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# Next-to-leading-log renormalization-group running in heavy-quarkonium creation and annihilation

Antonio Pineda

*Institut für Theoretische Teilchenphysik*
*Universität Karlsruhe, D-76128 Karlsruhe, Germany*

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In the framework of potential NRQCD, we obtain the next-to-leading-log renormalization-group running of the matching coefficients for the heavy quarkonium production currents near threshold. This allows to obtain S-wave heavy-quarkonium production/annihilation observables with next-to-leading-log accuracy within perturbative QCD. In particular, we give expressions for the decays of heavy quarkonium to $e^+e^-$ and to two photons.

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Heavy quark-antiquark systems near threshold are characterized by the small relative velocity $v$ of the heavy quarks in their center of mass frame. This small parameter produces a hierarchy of widely separated scales: $m$ (hard), $mv$ (soft), $mv^2$ (ultrasoft), ... . The factorization between them is efficiently achieved by using effective field theories, where one can organize the calculation as various perturbative expansions on the ratio of the different scales effectively producing an expansion in $v$. The terms in these series get multiplied by parametrically large logs: $\ln v$, which can also be understood as the ratio of the different scales appearing in the physical system. Again effective field theories are very efficient in the resummation of these large logs once a renormalization group (RG) analysis of them has been performed. This will be the aim of this paper for annihilation and production processes near threshold.

We will restrict ourselves, in this paper, to the situation where $\Lambda_{\rm QCD} \ll m\alpha_{\rm s}^2$ (to be implicit in what follows), which is likely to be relevant, at least, for $t$-$\bar{t}$ production near threshold.

NRQCD [1] has an ultraviolet (UV) cut-off $\nu_{NR} = \{\nu_p, \nu_s\}$ satisfying $mv \ll \nu_{NR} \ll m$. At this stage $\nu_p \sim \nu_s$. $\nu_p$ is the UV cut-off of the relative three-momentum of the heavy quark and antiquark, $\mathbf{p}$. $\nu_s$ is the UV cut-off of the three-momentum of the gluons and light quarks.

Potential NRQCD (pNRQCD) [2] (see [3,4] for details) is defined by its particle content and cut-off $\nu_{pNR} = \{\nu_p, \nu_{us}\}$, where $\nu_p$ is the cut-off of the relative three-momentum of the heavy quarks and is such that $|\mathbf{p}| \ll \nu_p \ll m$, and $\nu_{us}$ is the cut-off of the three-momentum of the gluons and light quarks with $\mathbf{p}^2/m \ll \nu_{us} \ll |\mathbf{p}|$. Note that no gluons or light quarks with momentum of $O(|\mathbf{p}|)$ are kept dynamical, since they do not appear as physical (on-shell) states near threshold. Nevertheless, these degrees of freedom can appear off-shell and, since their momentum is of the order of the relative three-momentum of the heavy quarks, their integration produces non-local terms (potentials) in three-momentum space. In particular, the integration of these degrees of freedom takes into account the non-analytical behavior in the transfer momentum of the heavy quark, $\mathbf{k} = \mathbf{p} - \mathbf{p}'$, of the order of the relative three-momentum of the heavy quarks.

The matching process, which basically means the computation of the potentials, is carried out for a given external incoming (outcoming) momentum $\mathbf{p}$ ($\mathbf{p}'$) and one has to sum over all them in the pNRQCD Lagrangian, since they are still physical degrees of freedom as far as their momentum is below $\nu_p$. In position space this means that an integral over $\mathbf{x}$, the relative distance between the heavy quarks, appears in the Lagrangian when written in terms of the heavy quark-antiquark bilinear.

Within pNRQCD, integrals over $\mathbf{p}$ (or $\mathbf{x}$) appear when solving the Schrödinger equation that dictates the dynamics of the heavy quarkonium near threshold. At lower orders these integrals are finite effectively replacing $\mathbf{p}$ by $\sim m\alpha_{\rm s}$. Nevertheless, at higher orders in quantum mechanics perturbation theory and/or if some singular enough operators are introduced (as it will be the case of the heavy quarkonium production currents) singularities proportional to $\nu_p$ appear that must be absorbed by the potentials or by the matching coefficients of the currents. We will describe how to resum the logarithms associated to this cutoff within pNRQCD.

