Large order behavior in perturbation theory of the pole mass and the singlet static potential

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Abstract. We discuss upon recent progress in our knowledge of the large order behavior in perturbation theory of the pole mass and the singlet static potential. We also discuss about the renormalon subtracted scheme, a matching scheme between QCD and any effective field theory with heavy quarks where, besides the usual perturbative matching, the first renormalon in the Borel plane of the pole mass is subtracted.

MASS NORMALIZATION CONSTANT

In this paper, we review some results obtained in Ref. [1].

The on-shell (OS) or pole mass can be related to the $\overline{\text{MS}}$ renormalized mass by the series

$$m_{\text{OS}} = m_{\overline{\text{MS}}} + \sum_{n=0}^{\infty} r_n \alpha_s^{n+1}\,, \tag{1}$$

where the normalization point $\nu = m_{\overline{\text{MS}}}$ is understood for $m_{\overline{\text{MS}}}$ and the first three coefficients r_0, r_1 and r_2 are known [2] ($\alpha_s = \alpha_s^{(n_l)}(\nu)$, where n_l is the number of light fermions). The pole mass is also known to be IR finite and scheme-independent at any finite order in α_s [3]. We then define the Borel transform

$$m_{\text{OS}} = m_{\overline{\text{MS}}} + \int_0^\infty \mathrm{d}t\, e^{-t/\alpha_s} B[m_{\text{OS}}](t)\,, \qquad B[m_{\text{OS}}](t) \equiv \sum_{n=0}^{\infty} r_n \frac{t^n}{n!}. \tag{2}$$

The behavior of the perturbative expansion of Eq. (1) at large orders is dictated by the closest singularity to the origin of its Borel transform, which happens to be located at $t = 2\pi/\beta_0$, where we define

$$\nu \frac{d\alpha_s}{d\nu} = -2\alpha_s \left\{ \beta_0 \frac{\alpha_s}{4\pi} + \beta_1 \left(\frac{\alpha_s}{4\pi}\right)^2 + \cdots \right\}.$$

Being more precise, the behavior of the Borel transform near the closest singularity at the origin reads (we define $u = \frac{\beta_0 t}{4\pi}$)

$$B[m_{\text{OS}}](t(u)) = N_m \nu \frac{1}{(1-2u)^{1+b}} \left(1 + c_1(1-2u) + c_2(1-2u)^2 + \cdots\right) + (\text{analytic term}), \tag{3}$$

where by *analytic term*, we mean a piece that we expect it to be analytic up to the next renormalon ($u = 1$). This dictates the behavior of the perturbative expansion at large orders to be

$$r_n \stackrel{n\to\infty}{=} N_m \nu \left(\frac{\beta_0}{2\pi}\right)^n \frac{\Gamma(n+1+b)}{\Gamma(1+b)} \left(1+\frac{b}{(n+b)}c_1+\frac{b(b-1)}{(n+b)(n+b-1)}c_2+\cdots\right). \quad (4)$$

The different b, c_1, c_2, etc ... can be obtained from the procedure used in [4] (see [4, 1] for the explicit expressions). We then use the idea of [5] and define the new function

$$\begin{aligned} D_m(u) &= \sum_{n=0}^{\infty} D_m^{(n)} u^n = (1-2u)^{1+b} B[m_{\rm OS}](t(u)) \quad (5)\\ &= N_m \nu \left(1+c_1(1-2u)+c_2(1-2u)^2+\cdots\right)+(1-2u)^{1+b}(\text{analytic term}). \end{aligned}$$

This function is singular but bounded at the first IR renormalon. Therefore, we can expect to obtain an approximate determination of N_m if we know the first coefficients of the series in u and by using

$$N_m \nu = D_m(u = 1/2). \quad (6)$$

The first three coefficients: $D_m^{(0)}$, $D_m^{(1)}$ and $D_m^{(2)}$ are known in our case. In order the calculation to make sense, we choose $\nu \sim m$. For the specific choice $\nu = m$, we obtain (up to $O(u^3)$ with $u = 1/2$)

$$\begin{aligned} N_m &= 0.424413+0.137858+0.0127029 = 0.574974 \quad (n_f = 3) \quad (7)\\ &= 0.424413+0.127505+0.000360952 = 0.552279 \quad (n_f = 4)\\ &= 0.424413+0.119930-0.0207998 = 0.523543 \quad (n_f = 5). \end{aligned}$$

The convergence is surprisingly good. The scale dependence is also quite mild (see [1]).

