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# Next-to-Next-to-Leading Logarithms in Four-Fermion Electroweak Processes at High Energy

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**Abstract**

We sum up the next-to-next-to-leading logarithmic virtual electroweak corrections to the high energy asymptotics of the neutral current four-fermion processes for light fermions to all orders in the coupling constants using the evolution equation approach. From this all order result we derive finite order expressions through next-to-next-to leading order for the total cross section and various asymmetries. We observe an amazing cancellation between the sizable leading, next-to-leading and next-to-next-to-leading logarithmic contributions at TeV energies.

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# 1 Introduction

Experimental and theoretical studies of electroweak interactions have traditionally explored the range from very low energies, e.g. through parity violation in atoms, up to energies comparable to the masses of the $W$- and $Z$-bosons, e.g. at LEP or the Tevatron. The advent of multi-TeV-colliders like the LHC or a future linear electron-positron collider during the present decade will give access to a completely new energy domain.

Once the characteristic energies $s$ are far larger than the masses of the $W$- and $Z$-bosons, $M_{W,Z}$, multiple soft and collinear gauge boson emission is kinematically possible. Conversely, exclusive reactions like electron-positron (or quark-antiquark) annihilation into a pair of fermions or gauge bosons will receive large negative corrections from virtual gauge boson emission. These double logarithmic "Sudakov" corrections [1, 2] proportional to powers of $g^2 \ln^2(s/M_{W,Z}^2)$ are dominant at high energies and thus have to be controlled in higher orders to arrive at reliable predictions.

The importance of large logarithmic corrections for electroweak reactions at high energies in one-loop approximation which may well amount to ten or even twenty percent was noticed already several years ago [3, 4]. The need for a resummation of higher orders of these double-logarithmic terms in the context of electroweak interactions was first emphasized in [5] which also contains a first discussion of this resummation. In particular, it was shown that the double logarithms do not depend on the details of the mechanism of the gauge boson mass generation. The issue is complicated by the appearance of massive ($W$, $Z$) and massless ($\gamma$) gauge bosons in the $SU_L(2) \times U(1)$ theory, which necessarily have to be treated on a different footing. A complete analysis of this problem in the double or leading logarithmic (LL) approximation by the systematic separation of soft ($\omega_\gamma \leq M$) and hard ($\omega_\gamma \geq M$) photons was given in [6]. In two loops the results of this approach essentially based on the concept of infrared evolution equations have been confirmed by explicit calculations in [7].

The large coefficient in front of the single logarithmic term in the one-loop corrections to the electroweak amplitudes (see, e.g. [8]) suggests that subleading terms play an important role also in higher orders, as long as realistic energies of order TeV are under consideration. Motivated by this observation a systematic evaluation of the next-to-leading logarithmic (NLL) terms for the neutral current massless four fermion process was performed in ref. [9]. Indeed one finds sizeable two-loop effects both for the total cross section, for the left-right and for the forward-backward asymmetry. A subclass of NLL corrections for general electroweak processes was subsequently evaluated in [10] without, however, the very important *angular dependent* contributions.

Various authors have also extracted the double and single logarithmic corrections from the complete one-loop calculations [11, 12]. The analysis for the general electroweak processes given in [12] is in full agreement with [9], whereas a different prescription to separate the QED contribution is adopted in [11]. Higher order heavy fermion mass effects on the asymptotic high energy behavior of the electroweak amplitudes were discussed in [13]. The incomplete cancellation of the real and virtual electroweak double logarithmic corrections in the inclusive cross sections was investigated in [14].

Following the approach of [9] which in turn is based on the investigations in the context of QCD [15, 16, 17, 18, 19, 20, 21, 22, 23], this paper is devoted to the derivation of the next-to-next-to-leading logarithmic (NNLL) terms for the massless neutral current four-fermion cross sections. In Section 2 we present as a first step the NNLL form factor which describes the scattering amplitude in an external Abelian field for the $SU(N)$ gauge theory. The derivation is based on the evolution equation derived in ref. [18, 19, 20]. In Section 3 we then generalize the result to the four fermion process in the $SU(N)$ gauge theory. After factoring off the collinear logarithms we use an evolution equation for the remaining amplitudes which is governed by an angular dependent soft anomalous dimension matrix [20, 21, 23].