Let us now describe the matching between QCD and pNRQCD within an RG framework. For the case where no divergences proportional to $\nu_p$ appear, the procedure reduces to the results of Ref. [5], of which we will closely follow the notation in the following.

We first address the procedure that gives the running of the potentials. One first does the matching from QCD to NRQCD. The latter depends on some matching coefficients: $c(\nu_s)$ and $d(\nu_p, \nu_s)$, which can be obtained order by order in $\alpha_s$ (with $\nu_p = \nu_s$) following the procedure described in Ref. [6]. The $c(\nu_s)$ stand for the coefficients of the operators that already exist in the theory with only one heavy quark (ie. HQET) and the $d(\nu_p, \nu_s)$ stand for the coefficients of the four heavy fermion operators. The starting point of the renormalization group equation can be obtained from these calculations by setting $\nu_p = \nu_s = m$ (up to a constant of order one). In principle, we should now compute the running of $\nu_p$ and $\nu_s$. The running of the $c(\nu_s)$ can be obtained using HQET techniques [7]. The running of the $d(\nu_p, \nu_s)$ is more complicated. At one-loop, $\nu_p$ does not appear and we effectively have $d(\nu_p, \nu_s) \simeq d(\nu_s)$, whose running can also be obtained using HQET-like techniques [5]. At higher orders the dependence on $\nu_p$ appears and the running of the $d(\nu_p, \nu_s)$ becomes more complicated. Fortunately, we need not to compute the running of $d$ in this more general case because, as we will see, the relevant running of the $d$ for near threshold observables can be obtained within pNRQCD.

The next step is the matching from NRQCD to pNRQCD. The latter depends on some matching coefficients (potentials). They typically have the following structure: $\tilde{V}(c(\nu_s), d(\nu_p, \nu_s), \nu_s, \nu_{us}, r)$. After matching, any dependence on $\nu_s$ disappears since the potentials have to be independent of $\nu_s$. Therefore, they could be formally written as $\tilde{V}(c(1/r), d(\nu_p, 1/r), 1/r, \nu_{us}, r)$. These potentials can be obtained order by order in $\alpha_s$ following the procedure of [2,3]. The integrals in the matching calculation would depend on a factorization scale $\mu$, which could be either $\nu_s$ or $\nu_{us}$. In the explicit calculation they could be distinguished by knowing the UV and infrared (IR) behavior of the diagrams: UV divergences are proportional to $\nu_s$, which should be such as to cancel the $\nu_s$ scale dependence inherited from the NRQCD matching coefficients, and IR divergences to $\nu_{us}$. In practice, however, as far as we only want to perform a matching calculation at some given scale $\mu = \nu_s = \nu_{us}$, it is not necessary to distinguish between $\nu_s$ and $\nu_{us}$ (or if working order by order in $\alpha_s$ without attempting any log resummation).

Before going into the rigorous procedure to obtain the RG equations of the potentials, let us first discuss their structure on physical grounds. As we have mentioned, the potential is independent of $\nu_s$. The independence of the potential with respect $\nu_s$ allows us to fix the latter to $1/r$ that, in a way, could be understood as the matching scale for $\nu_s$ [1]. Therefore $1/r$, the point where the multipole expansion starts, would also provide with the starting point of the renormalization group evolution of $\nu_{us}$ (up to a constant of order one). The running of $\nu_{us}$ can then be obtained following the procedure described in Ref. [8,5]. At the end of the day, we would have $\tilde{V}(c(1/r), d(\nu_p, 1/r), 1/r, \nu_{us}, r)$, where the running on $\nu_{us}$ is known and also the running in $1/r$ if the $d$ is $\nu_p$-independent. So far the only explicit dependence of the

[1] In practice, the potential is often first obtained in momentum space so that one could then set $\mu = k$. Note, however, that this is not equivalent to $\mu = 1/r$ since finite pieces will appear after performing the Fourier transform.

