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# The analytic value of the sunrise self-mass with two equal masses and the external invariant equal to the third squared mass.

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**Abstract**

We consider the two-loop self-mass sunrise amplitude with two equal masses $M$ and the external invariant equal to the square of the third mass $m$ in the usual $d$-continuous dimensional regularization. We write a second order differential equation for the amplitude in $x = m/M$ and show as solve it in close analytic form. As a result, all the coefficients of the Laurent expansion in $(d-4)$ of the amplitude are expressed in terms of harmonic polylogarithms of argument $x$ and increasing weight. As a by product, we give the explicit analytic expressions of the value of the amplitude at $x = 1$, corresponding to the on-mass-shell sunrise amplitude in the equal mass case, up to the $(d-4)^5$ term included.

# 1 Introduction

It is known that the sunrise graph of Figure 1 in the arbitrary mass case has four master integrals; they

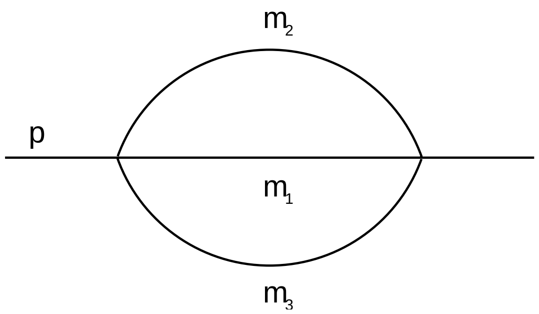


Figure 1: The sunrise graph for arbitrary masses.

can be taken to be the fully scalar amplitude

$$F_0(d,m_1^2,m_2^2,m_3^2,p^2) = \int \frac{d^dk_1}{(2\pi)^{d-2}}\frac{d^dk_2}{(2\pi)^{d-2}}\frac{1}{[(p-k_1)^2+m_1^2][(k_1-k_2)^2+m_2^2](k_2^2+m_3^2)}, \quad (1.1)$$

where $d$ is the continuous dimension, $m_i, i=1,3$ the masses, $p^2$ the square of the external momentum (all momenta being Euclidean), and the three amplitudes $F_i$ defined as

$$F_i(d,m_1^2,m_2^2,m_3^2,p^2) = -\frac{\partial}{\partial m_i^2}F_0(d,m_1^2,m_2^2,m_3^2,p^2) \ , \quad (1.2)$$

for $i=1,3$. The higher mass derivatives of the amplitudes can all be expressed, as a consequence of the integration by parts identities, in terms of the four master integrals. The four master integrals satisfy further a system of linear non homogeneous differential equations in $p^2$, derived in [1].

In the particular kinematical configuration $-p^2 = m_1^2 = m^2$ and $m_2^2 = m_3^2 = M^2$, Figure 2, one has

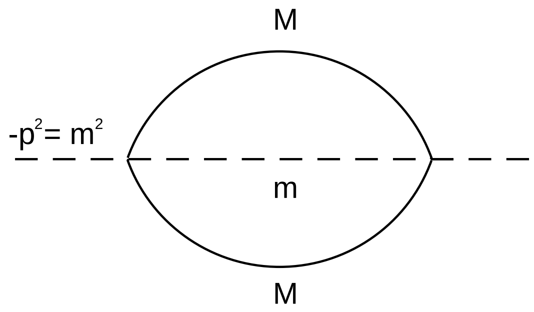


Figure 2: The sunrise graph with two equal masses and the external invariant equal to the third squared mass.

$$F_0(d,m^2,M^2,M^2,p^2=-m^2) = \int \frac{d^dk_1}{(2\pi)^{d-2}}\frac{d^dk_2}{(2\pi)^{d-2}}\frac{1}{(k_1^2-2pk_1)[(k_1-k_2)^2+M^2](k_2^2+M^2)} \ ; \quad (1.3)$$

introducing the dimensionless variables $x = m/M$ one can define the dimensionless function $\Phi(d,x)$ as

$$F_0(d,m^2,M^2,M^2,-m^2) = M^{2d-6}C^2(d)\Phi(d,x)\ , \tag{1.4}$$

*i.e.*

$$\Phi(d,x) = C^{-2}(d)M^{6-2d}F_0(d,M^2x^2,M^2,M^2,-M^2x^2)\ , \tag{1.5}$$

where $C(d) = (4\pi)^{\frac{4-d}{2}}\Gamma(3-d/2)$ is an overall normalization factor, with the limiting value $C(4) = 1$ at $d = 4$.