Finally we apply this result to electroweak processes in Section 4. To identify the pure QED infrared logarithms which are compensated by soft real photon radiation we combine the hard evolution equation which governs the dependence of the amplitudes on $s$ with the infrared evolution equation [6]. The latter describes the dependence of the amplitude on a fictitious photon mass which serves as an infrared regulator and drops out after including the effect of the soft photon emission. The hard and infrared evolutions are matched by fixing the initial conditions at the scale $M_{W,Z}$. Section 4 also contains a discussion of the numerical implications of our result. A brief summary and conclusions are given in Section 5.

# 2 The Abelian form factor in the Sudakov limit

Let us first consider the vector form factor which determines the fermion scattering amplitude in an external Abelian field for the $SU(N)$ gauge model. In the Born approximation,

$$\mathcal{F}_B = \bar{\psi}(p_2)\gamma_\mu\psi(p_1)\,, \tag{1}$$

where $p_1$ denotes the incoming and $p_2$ the outgoing momentum.

There are two "standard" regimes of the Sudakov limit $s = (p_1 - p_2)^2 \to -\infty$: (i) on-shell massless fermions, $p_1^2 = p_2^2 = 0$ and gauge bosons with a small non-zero mass $M^2 \ll Q^2$ [2], or (ii) slightly off-shell fermions $p_1^2 = p_2^2 = -M^2$, and massless gauge boson [1]. Let us consider the first case and choose, for convenience, $p_{1,2} = (Q/2, 0, 0, \mp Q/2)$ so that $2p_1p_2 = Q^2 = -s$. The asymptotic $Q$-dependence of the form factor in this limit is governed by the evolution equation [18, 19]

$$\frac{\partial}{\partial \ln Q^2}\mathcal{F} = \left[\int_{M^2}^{Q^2}\frac{\mathrm{d}x}{x}\gamma(\alpha(x)) + \zeta(\alpha(Q^2)) + \xi(\alpha(M^2))\right]\mathcal{F}\,. \tag{2}$$

Its solution is

$$\mathcal{F} = F_0(\alpha(M^2))\exp\left\{\int_{M^2}^{Q^2}\frac{\mathrm{d}x}{x}\left[\int_{M^2}^{x}\frac{\mathrm{d}x'}{x'}\gamma(\alpha(x')) + \zeta(\alpha(x)) + \xi(\alpha(M^2))\right]\right\}\mathcal{F}_B\,. \tag{3}$$

The LL approximation includes all the terms of the form $\alpha^n \log^{2n}(Q^2/M^2)$ and is determined by the one-loop value of $\gamma(\alpha)$. The NLL approximation includes all the terms of the form

$\alpha^n \log^{2n-m}(Q^2/M^2)$ with $m = 0,\ 1$. This requires the one-loop values of $\gamma(\alpha)$ and $\zeta(\alpha)+\xi(\alpha)$ and using the one-loop running of $\alpha$ in $\gamma(\alpha)$. The NNLL approximation includes all the terms of the form $\alpha^n \log^{2n-m}(Q^2/M^2)$ with $m = 0,\ 1,\ 2$. In this case $\gamma(\alpha)$ is required up to $\mathcal{O}(\alpha^2)$, $\zeta(\alpha)$, $\xi(\alpha)$ and $F_0(\alpha)$ up to $\mathcal{O}(\alpha)$ together with the two-loop running of $\alpha$ in $\gamma(\alpha)$ and one-loop running of $\alpha$ in $\zeta(\alpha)$.

The functions entering the evolution equation can be determined by comparing eq. (3) expanded in the coupling constant to the asymptotic, *i.e.* leading in $M^2/Q^2$, fixed order result for the form factor. To compute this fixed order asymptotic result we apply the expansion by regions approach formulated in [24] and discussed using characteristic two-loop examples in [25]. It consists of the following steps: (i) consider various regions of a loop momentum $k$ and expand, in every region, the integrand in Taylor series with respect to the parameters that are there considered small, (ii) integrate the expanded integrand over the whole integration domain of the loop momenta, (iii) put to zero any scaleless integral. In step (ii) dimensional regularization [26] with $d = 4 - 2\epsilon$ space-time dimensions is used to handle the divergences. The following regions are relevant in the considered version (i) of the Sudakov limit [27]:

$$
\begin{aligned}
\textit{hard}\ (\mathrm{h}): &\quad k \sim Q\,, \\
\textit{1-collinear}\ (\mathrm{1c}): &\quad k_+ \sim Q,\ k_- \sim M^2/Q\,,\ \underline{k} \sim M\,, \\
\textit{2-collinear}\ (\mathrm{2c}): &\quad k_- \sim Q,\ k_+ \sim M^2/Q\,,\ \underline{k} \sim M\,, \\
\textit{soft}\ (\mathrm{s}): &\quad k \sim M\,.
\end{aligned}
\tag{4}
$$