potential on $\nu_p$ appears in the $d$. Nevertheless, the potential is also implicitly dependent on the three-momentum of the heavy quarks through the requirement $1/r \sim \mathbf{p} \ll \nu_p$ and also through $\nu_{us}$ since the latter needs to fulfill $\mathbf{p}^2/m \ll \nu_{us} \ll |\mathbf{p}|$. This latter requirement is fulfilled if we fix $\nu_{us} = \nu_p^2/m$ (this constraint tells you how much you can run down $\nu_{us}$ in the potential before finding the cutoff $\nu_p^2/m$ caused by the cutoff of $\mathbf{p}$).

Within pNRQCD, the potentials should be introduced in the Schrödinger equation. This means that integrals over the relative three-momentum of the heavy quarks take place. When these integrals are finite one has $\mathbf{p} \sim 1/r \sim m\alpha_{\rm s}$ and $\mathbf{p}^2/m \sim m\alpha_{\rm s}^2$. Therefore, one can lower $\nu_{us}$ up to $\sim m\alpha_{\rm s}^2$ reproducing the results obtained in Ref. [5]. In some cases, in particular in heavy quarkonium creation, the integrals over $\mathbf{p}$ are divergent and the log structure is dictated by the ultraviolet behavior of $\mathbf{p}$ and $1/r$. This means that we can not replace $1/r$ and $\nu_{us}$ by its physical expectation values but rather by their cutoffs within the integral over $\mathbf{p}$. Therefore, for the RG equation of $\nu_p$, the anomalous dimensions will depend (at leading order) on $\tilde{V}(c(\nu_p), d(\nu_p, \nu_p), \nu_p, \nu_p^2/m, \nu_p)$ [2] and the running will go from $\nu_p \sim m$ down to $\nu_p \sim m\alpha_{\rm s}$. Note that at this stage a single cutoff, $\nu_p$, exists and the correlation of cutoffs can be seen. The importance of the idea that the cutoffs of the non-relativistic effective theory should be correlated was first realized by Luke, Manohar and Rothstein in Ref. [9] (for an application to QED see [10]). Note also that at the matching scale $\nu_p \sim m$ what it would be the ultrasoft cutoff is also of order $m$. In this sense it should be understood the statement in Ref. [9] that ultrasoft gluons appear at the scale $m$, a point that becomes relevant within a RG approach.

With the above discussion in mind the matching between NRQCD and pNRQCD could be thought as follows. One does the matching by computing the potentials order by order in $\alpha_{\rm s}$ at the matching scale $\nu_p = \nu_s = \nu_{us}$ following the procedure of [2,3] (by doing the matching at a generic $\nu_p$ some of the running is trivially obtained). The structure of the potential at this stage then reads $\tilde{V}(c(\nu_p), d(\nu_p, \nu_p), \nu_p, \nu_p, \nu_p)$ (and similarly for the derivatives with respect $\ln r$ of the potential). This provides the starting point of the renormalization group evolution of $\nu_{us}$ (up to a constant of order one). The running of $\nu_{us}$ can then be obtained following the procedure described in Ref. [8,5]. For the final point of the evolution of $\nu_{us}$ we choose $\nu_{us} = \nu_p^2/m$. At the end of the day, we obtain $\tilde{V}(c(\nu_p), d(\nu_p, \nu_p), \nu_p, \nu_p^2/m, \nu_p) \equiv \tilde{V}(\nu_p)$.