It will be shown in this paper that the integration by part identities for the master integrals imply a second order differential equation in $x$ for $\Phi(d,x)$; when functions and equations are systematically expanded around $d = 4$ as a Laurent series in $(d-4)$, one obtains a system of chained differential equations which can be easily solved recursively in closed analytical form, virtually up to any order in $(d-4)$; one finds in this way that the coefficients of the $(d-4)$ expansion of $\Phi(d,x)$ are a combination of harmonic polylogarithms [2] of increasing weight.

The differential equations themselves with the knowledge of the behaviour at $m = 0$ can be used to fix completely the integration constants; as a byproduct we give the value of the integral at $m = M$ (on mass shell equal mass case) up to the order $(d-4)^5$ included.

## 2 The differential equations.

For deriving the differential equation for $\Phi(d,x)$ we introduce the auxiliary functions

$$\begin{aligned}\Psi_0(d,\eta,p^2) &= m^{6-2d}\ F_0(d,m^2,m^2\eta,m^2\eta,p^2)\ , && (2.1)\\ \Psi_1(d,\eta,p^2) &= m^{8-2d}\left[F_2(d,m^2,m^2\eta,m^2\eta,p^2) + F_3(d,m^2,m^2\eta,m^2\eta,p^2)\right]\ . && (2.2)\end{aligned}$$

One finds at once

$$\frac{\partial}{\partial\eta}\Psi_0(d,\eta,p^2) = -\Psi_1(d,\eta,p^2)\ , \tag{2.3}$$

from which in turn

$$\begin{aligned}\frac{\partial}{\partial\eta}\Psi_1(d,\eta,p^2) &= -m^{(10-2d)}\\ &\left(\frac{\partial}{\partial m_2^2}+\frac{\partial}{\partial m_3^2}\right)\left[F_2(d,m^2,m_2^2,m_3^2,p^2)+F_3(d,m^2,m_2^2,m_3^2,p^2)\right]_{m_2^2=m_3^2=m^2\eta}\ . \quad (2.4)\end{aligned}$$

As already remarked, the mass derivatives of the master integrals $F_i, i = 1,3$ can be expressed, by using the integration by parts identities, in terms of the master integrals themselves (see for instance the Appendix of [1]), so that the above quantity is equal to a combination of $F_0$, corresponding to $\Psi_0(d,\eta,p^2)$, of the sum $F_2 + F_3$ (as required by the $m_2 = m_3$ symmetry), corresponding to $\Psi_1(d,\eta,p^2)$, and of the fourth master integral $F_1$, independent of $\Psi_0$ and $\Psi_1$, plus an inhomogeneous part given by a few products of tadpoles $T(d,m^2), T(d,M^2)$, with

$$T(d,m^2) = \int\frac{d^dk}{(2\pi)^{d-2}}\frac{1}{k^2+m^2} = C(d)\ \frac{m^{d-2}}{(d-2)(d-4)}\ . \tag{2.5}$$

The coefficients of the various terms are ratios of polynomials in $d$, $p^2$ and the masses; in the limit $p^2 \to -m^2$ the denominators of the coefficients develop a singular $1/(p^2+m^2)^2$ behaviour, which is however fully compensated by corresponding zeroes in the numerators, so that the limit $p^2 = -m^2$ is finite. In that limit, further, the coefficient of the master amplitude $F_1$ vanishes, so that $\frac{\partial}{\partial\eta}\Psi_1(d,\eta,p^2=-m^2)$, which is essentially the second $\eta$-derivative of $\Psi_0(d,\eta,p^2=-m^2)$, is expressed in terms of $\Psi_1(d,\eta,p^2=-m^2)$, its first $\eta$-derivative (up to a sign), and of $\Psi_0(d,\eta,p^2=-m^2)$ itself, plus an inhomogeneous term given by the tadpoles. For taking the $p^2 = -m^2$ limit, let us introduce the function $\Psi(d,\eta)$ through

$$F_0(d,m^2,M^2,M^2,-m^2) = m^{2d-6}\ C^2(d)\ \Psi(d,\eta)\ , \tag{2.6}$$

*i.e.*

$$\Psi(d,\eta) = C^{-2}(d)\ m^{6-2d}\ F_0(d,m^2,m^2\eta,m^2\eta,-m^2)\ . \tag{2.7}$$

In terms of $\Psi(d,\eta)$ the second order differential equation for $\Psi_0(d,\eta,p^2=-m^2)$ then reads

$$\begin{aligned} \eta(1-\eta)\frac{\partial^2}{\partial\eta^2}\Psi(d,\eta) - \left[\frac{1}{2} - \frac{3}{2}\eta + (1-2\eta)(d-4)\right]\frac{\partial}{\partial\eta}\Psi(d,\eta) \\ - \left[1 + \frac{7}{4}(d-4) + \frac{3}{4}(d-4)^2\right]\Psi(d,\eta) \quad &= \quad \frac{\eta^{(d-4)/2}}{4(d-4)^2} - \frac{\eta^{1+(d-4)}}{8(d-4)^2}\ . \end{aligned} \tag{2.8}$$

The equation, exact in the continuous dimension $d$, has been written in terms of powers of $(d-4)$ for convenience of later use.