Here $k_\pm = k_0 \pm k_3$, $\underline{k} = (k_1, k_2)$. By $k \sim Q$, etc. we mean that any component of $k_\mu$ is of order $Q$. In one loop this leads to the following decomposition [9]

$$
\mathcal{F}^{(1)} = (\Delta_h + \Delta_c + \Delta_s)\,\mathcal{F}_B\,, \tag{5}
$$

$$
\begin{aligned}
\Delta_h^{(1)} &= C_F\left(-\frac{2}{\epsilon^2} + \frac{1}{\epsilon}\left(2\ln(Q^2) - 3\right) - \ln^2(Q^2) + 3\ln(Q^2) + \frac{\pi^2}{6} - 8\right), \\
\Delta_c^{(1)} &= C_F\left(\frac{2}{\epsilon^2} - \frac{1}{\epsilon}\left(2\ln(Q^2) - 4\right) + 2\ln(Q^2)\ln(M^2) - \ln^2(M^2) - 4\ln(M^2) - \frac{5\pi^2}{6} + 4\right), \\
\Delta_s^{(1)} &= C_F\left(-\frac{1}{\epsilon} + \ln(M^2) + \frac{1}{2}\right).
\end{aligned}
\tag{6}
$$

where $C_F = (N^2-1)/(2N)$ is the quadratic Casimir operator of the fundamental representation of the $SU(N)$ group and the subscript $c$ denotes the contribution of both collinear regions. The 't Hooft scale $\mu$ has been dropped in the argument of the logarithms as well as the factor $(4\pi e^{-\gamma_E \epsilon}(\mu^2))^\epsilon$ per loop. For a perturbative function $f(\alpha)$ we define

$$
f(\alpha) = \sum_n \left(\frac{\alpha}{4\pi}\right)^n f^{(n)}\,. \tag{7}
$$

The contribution of all the regions add up to the well known result

$$\mathcal{F}^{(1)} = -C_F \left( \ln^2 \left( \frac{Q^2}{M^2} \right) - 3 \ln \left( \frac{Q^2}{M^2} \right) + \frac{7}{2} + \frac{2\pi^2}{3} \right) \mathcal{F}_B \,. \tag{8}$$

On the other hand the one-loop form factor can be written as

$$\mathcal{F}^{(1)} = \left( \frac{1}{2} \gamma^{(1)} \ln^2 \left( \frac{Q^2}{M^2} \right) + \left( \xi^{(1)} + \zeta^{(1)} \right) \ln \left( \frac{Q^2}{M^2} \right) + F_0^{(1)} \right) \mathcal{F}_B \,. \tag{9}$$

The expansion by regions is very efficient for determination of the parameters of the evolution equation. Indeed, it not necessary to compute the complete asymptotic result in order to obtain the functions parameterizing the logarithmic contributions. In the process of the scale separation through the expansion by regions the logarithmic contributions show up as the singularities of the contributions from different regions which cancel in the total result. Thus one can identify the regions relevant for determining a given parameter of the evolution equation and compute them separately up to the required accuracy. For example, the anomalous dimensions $\gamma(\alpha)$ and $\zeta(\alpha)$ are known to be independent on the infrared cutoff and are completely determined by the contribution from the hard loop momentum [18, 19]. If dimensional regularization is used for the infrared divergences of the hard loop momentum contribution, as in our approach, the anomalous dimensions $\gamma(\alpha)$ and $\zeta(\alpha)$ are given by the coefficients of the double and single poles of the hard contribution to the exponent (3), respectively [19, 22]. On the contrary, the functions $\xi(\alpha)$ and $F_0(\alpha)$ fix the initial conditions for the evolution equation. They are not universal and depend on the infrared sector of the model. Furthermore, the values of $\xi(\alpha)$ and $F_0(\alpha)$ depend on the definition of the lower integration limits in eq. (3). To determine the function $\xi(\alpha)$ one has to know also the singularities of the collinear region contribution while $F_0(\alpha)$ requires the complete information on the contributions of all the regions.