The running of $\nu_p$ goes from $\nu_p = m$ (this was fixed when the matching between QCD and NRQCD was done) up to the physical scale of the problem $\nu_p \sim m\alpha_{\rm s}$. As far as the running

[2] Roughly speaking, this result can be thought as expanding $\ln r$ around $\ln \nu_p$ in the potential ie.

$$\begin{aligned}\tilde{V}(c(1/r), d(\nu_p, 1/r), 1/r, \nu_p^2/m, r) \simeq\, & \tilde{V}(c(\nu_p), d(\nu_p, \nu_p), \nu_p, \nu_p^2/m, \nu_p) \\ & + \ln(\nu_p r) r \frac{d}{dr}\tilde{V}\Big|_{1/r=\nu_p} + \cdots \end{aligned} \quad (1)$$

The $\ln(\nu_p r)$ terms may give subleading contributions to the anomalous dimension when introduced in divergent integrals over $\mathbf{p}$. The discussion at this stage is not very rigorous and a more precise discussion would require a full detailed study within dimensional regularization. Nevertheless, we do not expect it to change the underlying idea although it deserves further investigations.

of the NRQCD matching coefficients is known, the above result gives the complete running of the potentials. The procedure to get the running of the $c$ is known at any finite order. For the $d$ it is only known at one-loop order, since, at this order, it is just $\nu_s$-dependent. Nevertheless, at higher orders, dependence on $\nu_p$ appears. Therefore, the above method is not complete unless an equation for the running of $\nu_p$ is provided. This is naturally given within pNRQCD. It appears through the iteration of potentials. One typical example would be the diagram in Fig. 1, which would contribute to the running of $D_{d,s}^{(2)}$, the matching coefficient of the delta potential, at next-to-leading-log (NLL) order as follows (see Ref. [5] for notation)

$$\nu_p \frac{d}{d\nu_p} D_{d,s}^{(2)}(\nu_p) \sim \alpha_{V_s}(\nu_p) D_{d,s}^{(2)2}(\nu_p) + \cdots \tag{2}$$

so that even without knowing the running of the $d$ (which need to be known at NLL order in this case) we can obtain the running of the potential (one can also think of trading Eq. (2) into an equation for $d$, which is the only unknown parameter within the potential).

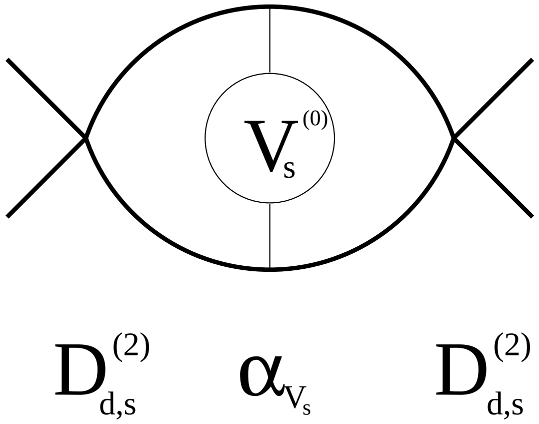


Fig. 1. *Contribution to the running of $D_{d,s}^{(2)}$ at NLL.*

The matching scale between QCD and NRQCD is $\nu_p \sim \nu_s \sim m$. On the other hand, the matching scale between NRQCD and pNRQCD is also the hard scale: $\nu_p \sim \nu_p^2/m \sim m$. Therefore, one could wonder about the necessity of using the intermediate theory NRQCD. This is indeed the attitude in Ref. [9–12], where they directly perform the matching between QCD to an effective field theory: vNRQCD that, once the RG evolution has been performed and the soft degrees of freedom have been integrated out, should be physically equivalent to pNRQCD with $\nu_p \sim m\alpha_s$. One motivation for going through NRQCD is that it allows to perform the factorization of the hard scale within an effective field theory framework. In fact, a full factorization of the different regions of momentum that ought to be integrated out is achieved within pNRQCD. This extremelly simplifies the matching process since one deals with only one scale (region of momentum) in the loops at each step. In the matching between QCD and NRQCD only hard loops need to be considered whereas in the matching between NRQCD and pNRQCD only soft loops need to be considered. Moreover, the structure of the UV cutoffs of the theory is better understood in this way. For instance, one can see that all the *explicit* dependence of the potentials on $\nu_p$ is inherited from the $d$ matching coefficients.

Within a diagrammatic approach the factorization of the different regions of momentum have been achieved using the threshold expansion [13].