As $\eta = 1/x^2$, by comparing the definitions Eq.s (1.4, 2.6) one has

$$\Psi\left(d,\eta=\frac{1}{x^2}\right) = x^{6-2d}\Phi(d,x)\ , \tag{2.9}$$

and the above equation for $\Psi(d,\eta)$ can be converted in the following equation for $\Phi(d,x)$

$$\begin{aligned} \frac{1}{4}x^2(x^2-1)\frac{\partial^2}{\partial x^2}\Phi(d,x) - \frac{1}{2}x\left[1+(d-4)x^2\right]\frac{\partial}{\partial x}\Phi(d,x) \\ + \left[\frac{1}{2}(1-x^2) + \left(\frac{3}{4}-\frac{1}{2}x^2\right)(d-4) + \frac{1}{4}(d-4)^2\right]\Phi(d,x) \quad &= \quad \frac{x^{2+(d-4)}}{4(d-4)^2} - \frac{1}{8(d-4)^2}\ . \end{aligned} \tag{2.10}$$

# 3 The expansion in $(d-4)$ around $d=4$ .

It is known [1] that in the $d\to 4$ limit $\Phi(d,x)$ develops a double pole in $(d-4)$, so that its Laurent expansion in $(d-4)$ reads

$$\Phi(d,x) = \sum_{n=-2}^{\infty}(d-4)^n\Phi^{(n)}(x)\ . \tag{3.1}$$

By inserting the above expansion in Eq. (2.10) one obtains a system of chained inhomogeneous equations in the functions $\Phi^{(n)}(x)$. For $n=-2,-1$ they read

$$\begin{aligned} \left[\frac{\partial^2}{\partial x^2} + \left(\frac{2}{x}+\frac{1}{1-x}-\frac{1}{1+x}\right)\frac{\partial}{\partial x} - \frac{2}{x^2}\right]\Phi^{(-2)}(x) = &\quad \frac{1}{2x^2} - \frac{1}{4(1-x)} - \frac{1}{4(1+x)} \\ \left[\frac{\partial^2}{\partial x^2} + \left(\frac{2}{x}+\frac{1}{1-x}-\frac{1}{1+x}\right)\frac{\partial}{\partial x} - \frac{2}{x^2}\right]\Phi^{(-1)}(x) = &\ -\left(\frac{1}{1-x}-\frac{1}{1+x}\right)\frac{\partial}{\partial x}\Phi^{(-2)}(x) \\ + \left(\frac{3}{x^2}+\frac{1}{2(1-x)}+\frac{1}{2(1+x)}\right)\Phi^{(-2)}(x) &\ - \frac{1}{2}\left(\frac{1}{1-x}+\frac{1}{1+x}\right)\ln x\ , \end{aligned} \tag{3.2}$$

while for $n\ge 0$ one has

$$\begin{aligned} \left[\frac{\partial^2}{\partial x^2} + \left(\frac{2}{x}+\frac{1}{1-x}-\frac{1}{1+x}\right)\frac{\partial}{\partial x} - \frac{2}{x^2}\right]\Phi^{(n)}(x) = &\ -\left(\frac{1}{1-x}-\frac{1}{1+x}\right)\frac{\partial}{\partial x}\Phi^{(n-1)}(x) \\ + \left(\frac{3}{x^2}+\frac{1}{2(1-x)}+\frac{1}{2(1+x)}\right)\Phi^{(n-1)}(x) &\ + \left(\frac{1}{x^2}+\frac{1}{2(1-x)}+\frac{1}{2(1+x)}\right)\Phi^{(n-2)}(x) \\ &\ - \frac{1}{2}\left(\frac{1}{1-x}+\frac{1}{1+x}\right)\frac{\ln^{n+2}x}{(n+2)!}\ . \end{aligned} \tag{3.3}$$