By using Eq. (4), we can now go backwards and give some estimates for the r_n. They are displayed in Table 1. We can see that they go closer to the exact values of r_n when increasing n. This makes us feel confident that we are near the asymptotic regime dominated by the first IR renormalon and that for higher n our predictions will become an accurate estimate of the exact values. In fact, they are quite compatible with the results obtained by other methods like the large β_0 approximation (see Table 1).

We can now try to see how the large β_0 approximation works in the determination of N_m. In order to do so, we study the one chain approximation from which we obtain the value [6]

$$N_m^{(\text{large } \beta_0)} = \frac{C_f}{\pi} e^{\frac{5}{6}} = 0.976564. \quad (8)$$

By comparing with Eq. (7), we can see that it does not provide an accurate determination of N_m. This may seem to be in contradiction with the accurate values that the large β_0 approximation provides for the r_n (starting at $n = 2$) in Table 1. Lacking of any physical explanation for this fact, it may just be considered to be a numerical accident. In fact, the agreement between our determination and the large β_0 results does not hold at very high orders in the perturbative expansion, whereas we believe, on physical grounds since our approach incorporates the exact nature of the renormalon, that our determination should go closer to the exact result at high orders in perturbation theory. Nevertheless, the large β_0 approximation remains accurate up to relative high orders.

TABLE 1. Values of r_n for $\nu = m_{\overline{\rm MS}}$. Either the exact result, the estimate using Eq. (4), or the estimate using the large β_0 approximation [7].

$\tilde{r}_n = r_n/m_{\overline{\rm MS}}$	$\tilde{r}_0$	$\tilde{r}_1$	$\tilde{r}_2$	$\tilde{r}_3$	$\tilde{r}_4$
exact ($n_f = 3$)	0.424413	1.04556	3.75086	—	—
Eq. (4) ($n_f = 3$)	0.617148	0.977493	3.76832	18.6697	118.441
large β_0 ($n_f = 3$)	0.424413	1.42442	3.83641	17.1286	97.5872
exact ($n_f = 4$)	0.424413	0.940051	3.03854	—	—
Eq. (4) ($n_f = 4$)	0.645181	0.848362	3.03913	13.8151	80.5776
large β_0 ($n_f = 4$)	0.424413	1.31891	3.28911	13.5972	71.7295
exact ($n_f = 5$)	0.424413	0.834538	2.36832	—	—
Eq. (4) ($n_f = 5$)	0.706913	0.713994	2.36440	9.73117	51.5952
large β_0 ($n_f = 5$)	0.424413	1.21339	2.78390	10.5880	51.3865

STATIC SINGLET POTENTIAL NORMALIZATION CONSTANT

One can think of playing the same game with the singlet static potential in the situation where $\Lambda_{\rm QCD} \ll 1/r$. Its perturbative expansion reads

$$V_s^{(0)}(r;\nu_{us}) = \sum_{n=0}^{\infty} V_{s,n}^{(0)} \alpha_s^{n+1}. \tag{9}$$

The first three coefficients $V_{s,0}^{(0)}$, $V_{s,1}^{(0)}$ and $V_{s,2}^{(0)}$ are known [8]. At higher orders in perturbation theory the log dependence on the IR cutoff ν_{us} appears [9]. Nevertheless, these logs are not associated to the first IR renormalon (see [1]), so we will not consider them further in this section. We now use the observation that the first IR renormalon of the singlet static potential cancels with the renormalon of (twice) the pole mass. We can then read the asymptotic behavior of the static potential from the one of the pole mass and work analogously to the previous section. We define the Borel transform

$$V_s^{(0)} = \int_0^\infty {\rm d}t\, e^{-t/\alpha_s} B[V_s^{(0)}](t)\,, \qquad B[V_s^{(0)}](t) \equiv \sum_{n=0}^{\infty} V_{s,n}^{(0)} \frac{t^n}{n!}. \tag{10}$$

The closest singularity to the origen is located at $t = 2\pi/\beta_0$. This dictates the behavior of the perturbative expansion at large orders to be

$$V_{s,n}^{(0)} \stackrel{n\to\infty}{=} N_V\, \nu \left(\frac{\beta_0}{2\pi}\right)^n \frac{\Gamma(n+1+b)}{\Gamma(1+b)} \left(1 + \frac{b}{(n+b)} c_1 + \frac{b(b-1)}{(n+b)(n+b-1)} c_2 + \cdots\right), \tag{11}$$

and the Borel transform near the singularity reads

$$B[V_s^{(0)}](t(u)) = N_V \nu \frac{1}{(1-2u)^{1+b}} \left(1 + c_1(1-2u) + c_2(1-2u)^2 + \cdots\right) + (\text{analytic term}). \tag{12}$$

In this case, by *analytic term*, we mean an analytic function up to the next IR renormalon at $u = 3/2$.