Let us now consider the case of the electromagnetic current, which will provide an example where to apply the above discussion. The procedure is analogous to the potentials. We first do the matching from QCD to NRQCD:

$$\bar{Q}\gamma^\mu Q(0)\Big|_{\rm QCD} \doteq c_{1,\rm NR}\psi^\dagger\sigma^i\chi(0) + O(1/m)\Big|_{\rm NRQCD}. \tag{3}$$

We will just concentrate in the coefficient $c_{1,\rm NR}$. Within NRQCD, it should be understood as a function of $\nu_p$ and $\nu_s$, ie. $c_{1,\rm NR}(\nu_p,\nu_s)$. One should first obtain the matching conditions at the hard scale. This has been computed up to two loops [14] but we will only need the one-loop expression [15]:

$$c_{1,\rm NR}(m,m) = 1 - 2C_f\frac{\alpha_{\rm s}(m)}{\pi}, \tag{4}$$

since we only aim to a NLL resummation in this paper. If we compare with the previous discussion of the potentials, the matching coefficients $d$ play the role of $c_1$. Therefore, within pNRQCD, we will need $c_{1,\rm NR}(\nu_p,\nu_p) \equiv c_{1,\rm NR}(\nu_p)$. The matching from NRQCD to pNRQCD creates the potentials but let $c_1$ unchanged since soft loops or HQET-like calculations give zero correction to $c_1$. Formally,

$$c_{1,\rm NR}\psi^\dagger\sigma^i\chi(0)\Big|_{\rm NRQCD} = c_{1,\rm pNR}\psi^\dagger\sigma^i\chi(0)\Big|_{\rm pNRQCD}, \tag{5}$$

or in other words the matching condition reads $c_{1,\rm pNR}(c_{1,\rm NR}(\nu_p),\nu_{us}=\nu_p) = c_{1,\rm NR}(\nu_p)$. The running of $\nu_{us}$ is also trivial as there is none (this has to do with the fact that we are dealing with an annihilation process). Therefore, we finally have $c_1(\nu_p) \equiv c_{1,\rm pNR}(c_{1,\rm NR}(\nu_p),\nu_p^2/m) = c_{1,\rm NR}(\nu_p)$. We can see that we are in the analogous situation to the running of $D^{(2)}_{d,s}(\nu_p)$ versus the running of $d(\nu_p)$. We now need the RG equation for $c_1(\nu_p)$. This demands to obtain the ultraviolet corrections to the current within pNRQCD keeping track of the contributions due to the different potentials. Fortunately, this calculation has already been done and we can extract the relevant information from [16]. The RG equation reads

$$\nu_p\frac{d}{d\nu_p}c_s = c_s\left[-\frac{C_AC_f}{2}D^{(1)}_s - \frac{C_f^2}{4}\alpha_{V_s}\left\{\alpha_{V_s} - \frac{4}{3}s(s+1)D^{(2)}_{S^2,s} - D^{(2)}_{d,s} + 4D^{(2)}_{1,s}\right\}\right], \tag{6}$$

where $C_f=(N_c^2-1)/(2N_c)$, $C_A=N_c$, and the RG-improved potential matching coefficients can be read from Ref. [8,5] with the assignment $1/r\to\nu_p$ and $\nu_{us}\to\nu_p^2/m$ (see also [17,4] for calculations of the potential at finite orders in $\alpha_{\rm s}$). We have kept the spin explicit so our results will also be valid for the pseudoscalar current:

$$\bar{Q}\gamma^0\gamma_5 Q(0)\Big|_{\rm QCD} \doteq c_{0,{\rm NR}}\psi^\dagger\chi(0) + O(1/m)\Big|_{\rm NRQCD}, \tag{7}$$

with the matching condition [18]:

$$c_0(m) = 1 + \left(\frac{\pi^2}{4} - 5\right)\frac{C_f}{2}\frac{\alpha_{\rm s}(m)}{\pi}. \tag{8}$$