A few comments are in order. Quite in general, all the equations have the form

$$\left[\frac{\partial^2}{\partial x^2} + \left(\frac{2}{x}+\frac{1}{1-x}-\frac{1}{1+x}\right)\frac{\partial}{\partial x} - \frac{2}{x^2}\right]\Phi^{(n)}(x) = R^{(n)}(x)\ ; \tag{3.4}$$

the homogeneous part of the equations is always

$$\left[\frac{\partial^2}{\partial x^2}+\left(\frac{2}{x}+\frac{1}{1-x}-\frac{1}{1+x}\right)\frac{\partial}{\partial x}-\frac{2}{x^2}\right]\phi(x)=0\ , \tag{3.5}$$

while the inhomogeneous terms $R^{(n)}(x)$ vary from equation to equation and involve, besides the coefficients of the $(d-4)$-expansion of the inhomogeneous term of Eq.(2.10) which are explicitly known since the beginning, also the coefficients $\Phi^{(n-1)}(x)$ and $\Phi^{(n-2)}(x)$ of the $(d-4)$-expansion of the unknown function $\Psi(d,x)$. However, in a systematic bottom-up approach to the solution starting from the smallest value of $n$, *i.e.* $n=-2$, the inhomogeneous part can also be considered as known.

Finally, all the coefficients appearing in the equation, both in the homogeneous and the inhomogeneous part, are simple combinations of the rational factors $1/x$, $1/(1-x)$ and $1/(1+x)$; powers of $\ln x$ are also present, from the expansion of the original inhomogeneous term.

# 4 The behaviour of the equation for $x\to 0^+$.

As $x$ is proportional to a mass and therefore is an essentially real and positive variable, we study Eq.(2.10) in the $x\to 0^+$ limit; to that aim, we try the most general expansion

$$\Phi(d,x)=\sum_i x^{\alpha_i}\left(\sum_{n=0}^{\infty}a_n^{(i)}(d)x^n\right)\ . \tag{4.1}$$

By substituting the above expansion in Eq.(2.10), we find for the leading exponents $\alpha_i$ the four allowed values $\alpha_1=(2-d)$, $\alpha_2=(d-3)$, corresponding to the two independent solutions of the associated homogeneous equation and $\alpha_3=(d-2)$, $\alpha_4=0$, as required by the inhomogeneous terms appearing in the *r.h.s.* of Eq.(2.10).

By direct inspection of Eq.(1.3) and Eq.(1.4), one sees that $\Phi(d,x)$ is finite for $d>2$ in the $x\to 0^+$ limit, so that the first two behaviours, with exponents $(2-d)$ and $(d-3)$, are ruled out (*i.e.* the coefficients $a_n^{(i)}(d)$ are all vanishing for $i=1,2$ and any value of $n$ ). The coefficients of the other terms are then completely fixed by the inhomogeneous part of the equation, and an elementary calculation gives the following explicit behaviour in the $x\to 0^+$ limit

$$\begin{aligned}\Phi(d,x) &= -\frac{1}{2(d-2)(d-4)^2}\left[\frac{1}{d-3}+\frac{2x^2}{d(d-5)}\ +x^2x^{d-4}\right]\\ &+\ \mathcal{O}(x^3)\ .\end{aligned} \tag{4.2}$$

When expanding in $(d-4)$ according to Eq.(3.1), the above equation reads

$$\begin{aligned}\Phi^{(-2)}(x) &= -\frac{1}{2}-\frac{1}{4}x^2+\mathcal{O}(x^3)\ ,\\ \Phi^{(-1)}(x) &= +\frac{3}{4}+\frac{5}{16}x^2-\frac{x^2}{2}\ln x+\mathcal{O}(x^3)\ ,\\ \Phi^{(0)}(x) &= -\frac{7}{8}+\frac{3}{64}x^2+\frac{x^2}{4}(\ln x-\ln^2 x)+\mathcal{O}(x^3)\ ,\end{aligned} \tag{4.3}$$

and so on up to any desired order in $(d-4)$.

It is to be observed that by using the differential equation Eq.(2.10) one can transform a qualitative information, the finiteness of $\Phi(d,x)$ at $x=0$, into a quantitative information, such as the actual value of the $x\to 0^+$ behaviour of $\Phi(d,x)$ given by Eq.(4.2), without carrying out any explicit loop integration in Eq.(1.3).

## 5 The solution.

The algorithm for solving Eq.(3.4) goes back to Euler. Let $\phi_1(x)$, $\phi_2(x)$ be two independent solutions of the associated homogeneous equation (3.5), such that their Wronskian $W(x)$, defined as

$$W(x) = \phi_1(x)\phi_2'(x) - \phi_1'(x)\phi_2(x) \ , \tag{5.1}$$

does not vanish; the solution of the second order Eq.(3.4) can then be written as

$$\Phi^{(n)}(x) = \phi_1(x)\left(\Phi_1^{(n)} - \int^x \frac{dx'\phi_2(x')}{W(x')}R^{(n)}(x')\right) + \phi_2(x)\left(\Phi_2^{(n)} + \int^x \frac{dx'\phi_1(x')}{W(x')}R^{(n)}(x')\right) \ , \tag{5.2}$$

where $\Phi_1^{(n)}, \Phi_2^{(n)}$ are two integration constants, to be fixed by the boundary conditions Eq.(4.2) expanded in $(d-4)$, *i.e.* by Eq.s(4.3).