As in the previous section, we define the new function

$$D_V(u) = \sum_{n=0}^{\infty} D_V^{(n)} u^n = (1-2u)^{1+b} B[V_s^{(0)}](t(u)) \tag{13}$$
$$= N_V \nu \left(1 + c_1(1-2u) + c_2(1-2u)^2 + \cdots\right) + (1-2u)^{1+b}(\text{analytic term})$$

and try to obtain an approximate determination of N_V by using the first three (known) coefficients of this series. By a discussion analogous to the one in the previous section, we fix $\nu = 1/r$. We obtain (up to $O(u^3)$ with $u = 1/2$)

$$\begin{aligned} N_V &= -1.33333 + 0.571943 - 0.345222 = -1.10661 \quad (n_f = 3) \\ &= -1.33333 + 0.585401 - 0.329356 = -1.07729 \quad (n_f = 4) \\ &= -1.33333 + 0.586817 - 0.295238 = -1.04175 \quad (n_f = 5)\,. \end{aligned} \tag{14}$$

The convergence is not as good as in the previous section. Nevertheless, it is quite acceptable and, in this case, apparently, we have a sign alternating series. In fact, the scale dependence is quite mild (see [1]). Overall, up to small differences, the same picture than for N_m applies.

So far we have not made use of the fact that $2N_m + N_V = 0$. We use this equality as a check of the reliability of our calculation. We can see that the cancellation is quite dramatic. We obtain

$$2\frac{2N_m + N_V}{2N_m - N_V} = \begin{cases} 0.038 & , n_f = 3 \\ 0.025 & , n_f = 4 \\ 0.005 & , n_f = 5. \end{cases}$$

We can now obtain estimates for $V_{s,n}^{(0)}$ by using Eq. (11). They are displayed in Table 2. Note that in Table 2 no input from the static potential has been used since even N_V have been fixed by using the equality $2N_m = -N_V$. We can see that the exact results are reproduced fairly well (the same discussion than for the r_n determination applies). This makes us feel confident that we are near the asymptotic regime dominated by the first IR renormalon and that for higher n our predictions will become an accurate estimate of the exact results. The comparison with the values obtained with the large β_0 approximation would go (roughly) along the same lines than for the mass case, although the large β_0 results seem to be less accurate in this case (see Table 2).

In order to avoid large corrections from terms depending on ν_{us}, the predictions should be understood with $\nu_{us} = 1/r$.

RENORMALON SUBTRACTED SCHEME

In effective theories with heavy quarks, the inverse of the heavy quark mass becomes one of the expansion parameters (and matching coefficients). A natural choice in the past (within the infinitely many possible definitions of the mass) has been the pole mass because it is the natural definition in OS processes where the particles finally measured in

TABLE 2. Values of $V_{s,n}^{(0)}$ with $\nu = 1/r$. Either the exact result (when available), the estimate using Eq. (11), or the estimate using the large β_0 approximation [10].

$\tilde{V}_{s,n}^{(0)} = rV_{s,n}^{(0)}$	$\tilde{V}_{s,0}^{(0)}$	$\tilde{V}_{s,1}^{(0)}$	$\tilde{V}_{s,2}^{(0)}$	$\tilde{V}_{s,3}^{(0)}$	$\tilde{V}_{s,4}^{(0)}$
exact ($n_f = 3$)	-1.33333	-1.84512	-7.28304	—	—
Eq. (11) ($n_f = 3$)	-1.23430	-1.95499	-7.53665	-37.3395	-236.882
large β_0 ($n_f = 3$)	-1.33333	-2.69395	-7.69303	-34.0562	—
exact ($n_f = 4$)	-1.33333	-1.64557	-5.94978	—	—
Eq. (11) ($n_f = 4$)	-1.29036	-1.69672	-6.07826	-27.6301	-161.155
large β_0 ($n_f = 4$)	-1.33333	-2.49440	-6.59553	-27.0349	—
exact ($n_f = 5$)	-1.33333	-1.44602	-4.70095	—	—
Eq. (11) ($n_f = 5$)	-1.41383	-1.42799	-4.72881	-19.4623	-103.190
large β_0 ($n_f = 5$)	-1.33333	-2.29485	-5.58246	-21.0518	—