Eq. (6) gives subleading effects within an strict expansion in $\alpha_{\rm s}$. Therefore, it can be approximated to

$$\nu_p \frac{d}{d\nu_p} c_s = -\frac{C_A C_f}{2} D_s^{(1)} - \frac{C_f^2}{4}\alpha_{\rm s}\left\{\alpha_{\rm s} - \frac{4}{3}s(s+1)D^{(2)}_{S^2,s} - D^{(2)}_{d,s} + 4D^{(2)}_{1,s}\right\}, \tag{9}$$

and the solution reads

$$\begin{aligned} c_s(\nu_p) = & c_s(m) + A_1\frac{\alpha_{\rm s}(m)}{w^{\beta_0}}\ln(w^{\beta_0}) + A_2\alpha_{\rm s}(m)\left[z^{\beta_0} - 1\right] + A_3\alpha_{\rm s}(m)\left[z^{\beta_0-2C_A} - 1\right] \\ & + A_4\alpha_{\rm s}(m)\left[z^{\beta_0-13C_A/6} - 1\right] + A_5\alpha_{\rm s}(m)\ln(z^{\beta_0}), \end{aligned} \tag{10}$$

where $\beta_0 = \frac{11}{3}C_A - \frac{4}{3}T_F n_f$, $z = \left[\frac{\alpha_{\rm s}(\nu_p)}{\alpha_{\rm s}(m)}\right]^{\frac{1}{\beta_0}}$ and $w = \left[\frac{\alpha_{\rm s}(\nu_p^2/m)}{\alpha_{\rm s}(\nu_p)}\right]^{\frac{1}{\beta_0}}$. The coefficients $A_i$ in Eq. (10) read

$$\begin{aligned} A_1 = & \frac{8\pi C_f}{3\beta_0^2}\left(C_A^2 + 2C_f^2 + 3C_f C_A\right), \\ A_2 = & \frac{\pi C_f[3\beta_0(26C_A^2 + 19C_A C_f - 32C_f^2) - C_A(208C_A^2 + 651C_A C_f + 116C_f^2)]}{78\,\beta_0^2\, C_A}, \\ A_3 = & -\frac{\pi C_f^2\left[\beta_0(4s(s+1) - 3) + C_A(15 - 14s(s+1))\right]}{6(\beta_0 - 2C_A)^2}, \\ A_4 = & \frac{24\pi C_f^2(3\beta_0 - 11C_A)(5C_A + 8C_f)}{13\, C_A(6\beta_0 - 13C_A)^2}, \\ A_5 = & \frac{-\pi C_f^2}{\beta_0^2\,(6\beta_0 - 13C_A)(\beta_0 - 2C_A)}\left\{C_A^2(-9C_A + 100C_f)\right. \\ & \left. +\beta_0 C_A(-74C_f + C_A(42 - 13s(s+1))) + 6\beta_0^2(2C_f + C_A(-3 + s(s+1)))\right\}. \end{aligned} \tag{11}$$

Our evaluation can be compared with the result obtained using the vNRQCD formalism [12]. We agree for the spin-dependent terms but differ for the spin-independent ones. The disagreement still holds if we consider QED with light fermions ($C_f \to 1$, $C_A \to 0$, $T_F \to 1$). Agreement is found if we consider QED without light fermions ($C_f \to 1$, $C_A \to 0$, $n_f \to 0$, $T_F \to 1$). If we expand our results in $\alpha_s$ we can compare with earlier results in the literature. By following the discussion in Ref. [12], we can relate our results with the correction to the wave-function at the origin as defined in Ref. [16]. We obtain

$$\Delta\psi^2(0) = \left|\frac{c_s(\nu_p)}{c_s(1)}\right|^2 - 1 = -C_f\alpha_s^2\ln(\alpha_s)\Big\{\big[2-\frac{2}{3}s(s+1)\big]C_f + C_A\Big\} \tag{12}$$
$$-\frac{C_f}{\pi}\alpha_s^3\ln^2(\alpha_s)\Big\{\frac{3}{2}C_f^2 + \big[\frac{41}{12}-\frac{7}{12}s(s+1)\big]C_fC_A + \frac{2}{3}C_A^2$$
$$+\frac{\beta_0}{2}\Big[\Big(2-\frac{2}{3}s(s+1)\Big)C_f + C_A\Big]\Big\} + \dots ,$$