For the actual use of Euler's algorithm, the solutions of the homogeneous equation (3.5) are needed. It turns out that the homogeneous equation associated to Eq.(2.8) at $d=4$

$$\left[\eta(1-\eta)\frac{\partial^2}{\partial\eta^2} - \left(\frac{1}{2} - \frac{3}{2}\eta\right)\frac{\partial}{\partial\eta} - 1\right]\psi(\eta) = 0 \ , \tag{5.3}$$

is a hypergeometric equation, whose solution regular at $\eta = 0$ is the very simple hypergeometric function

$${}_2F_1\left(-\frac{1}{2}, -2, -\frac{1}{2}; \eta\right) = (1-\eta)^2 \ . \tag{5.4}$$

With that input and the transformation (2.9) it is easy to obtain the following solutions of the homogeneous equation (3.5):

$$\begin{aligned} \phi_1(x) &= \frac{(1-x^2)^2}{x^2} \ , \\ \phi_2(x) &= \frac{(1-x^2)^2}{x^2}\ln\left(\frac{1+x}{1-x}\right) - 2\frac{1+x^2}{x} \ , \end{aligned} \tag{5.5}$$

having the Wronskian

$$W(x) = 16\frac{1-x^2}{x^2} \ , \tag{5.6}$$

and the $x \to 0$ behaviour

$$\begin{aligned} \phi_1(x) &\to \frac{1}{x^2} \ , \\ \phi_2(x) &\to -6x \ . \end{aligned} \tag{5.7}$$

With the above explicit homogeneous solutions one can solve recursively the system of the non homogeneous equations (3.3) for $\Phi^{(n)}(x)$, starting from $n=-2$, by carrying out repeatedly the integration in $x'$ of (5.2) in close analytic form; one obtains in that way a combination of Harmonic PolyLogarithms (HPLs) of increasing weight [2] times simple coefficients involving the rational factors $1/x$, $1/(1-x)$ and $1/(1+x)$.

To illustrate this statement, which is the central point of this paper, let us recall for convenience of the reader the definition of the HPL's. The HPL's form a family of functions depending on an argument, say $x$, and on a set of indices, say $a_i, i = 1,..,w$ or $\vec{a}$ in more compact notation, where each of the $a_i$ can take one of the three values $1, 0, -1$ and whose number $w$ is called the weight of the HPL. At weight $w=1$ there are 3 HPL's, defined as

$$\begin{aligned} \mathrm{H}(1;x) &= -\ln(1-x) \ , \\ \mathrm{H}(0;x) &= \ln x \ , \\ \mathrm{H}(-1;x) &= \ln(1+x) \ , \end{aligned} \tag{5.8}$$

whose derivatives can be written as

$$\frac{d}{dx}\mathrm{H}(a;x) = \mathrm{f}(a,x) \ , \tag{5.9}$$

where the index $a$ can take one of the three values $(1,0,-1)$ and the rational factors $\mathrm{f}(a;x)$ are given by

$$\begin{aligned} \mathrm{f}(1;x) &= \frac{1}{1-x} \ , \\ \mathrm{f}(0;x) &= \frac{1}{x} \ , \\ \mathrm{f}(-1;x) &= \frac{1}{1+x} \ . \end{aligned} \tag{5.10}$$

At weight $w > 1$, if all the $w$ indices are equal to 0 let us indicate them by $\vec{0}_w$ and define correspondingly

$$\mathrm{H}(\vec{0}_w;x) = \frac{1}{w!}\ln^w x \ ; \tag{5.11}$$

in all the other cases (*i.e.* when the indices are not all equal to zero), let us indicate any set of $w$ indices by $(a,\vec{b})$, where $a$ can take one of the three values $(1,0,-1)$ and $\vec{b}$ stands for the set of other $w-1$ indices, and define correspondingly

$$\mathrm{H}(a,\vec{b};x) = \int_0^x dx'\mathrm{f}(a;x')\mathrm{H}(\vec{b};x') \ . \tag{5.12}$$

Note that in full generality one has

$$\frac{d}{dx}\mathrm{H}(a,\vec{b};x) = \mathrm{f}(a,x)\mathrm{H}(\vec{b};x) \ , \tag{5.13}$$

which can also be written as the equivalent indefinite integration formula

$$\int^x dx'\mathrm{f}(a;x')\mathrm{H}(\vec{b};x') = A + \mathrm{H}(a,\vec{b};x) \ , \tag{5.14}$$

where $A$ is an integration constant.