From the first line of eq. (6) we find the one-loop anomalous dimensions

$$\begin{aligned} \gamma^{(1)} &= -2C_F \,, \\ \zeta^{(1)} &= 3C_F \,. \end{aligned} \tag{10}$$

With the above values of $\gamma^{(1)}$ and $\zeta^{(1)}$ it is straightforward to obtain the one-loop result for the remaining functions

$$\begin{aligned} \xi^{(1)} &= 0 \,, \\ F_0^{(1)} &= -C_F \left( \frac{7}{2} + \frac{2\pi^2}{3} \right) \end{aligned} \tag{11}$$

by comparing eqs. (9) and (8). Note that in the Born approximation $F_0^{(0)} = 1$.

A similar decomposition can be performed in two loops

$$\mathcal{F}^{(2)} = \left( \Delta_{hh} + \Delta_{hc} + \Delta_{cc} + \ldots \right) \mathcal{F}_B \,, \tag{12}$$

Only the hard-hard part is now available [28]. However, this information is sufficient to determine the two-loop value $\gamma^{(2)}$. Beside the running of the coupling constant, $\gamma^{(2)}$ is the only two-loop quantity we need for the NNLL approximation. It reads [29]

$$\gamma^{(2)} = -2C_F\left[\left(\frac{67}{9}-\frac{\pi^2}{3}\right)C_A-\frac{20}{9}T_F n_f\right], \tag{13}$$

for $\alpha$ defined in the $\overline{MS}$ scheme. Here $C_A = N$ is the quadratic Casimir operator of the adjoint representation, $T_F = 1/2$ is the index of the fundamental representation and $n_f$ is the number of light (Dirac) fermions.

Let us consider the two-loop corrections. The LL, NLL and NNLL approximations are given respectively by

$$\mathcal{F}^{(2)}_{LL} = \frac{1}{8}(\gamma^{(1)})^2\ln^4\left(\frac{Q^2}{M^2}\right)\mathcal{F}_B\,, \tag{14}$$

$$\mathcal{F}^{(2)}_{NLL} = \frac{1}{2}\left(\zeta^{(1)}-\frac{1}{3}\beta_0\right)\gamma^{(1)}\ln^3\left(\frac{Q^2}{M^2}\right)\mathcal{F}_B\,, \tag{15}$$

$$\mathcal{F}^{(2)}_{NNLL} = \frac{1}{2}\left(\gamma^{(2)}+\left(\zeta^{(1)}-\beta_0\right)\zeta^{(1)}+F_0^{(1)}\gamma^{(1)}\right)\ln^2\left(\frac{Q^2}{M^2}\right)\mathcal{F}_B\,, \tag{16}$$

where $\beta_0 = 11C_A/3 - 4T_F n_f/3$ is the one-loop $\beta$-function provided the normalization point of $\alpha$ is $M$. The two-loop running of the coupling constant in the leading order $\gamma(\alpha)$ starts to contribute in the three-loop NNLL approximation.

The presence of a scalar particle in the fundamental representation with no Yukawa coupling to fermions leads only to a modification of the $\gamma$- and $\beta$-functions. One scalar boson with the mass much less than $Q$ gives the additional contribution of $16C_FT_F/9$ to $\gamma^{(2)}$ and the additional contribution of $-T_F/3$ to $\beta_0$.

For the standard model inspired case of the $SU(2)_L$ and $U(1)$ gauge groups with $n_f = 6$ and one charged scalar boson either in the fundamental representation of $SU(2)$ or of the unit $U(1)$ charge up to NNLL approximation we have

$$\begin{aligned}
\mathcal{F}^{(1)} &= \left[-\frac{3}{4}\ln^2\left(\frac{Q^2}{M^2}\right)+\frac{9}{4}\ln\left(\frac{Q^2}{M^2}\right)-\left(\frac{21}{8}+\frac{\pi^2}{2}\right)\right]\mathcal{F}_B\,,\\
\mathcal{F}^{(2)} &= \left[\frac{9}{32}\ln^4\left(\frac{Q^2}{M^2}\right)-\frac{19}{48}\ln^3\left(\frac{Q^2}{M^2}\right)-\left(\frac{463}{48}-\frac{7\pi^2}{8}\right)\ln^2\left(\frac{Q^2}{M^2}\right)\right]\mathcal{F}_B\,,
\end{aligned} \tag{17}$$