the detectors correspond to the fields in the Lagrangian (as in QED). Unfortunately, this is not the case in QCD and one reflection of this fact is that the pole mass suffers from renormalon singularities. Moreover, these renormalon singularities lie close together to the origin and perturbative calculations have gone very far for systems with heavy quarks. At the practical level, this has reflected in the worsening of the perturbative expansion in processes where the pole mass was used as an expansion parameter. It is then natural to try to define a new expansion parameter replacing the pole mass but still being an adequate definition for threshold problems. This idea is not new and has already been pursued in the literature, where several definitions have arisen [11]. We can not resist the tentation of trying our own definition. We believe that, having a different systematics than the other definitions, it could further help to estimate the errors in the more recent determinations of the $\overline{\mathrm{MS}}$ quark mass. Our definition, as the definitions above, try to cancel the bad perturbative behavior associated to the renormalon. On the other hand, we would like to understand this problem within an effective field theory perspective. From this point of view what one is seeing is that the coefficients multiplying the (small) expansion parameters in the effective theory calculation are not of natural size (of $O(1)$). The natural answer to this problem is that we are not properly separating scales in our effective theory and some effects from small scales are incorporated in the matching coefficients. These small scales are dynamically generated in n-loop calculations (n being large) and are of $O(me^{-n})$ (we are having in mind a large β_0 evaluation) producing the bad (renormalon associated) perturbative behavior. In order to overcome this problem, we may think of doing the Borel transform. In that case, the renormalon singularities correspond to the non-analytic terms in $1-2u$. These terms also exist in the effective theory. Therefore, our procedure will be to subtract the pure renormalon contribution in the new mass definition, which we will call renormalon subtracted (RS) mass, m_{RS}. We define the Borel transform of m_{RS} as follows

$$B[m_{\mathrm{RS}}] \equiv B[m_{\mathrm{OS}}] - N_m \nu_f \frac{1}{(1-2u)^{1+b}} \left(1 + c_1(1-2u) + c_2(1-2u)^2 + \cdots\right), \qquad (15)$$

where ν_f could be understood as a factorization scale between QCD and NRQCD (or HQET) and, at this stage, should be smaller than m. The expression for $m_{\rm RS}$ reads

$$m_{\rm RS}(\nu_f) = m_{\rm OS} - \sum_{n=0}^{\infty} N_m \nu_f \left(\frac{\beta_0}{2\pi}\right)^n \alpha_s^{n+1}(\nu_f) \sum_{k=0}^{\infty} c_k \frac{\Gamma(n+1+b-k)}{\Gamma(1+b-k)}\,, \qquad (16)$$

where $c_0 = 1$. We expect that with this renormalon free definition the coefficients multiplying the expansion parameters in the effective theory calculation will have a natural size and also the coefficients multiplying the powers of α_s in the perturbative expansion relating $m_{\rm RS}$ with $m_{\overline{\rm MS}}$. Therefore, we do not loose accuracy if we first obtain $m_{\rm RS}$ and later on we use the perturbative relation between $m_{\rm RS}$ and $m_{\overline{\rm MS}}$ in order to obtain the latter. Nevertheless, since we will work order by order in α_s in the relation between $m_{\rm RS}$ and $m_{\overline{\rm MS}}$, it is important to expand everything in terms of α_s, in particular $\alpha_s(\nu_f)$, in order to achieve the renormalon cancellation order by order in α_s. Then, the perturbative expansion in terms of the $\overline{\rm MS}$ mass reads

$$m_{\rm RS}(\nu_f) = m_{\overline{\rm MS}} + \sum_{n=0}^{\infty} r_n^{\rm RS} \alpha_s^{n+1}\,, \qquad (17)$$

where $r_n^{\rm RS} = r_n^{\rm RS}(m_{\overline{\rm MS}}, \nu, \nu_f)$. These $r_n^{\rm RS}$ are the ones expected to be of natural size (or at least not to be artificially enlarged by the first IR renormalon).

In Ref. [1], we have applied this scheme to potential NRQCD and HQET. For the former, by using the $\Upsilon(1S)$ mass, we have obtained a determination of the $\overline{\rm MS}$ bottom quark mass. For the latter, we have obtained a value of the charm mass by using the difference between the D and B meson mass. In both cases the convergence is significantly improved if compared with the analogous OS evaluations.

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