Further, the product of two HPL's of a same argument $x$ and weights $p,q$ can be expressed as a combination of HPL's of argument $x$ and weight $r = p+q$, according to the product identity

$$\mathrm{H}(\vec{p};x)\mathrm{H}(\vec{q};x) = \sum_{\vec{r}=\vec{p}\uplus\vec{q}} \mathrm{H}(\vec{r};x) \ , \tag{5.15}$$

where $\vec{p},\vec{q}$ stand for the $p$ and $q$ components of the indices of the two HPL's, while $\vec{p}\uplus\vec{q}$ represents all possible mergers of $\vec{p}$ and $\vec{q}$ into the vector $\vec{r}$ with $r$ components, in which the relative orders of the elements of $\vec{p}$ and $\vec{q}$ are preserved. The simplest cases of the above identities are

$$\begin{aligned} \mathrm{H}(a;x)\mathrm{H}(b;x) &= \mathrm{H}(a,b;x) + \mathrm{H}(b,a;x) \ , \\ \mathrm{H}(a;x)\mathrm{H}(b,c;x) &= \mathrm{H}(a,b,c;x) + \mathrm{H}(b,a,c;x) + \mathrm{H}(b,c,a;x) \ ; \end{aligned} \tag{5.16}$$

more complicated cases are immediately established recursively (all the above formulae can indeed easily be checked by differentiating, repeatedly when needed, with respect to $x$).

After this digression, let us come back to Eq.(5.2), which gives the solution as an indefinite integral. The original $n=-2$ integrand consists of powers of $x'$, $1/(1-x')$ and $1/(1+x')$, (without any loss of generality $(1-x')$ and $(1+x')$ can be taken to occur only as negative powers), which are present in all the factors (see the first of Eq.s(3.2) for the definition of $R^{(-2)}(x')$), and of the two HPL's of weight 1, $\mathrm{H}(1;x') = -\ln(1-x')$ and $\mathrm{H}(-1;x') = \ln(1+x')$, which occur in $\phi_2(x')$, Eq.(5.5). The terms containing powers of the rational factors different from the mere inverse, *i.e.* different from the first powers of $1/x'$, $1/(1-x')$ and $1/(1+x')$, can be integrated by parts; in so doing they give rise at most to derivatives of the occurring HPL's, *i.e.* again products of the three rational factors $x'$, $1/(1-x')$ and $1/(1+x')$, and, according to Eq.s(5.13, 5.9), of HPL's of lower weight if the HPL's in the integrand have weight greater than 1 (as it is in the general case $n > -2$). The result can be partial fractioned, and the integration by parts repeated until all powers of the rational fractions different from the mere first power are processed. The terms with the mere inverse of the three rational fractions and HPL's, finally, can be integrated at once according to Eq.(5.14), giving rise to HPL's of higher weight. The integration constants are then fixed by imposing the $x \to 0^+$ behaviour (4.3). It is clear that the argument applies to any of the subsequent integrations as well, recursively in $n$.

For $n = -2$ the explicit calculation gives $\Phi_1^{(-2)} = 0$, $\Phi_2^{(-2)} = 0$ and

$$\Phi^{(-2)}(x) = -\,\frac{2+x^2}{8}\ . \tag{5.17}$$

Note that, in this case, all the HPL's cancel out from the final result, which is of course in agreement with the general mass case given in the literature [1].

One can then proceed to $n = -1$; the term $\ln x$ appearing in the inhomogeneous part of the second of Eq.s(3.2) can be written as $H(0;x)$, see (5.8), the products of the HPL's already present in $R^{(-1)}(x')$ and the HPL's of the $\phi_i(x')$ can be rewritten as combinations of single HPL's of suitable indices according to 5.15, the integration performed as explained above – and so on for the higher values of $n$, up to any required order. An explicit calculation gives for the integration constants $\Phi_1^{(-1)} = -5/64$, $\Phi_1^{(0)} = 29/256$, $\Phi_1^{(1)} = -107/1024$, $\Phi_1^{(2)} = 185/4096$, $\Phi_1^{(3)} = -1285/16384$, $\Phi_1^{(4)} = -18991/65536$, $\Phi_1^{(5)} = 164173/262144$, while, for any $n$, $\Phi_2^{(n)} = 0$. The first $\Phi^{(n)}(x)$ are