and

$$\begin{aligned}
\mathcal{F}^{(1)} &= \left[-\ln^2\left(\frac{Q^2}{M^2}\right)+3\ln\left(\frac{Q^2}{M^2}\right)-\left(\frac{7}{2}+\frac{2\pi^2}{3}\right)\right]\mathcal{F}_B\,,\\
\mathcal{F}^{(2)} &= \left[\frac{1}{2}\ln^4\left(\frac{Q^2}{M^2}\right)-\frac{52}{9}\ln^3\left(\frac{Q^2}{M^2}\right)+\left(\frac{625}{18}+\frac{2\pi^2}{3}\right)\ln^2\left(\frac{Q^2}{M^2}\right)\right]\mathcal{F}_B\,,
\end{aligned} \tag{18}$$

respectively. The relatively small coefficient of the LL terms and the large coefficient of the NNLL terms in the form factor are clearly indicative of the importance of the NNLL corrections and, as we will see, reflect the general structure of the logarithmically enhanced electroweak corrections.

# 3 The four fermion amplitude

Let us now investigate the four-fermion scattering at fixed angles in the limit when all the invariant energy and momentum transfers of the process are far larger than the gauge boson mass, $|s| \sim |t| \sim |u| \gg M^2$. The analysis of the four fermion amplitude is complicated by the extra kinematical variable and the presence of different "color" and Lorentz structures. We adopt the following notation

$$\begin{aligned} \mathcal{A}^\lambda &= \bar{\psi}_2 t^a \gamma_\mu \psi_1 \bar{\psi}_4 t^a \gamma_\mu \psi_3 \,, \\ \mathcal{A}_5^d &= \bar{\psi}_2 \gamma_\mu \gamma_5 \psi_1 \bar{\psi}_4 \gamma_\mu \gamma_5 \psi_3 \,, \\ \mathcal{A}_{LL}^\lambda &= \bar{\psi}_{2L} t^a \gamma_\mu \psi_{1L} \bar{\psi}_{4L} t^a \gamma_\mu \psi_{3L} \,, \\ \mathcal{A}_{LR}^d &= \bar{\psi}_{2L} \gamma_\mu \psi_{1L} \bar{\psi}_{4R} \gamma_\mu \psi_{3R} \,, \end{aligned} \tag{19}$$

etc. Here $t^a$ denotes the $SU(N)$ generator, $p_i$ the momentum of the $i$th fermion and $p_1$, $p_3$ are incoming, and $p_2$, $p_4$ outgoing momenta respectively. Hence $t = (p_1 - p_4)^2 = -sx_-$ and $u = (p_1 + p_3)^2 = -sx_+$ where $x_\pm = (1 \pm \cos\theta)/2$ and $\theta$ is the angle between the spatial components of $p_1$ and $p_4$. The complete basis consists of four independent chiral amplitudes, each of them of two possible color structure. For the moment we consider a parity conserving theory, hence only two chiral amplitudes are not degenerate. The Born amplitude is given by

$$\mathcal{A}_B = \frac{ig^2}{s} \mathcal{A}^\lambda \,. \tag{20}$$

The collinear divergences in the hard part of the virtual corrections and the corresponding "collinear" logarithms are known to factorize. They are responsible, in particular, for the double logarithmic contribution and depend only on the properties of the external on-shell particles but not on a specific process [15, 16, 17, 18, 19, 20]. This fact is especially clear if a physical (Coulomb or axial) gauge is used for the calculation. In this gauge the collinear divergences are present only in the self energy insertions to the external particles [16, 19, 20]. Thus, for each fermion-antifermion pair of the four-fermion amplitude the collinear logarithms are the same as for the form factor $\mathcal{F}$ discussed in the previous section. Let us denote by $\tilde{\mathcal{A}}$ the amplitude with the collinear logarithms factored out. For convenience we separate from $\tilde{\mathcal{A}}$ all the corrections entering eq. (3) so that

$$\mathcal{A} = \frac{ig^2}{s} \left( \frac{\mathcal{F}}{\mathcal{F}_B} \right)^2 \tilde{\mathcal{A}} \,. \tag{21}$$

The resulting amplitude $\tilde{\mathcal{A}}$ contains the logarithms of the "soft" nature corresponding to the soft divergences of the hard region contribution and the renormalization group logarithms.