where we have expanded up to second order in $\ln(\nu_p) = \ln(m\alpha_s)$ with $\alpha_s \equiv \alpha_s(\nu_p)$. The first term reproduces the leading log term [14,19] (see also [20]), the $\beta_0$-independent $O(\alpha_s^3\ln^2\alpha_s)$ terms reproduce Kniehl and Penin results [16] and we agree with the complete $O(\alpha_s^3\ln^2\alpha_s)$ term computed by Manohar and Stewart [12] (the sign of difference for the $\beta_0$-dependent terms displayed in Ref. [12] is due to the fact that in Ref. [12] the expansion was made with $\alpha_s(m)$ whereas here we have chosen $\alpha_s(m\alpha_s)$). Nevertheless, disagreement with this last evaluation appears at higher orders in the expansion in $\alpha_s$ (we have explicitly checked this for the $O(\alpha_s^4\ln^3\alpha_s)$ terms). As far as we can see the disagreement seems to be due to the fact that they have different expressions for the RG improved potentials [11,12].

By setting $\nu_p \sim m\alpha_s$, $c_s(\nu_p)$ includes all the large logs at NLL order in any S-wave heavy-quarkonium production observable we can think of. For instance, the decays to $e^+e^-$ and to two photons at NLL order read

$$\Gamma(V_Q(nS)\to e^+e^-) = 2\left[\frac{\alpha_{em}Q}{M_{V_Q(nS)}}\right]^2\left(\frac{m_QC_f\alpha_s}{n}\right)^3\{c_1(\nu_p)(1+\delta\phi_n)\}^2 \tag{13}$$
$$\simeq 2\left[\frac{\alpha_{em}Q}{M_{V_Q(nS)}}\right]^2\left(\frac{m_QC_f\alpha_s}{n}\right)^3\{1+2(c_1(\nu_p)-1)+2\delta\phi_n\} ,$$
$$\Gamma(P_Q(nS)\to\gamma\gamma) = 6\left[\frac{\alpha_{em}Q^2}{M_{P_Q(nS)}}\right]^2\left(\frac{m_QC_f\alpha_s}{n}\right)^3\{c_0(\nu_p)(1+\delta\phi_n)\}^2 \tag{14}$$
$$\simeq 6\left[\frac{\alpha_{em}Q^2}{M_{P_Q(nS)}}\right]^2\left(\frac{m_QC_f\alpha_s}{n}\right)^3\{1+2(c_0(\nu_p)-1)+2\delta\phi_n\} ,$$

where $V$ and $P$ stand for the vector and pseudoscalar heavy quarkonium, we have fixed $\nu_p = m_QC_f\alpha_s/n$, $\alpha_s = \alpha_s(\nu_p)$, and ($\Psi_n(z) = \frac{d^n\ln\Gamma(z)}{dz^n}$ and $\Gamma(z)$ is the Euler $\Gamma$-function)

$$\delta\phi_n = \frac{\alpha_s}{\pi}\left[-C_A + \frac{\beta_0}{4}\left(\Psi_1(n+1) - 2n\Psi_2(n) + \frac{3}{2} + \gamma_E + \frac{2}{n}\right)\right] , \tag{15}$$

which has been read from Ref. [19] (see also [21]). Working along similar lines one could easily obtain NLL expressions for other heavy quarkonium observables in the study of $t$-$\bar{t}$ production near threshold or in sum rules of bottomonium.

In conclusion, we have shown that pNRQCD, if necessary, can comprehensively incorporate a RG framework by using the method described in Ref. [5] plus incorporating the idea [9] of correlating the cutoffs of the effective theory. We have used this formalism to compute the running of the matching coefficients of the vector and pseudoscalar currents and disagreement with the results obtained using the vNRQCD framework have been found [12]. Our results allow to obtain S-wave heavy quarkonium production observables with NLL accuracy. We have explicitely illustrated this point for heavy-quarkonium decays to $e^+e^-$ and to two photons.

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