$$\Phi^{(-1)}(x) = \frac{3}{8} + x^2\left(\frac{5}{32} - \frac{1}{4}H(0;x)\right)\ , \tag{5.18}$$

and

$$\begin{aligned}\Phi^{(0)}(x) \;=\;& -\frac{3}{8} - \frac{1}{8}H(0;x) + x^2\left(-\frac{11}{128} + \frac{5}{16}H(0;x) - \frac{1}{8}H(0;x)H(0;x)\right)\\ & -\frac{(1-x^2)^2}{8x^2}\Big(H(-1;x)H(0;x) - H(0;x)H(1;x) - H(0,-1;x) + H(0,1;x)\Big)\ . \end{aligned}\tag{5.19}$$

As matter of fact we evaluated explicitly, by means of an integration routine written in `FORM` [3], all the $\Phi^{(n)}(x)$ up to $n = 5$ included, which is found to involve HPL's of weight **7**. The explicit expressions obtained are however too long to be listed here. It is found that, in general, $\Phi^{(n)}(x)$ involves HPL's of weight $(n+2)$.

The values of the HPL's of argument equal to 1 are known [4]; we can then take our explicit analytic results for the $\Phi^{(n)}(x)$ up to $n = 5$ and evaluate them at $x = 1$ by using [4], obtaining the analytical on-shell value of the sunrise amplitude Eq. 1.3 in the equal mass case limit up to the 5th order in the $(d-4)$ expansion. At $x = 1$ the terms of highest weight disappear from $\Phi^{(n)}(x)$, due to the general structure of the solution, Eq.(5.5,5.2), and the coefficient of $(d-4)^n$ involves constants up to weight $n+1$ only. The result reads

$$\begin{aligned}\Phi(d,x=1) \;=\;& -\frac{3}{8}\frac{1}{(d-4)^2} + \frac{17}{32}\frac{1}{(d-4)} - \frac{59}{128} + (d-4)\left(\frac{65}{512} + \frac{1}{24}\pi^2\right)\\ & +(d-4)^2\left(\frac{1117}{2048} - \frac{13}{96}\pi^2 + \frac{1}{8}\pi^2\ln 2 - \frac{7}{16}\zeta(3)\right)\\ & +(d-4)^3\left(-\frac{13783}{8192} + \frac{115}{384}\pi^2 - \frac{13}{32}\pi^2\ln 2 + \frac{91}{64}\zeta(3) + \frac{1}{8}\pi^2\ln^2 2 - \frac{31}{2880}\pi^4 + \frac{1}{16}\ln^4 2 + \frac{3}{2}a_4\right)\\ & +(d-4)^4\left(\frac{114181}{32768} - \frac{865}{1536}\pi^2 + \frac{115}{128}\pi^2\ln 2 - \frac{805}{256}\zeta(3) - \frac{13}{32}\pi^2\ln^2 2 + \frac{403}{11520}\pi^4 - \frac{13}{64}\ln^4 2\right.\\ & \qquad\left. -\frac{39}{8}a_4 - \frac{31}{960}\pi^4\ln 2 + \frac{1}{8}\pi^2\ln^3 2 + \frac{5}{96}\pi^2\zeta(3) + \frac{3}{80}\ln^5 2 - \frac{9}{2}a_5 + \frac{465}{128}\zeta(5)\right)\end{aligned}$$

$$
\begin{aligned}
&+(d-4)^5\left(-\frac{820495}{131072}+\frac{5971}{6144}\pi^2-\frac{865}{512}\pi^2\ln 2+\frac{6055}{1024}\zeta(3)+\frac{115}{128}\pi^2\ln^2 2-\frac{713}{9216}\pi^4+\frac{115}{256}\ln^4 2\right.\\
&\qquad+\frac{345}{32}a_4-\frac{65}{384}\pi^2\zeta(3)-\frac{13}{32}\pi^2\ln^3 2+\frac{403}{3840}\pi^4\ln 2-\frac{39}{320}\ln^5 2-\frac{6045}{512}\zeta(5)+\frac{117}{8}a_5\\
&\qquad-\frac{9}{160}\pi^4\ln^2 2+\frac{1}{64}\pi^2\ln^4 2+\frac{79}{34560}\pi^6+\frac{15}{8}\zeta(3)\ln^3 2-\frac{25}{32}\zeta(3)\pi^2\ln 2+\frac{595}{128}\zeta^2(3)\\
&\qquad\left.+\frac{45}{4}\zeta(5)\ln 2-\frac{45}{4}a_5\ln 2+\frac{21}{320}\ln^6 2-9a_6+\frac{45}{4}b_6\right)\\
&+\mathcal{O}((d-4)^6)\ , \qquad (5.20)
\end{aligned}
$$

