R}(x_1,x_2)$, which is well known from [16],

$$
\mathrm{R}(x_1,x_2) = \left(\frac{1}{2}\ln(x_1)\ln(x_2) - \ln(x_1)\ln(1-x_1) + \frac{1}{2}\zeta(2) - \mathrm{Li}_2(x_1)\right) + (x_1 \leftrightarrow x_2)\,. \tag{10}
$$

In addition, it is convenient, to define the symmetric function $\mathrm{R}_1(x_1,x_2)$, which contains a particular combination of multiple polylogarithms [11],

$$
\begin{aligned}
\mathrm{R}_1(x_1,x_2) = \Bigg(&\ln(x_1)\mathrm{Li}_{1,1}\left(\frac{x_1}{x_1+x_2}, x_1+x_2\right) - \frac{1}{2}\zeta(2)\ln(1-x_1-x_2) \\
&+\mathrm{Li}_3(x_1+x_2) - \ln(x_1)\mathrm{Li}_2(x_1+x_2) - \frac{1}{2}\ln(x_1)\ln(x_2)\ln(1-x_1-x_2) \\
&-\mathrm{Li}_{1,2}\left(\frac{x_1}{x_1+x_2}, x_1+x_2\right) - \mathrm{Li}_{2,1}\left(\frac{x_1}{x_1+x_2}, x_1+x_2\right)\Bigg) + (x_1 \leftrightarrow x_2)\,.
\end{aligned} \tag{11}
$$

We have made the following checks on our result. As remarked, the infrared poles agree with the structure predicted by Catani [15]. This provides a strong check of the complete pole structure of our result. In addition, we have tested various relations between the c_i. For instance, the combination $x_1 c_6$ is symmetric under exchange of x_1 with x_2. Finally, we could compare with the result for the squared matrix elements, i.e. the interference of the two-loop amplitude with the Born amplitude, and the interference of the one-loop amplitude with itself. The results of [5] are given in terms of one- and two-dimensional harmonic polylogarithms, which form a subset of the multiple polylogarithms [11]. Thus, we have performed the comparision analytically and we agree with the results of [5].

3. Conclusions

Our result represents one contribution to the full next-to-next-to-leading order calculation of $e^+e^- \to$ 3-jets. It has been obtained by means of an efficient method based on nested sums and is expressed in terms of multiple polylogarithms with simple arguments. As a consequence, our result can be continued analytically and applies also to $(2+1)$-jet production in deep-inelastic scattering or to the production of a massive vector boson in hadron-hadron collisions. At the same time, it provides an important cross check on the results for the squared matrix elements [5] with a completely independent method.

After the results of section 2 had been presented at this conference, Garland et al. published results for the complete two-loop amplitude for $e^+e^- \to q\bar{q}g$. Our results are in agreement with ref. [17].

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