It can be represented as a vector in the color/chiral basis and satisfies the following evolution equation [20, 21, 23]:

$$\frac{\partial}{\partial \ln Q^2}\tilde{\mathcal{A}} = \chi(\alpha(Q^2))\tilde{\mathcal{A}}\,, \tag{22}$$

where $\chi(\alpha)$ is the matrix of the soft anomalous dimensions. Note that we do not include to eq. (22) the pure renormalization group logarithms which can be absorbed by fixing the normalization scale of $g$ in the Born amplitude (20) to be $Q$. The solution of eq. (22) reads

$$\tilde{\mathcal{A}} = \sum_i \tilde{\mathcal{A}}_{0i}(\alpha(M^2)) \exp\left[\int_{M^2}^{Q^2} \frac{\mathrm{d}x}{x}\chi_i(\alpha(x))\right], \tag{23}$$

where $\chi_i(\alpha)$ are eigenvalues of $\chi(\alpha)$ and $\tilde{\mathcal{A}}_{0i}(\alpha)$ are $Q$-independent eigenvectors of $\chi(\alpha)$ which determine the initial conditions for the evolution equation at $Q = M$. Similar to the function $F_0(\alpha)$ they get contributions from all the regions while the matrix of the soft anomalous dimensions is given by the coefficients of the single pole of the hard region contribution to the exponent (23) [9, 20]. Strictly speaking the matrices $\chi(\alpha(Q^2))$ for different values of $Q$ do not commute and the solution is given by the path-ordered exponent [20]. The NLL approximation is given by the one-loop value of $\chi(\alpha)$ while the NNLL approximation requires $\tilde{\mathcal{A}}_{0i}(\alpha)$ up to $\mathcal{O}(\alpha)$ together with the one-loop running of $\alpha$ in $\chi(\alpha)$.

In one loop the elements of the matrix $\chi(\alpha)$ do not depend on chirality and read [9]

$$\begin{aligned}
\chi^{(1)}_{\lambda\lambda} &= -2C_A\left(\ln(x_+) + i\pi\right) + 4\left(C_F - \frac{T_F}{N}\right)\ln\left(\frac{x_+}{x_-}\right),\\
\chi^{(1)}_{\lambda d} &= 4\frac{C_F T_F}{N}\ln\left(\frac{x_+}{x_-}\right),\\
\chi^{(1)}_{d\lambda} &= 4\ln\left(\frac{x_+}{x_-}\right),\\
\chi^{(1)}_{dd} &= 0\,.
\end{aligned} \tag{24}$$

In higher orders the matrix $\chi(\alpha)$ may be non-degenerate for the different chiral components of the basis. In the Abelian case, there are no different color amplitudes and there is only one anomalous dimension

$$\chi^{(1)} = 4\ln\left(\frac{x_+}{x_-}\right). \tag{25}$$

In terms of the functions introduced above the one-loop correction reads

$$\begin{aligned}
\mathcal{A}^{(1)} &= \frac{ig^2}{s}\left[\left(\gamma^{(1)}\ln^2\left(\frac{Q^2}{M^2}\right) + \left(2\xi^{(1)} + 2\zeta^{(1)} + \chi^{(1)}_{\lambda\lambda}\right)\ln\left(\frac{Q^2}{M^2}\right) + 2F_0^{(1)}\right)\mathcal{A}^\lambda\right.\\
&+ \left.\chi^{(1)}_{\lambda d}\ln\left(\frac{Q^2}{M^2}\right)\mathcal{A}^d + \tilde{\mathcal{A}}_0^{(1)}\right],
\end{aligned} \tag{26}$$

where $\tilde{\mathcal{A}}_0^{(1)} = \sum_i \tilde{\mathcal{A}}_{0i}^{(1)} = \tilde{\mathcal{A}}^{(1)}|_{Q^2=M^2}$ has the following decomposition

$$\tilde{\mathcal{A}}_0^{(1)} = \tilde{A}_0^{(1)\lambda}{}_{LL}\mathcal{A}_{LL}^{\lambda} + \tilde{A}_0^{(1)\lambda}{}_{LR}\mathcal{A}_{LR}^{\lambda} + \ldots . \tag{27}$$