where we have expressed the constant `s6` appearing in [4] as

$$
\begin{aligned}
\mathtt{s6} \;=\; & -\frac{1}{6}\pi^2\zeta(3)\ln 2-\frac{1}{72}\pi^2\ln^4 2-\frac{1}{720}\pi^4\ln^2 2+\frac{1}{540}\pi^6+2\zeta(5)\ln 2\\
& -2a_5\ln 2+\frac{1}{3}\zeta(3)\ln^3 2+\frac{1}{120}\ln^6 2+\frac{5}{4}\zeta^2(3)-4a_6+2b_6\ , \qquad (5.21)
\end{aligned}
$$

and the definitions of all the previous constants in terms of HPL's or Nielsen's polylogarithms (NPl) is given in Table 1, together with their numerical values.

Our result, Eq.(5.20), agrees with previous results in the literature (apart from obvious normalization factors), such as [5], which goes up the $(d-4)^3$ of present work, and [6] which arrives at $(d-4)^4$.

Numerically, Eq.(5.20) gives

$$
\begin{aligned}
\Phi(d,x=1) \;=\; & -0.37500000000000000000\frac{1}{(d-4)^2}+0.53125000000000000000\frac{1}{(d-4)}\\
& -0.46093750000000000000+0.53818664171204166667(d-4)\\
& -0.46186261021407291667(d-4)^2+0.53810855601624843750(d-4)^3\\
& -0.46190033063946289062(d-4)^4+0.53809670843016515942(d-4)^5\\
& +\mathcal{O}((d-4)^6)\ , \qquad (5.22)
\end{aligned}
$$

in perfect agreement with the numerical results of [7], which moreover gives two further terms in the expansion.

As a last curiosity, let us observe that for $d\to 4$ Eq.(5.22) can be rewritten as

$$
\begin{aligned}
\Phi(d,x=1) \;=\; & \frac{1}{(d-5)}\left(+0.37500000000000000000\frac{1}{(d-4)^2}-0.90625000000000000000\frac{1}{(d-4)}\right.\\
& +0.99218750000000000000-0.99912414171204166667(d-4)\\
& +1.00004925192611145833(d-4)^2-0.99997116623032135417(d-4)^3\\
& +1.00000888665571132 81(d-4)^4-0.99999703906962805005(d-4)^5\\
& \left.+\,\mathcal{O}((d-4)^6)\right)\ , \qquad (5.23)
\end{aligned}
$$

showing that the coefficients of the powers of $(d-4)^n$ approach quickly $(-1)^n$ for increasing $n$.

# 6 Conclusions.

We have shown that the two-loop self-mass sunrise amplitude, in the usual $d$-continuous dimensional regularization and particular kinematical configuration corresponding to two equal masses $M$ and the external invariant equal to the square of the third mass $m$, satisfies a second order differential equation in the dimensionless ratio $x = m/M$. The equation can be expanded in Laurent series at $d = 4$, and solved in closed analytical form. As a results, all the $n$-th coefficient of the Laurent expansion can be expressed in

| constant | HPL | NPl | numerical value |
|---|---|---|---|
| $\zeta(3)$ | $H(0,0,1;1)$ | $S_{2,1}(1)$ | 1.2020569031595942854 |
| $a_4$ | $H(0,0,0,1;1/2)$ | $S_{3,1}(1/2)$ | 0.5174790616738993863 |
| $\zeta(5)$ | $H(0,0,0,0,1;1)$ | $S_{4,1}(1)$ | 1.0369277551433699263 |
| $a_5$ | $H(0,0,0,0,1;1/2)$ | $S_{4,1}(1/2)$ | 0.5084005792422687074 |
| $a_6$ | $H(0,0,0,0,0,1;1/2)$ | $S_{5,1}(1/2)$ | 0.5040953978039885506 |
| $b_6$ | $H(0,0,0,0,1,1;1/2)$ | $S_{4,2}(1/2)$ | 0.0087230030575968884272 |

Table 1: Constants up to weight 6 appearing in the calculation

terms of harmonic polylogarithms of argument $x$ and weight up to $n+2$. As a by product, we give the explicit analytic expressions of the value of the amplitude at $x=1$, corresponding to the on-mass-shell sunrise amplitude in the equal mass case, up to the $(d-4)^5$ term included.

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[7] S. Laporta, arXiv:hep-ph/0111123. See in particular Eq.(14) for the quantity $I_{15}$; in order to match the normalization our result must be multiplied by $64/(d-2)$ and then expanded in $(4-d)/2$.