For the present two-loop analysis of the annihilation cross section only the real part of the coefficients $\tilde{A}_0^{(1)}$ is needed,

$$\begin{aligned}
\mathrm{Re}\left[\tilde{A}_0^{(1)\lambda}{}_{LL}\right] &= \left(C_F - \frac{T_F}{N}\right) f(x_+, x_-) + C_A\left(\frac{85}{9} + \pi^2\right) - \frac{20}{9}T_F n_f\,, \\
\mathrm{Re}\left[\tilde{A}_0^{(1)\lambda}{}_{LR}\right] &= -\left(C_F - \frac{T_F}{N} - \frac{C_A}{2}\right) f(x_-, x_+) + C_A\left(\frac{85}{9} + \pi^2\right) - \frac{20}{9}T_F n_f\,, \\
\mathrm{Re}\left[\tilde{A}_0^{(1)d}{}_{LL}\right] &= \frac{C_F T_F}{N} f(x_+, x_-)\,, \\
\mathrm{Re}\left[\tilde{A}_0^{(1)d}{}_{LR}\right] &= -\frac{C_F T_F}{N} f(x_-, x_+)\,,
\end{aligned} \tag{28}$$

where

$$f(x_+, x_-) = \frac{2}{x_+}\ln x_- + \frac{x_- - x_+}{x_+^2}\ln^2 x_-\,. \tag{29}$$

A scalar particle in the fundamental representation with no Yukawa coupling to fermions gives the additional contribution of $-8T_F/9$ to the first two lines of eq. (28).

The two-loop correction is obtained by the direct generalization of the form factor analysis. The only complication is related to the matrix structure of eq. (23):

$$\mathcal{A}_{LL}^{(2)} = \frac{ig^2}{s}\frac{(\gamma^{(1)})^2}{2}\ln^4\left(\frac{Q^2}{M^2}\right)\mathcal{A}^{\lambda}\,, \tag{30}$$

$$\mathcal{A}_{NLL}^{(2)} = \frac{ig^2(Q^2)}{s}\left[\left(2\zeta^{(1)} + \chi_{\lambda\lambda}^{(1)} - \frac{1}{3}\beta_0\right)\mathcal{A}^{\lambda} + \chi_{\lambda d}^{(1)}\mathcal{A}^{d}\right]\gamma^{(1)}\ln^3\left(\frac{Q^2}{M^2}\right)\,, \tag{31}$$

$$\begin{aligned}
\mathcal{A}_{NNLL}^{(2)} = \frac{ig^2(Q^2)}{s}&\left[\left(\gamma^{(2)} + \left(2\zeta^{(1)} - \beta_0\right)\zeta^{(1)} + 2F_0^{(1)}\gamma^{(1)} + \frac{1}{2}\left(\left(4\zeta^{(1)} - \beta_0\right)\chi_{\lambda\lambda}^{(1)} + \chi_{\lambda\lambda}^{(1)2}\right.\right.\right. \\
&\left.\left. + \chi_{d\lambda}^{(1)}\chi_{\lambda d}^{(1)}\right)\right)\mathcal{A}^{\lambda} + \frac{1}{2}\left(\left(4\zeta^{(1)} - \beta_0\right)\chi_{\lambda d}^{(1)} + \chi_{\lambda d}^{(1)}\chi_{\lambda\lambda}^{(1)} + \chi_{\lambda d}^{(1)}\chi_{dd}^{(1)}\right)\mathcal{A}^{d} + \gamma^{(1)}\tilde{\mathcal{A}}_0^{(1)}\Big] \\
&\times\ln^2\left(\frac{Q^2}{M^2}\right)\,.
\end{aligned} \tag{32}$$

The structure of the infrared singularities of the hard part of the two-loop corrections presented here is in full agreement with the result of [30] which was confirmed by explicit calculation [31].

To illustrate the significance of the subleading contributions let us again discuss the standard model inspired example considered in the previous section. Having the result for the amplitudes it is straightforward to compute the one- and two-loop corrections to the total cross section of the four-fermion annihilation process using the standard formulae. For the annihilation process one has to make the analytical continuation of the above result to the Minkowskian region of negative $Q^2 = -s$ according to $s + i0$ prescription. Although the above approximation is formally not valid for small angles $\theta < M/\sqrt{s}$ we can integrate the differential cross section over all angles to get a result with the logarithmic accuracy. In this way we obtain for the case of $SU(2)_L$ group

$$\begin{aligned}\sigma^{(1)} &= \left[-3\ln^2\left(\frac{s}{M^2}\right) + \frac{80}{3}\ln\left(\frac{s}{M^2}\right) - \left(\frac{25}{9} + 3\pi^2\right)\right]\sigma_B\,,\\ \sigma^{(2)} &= \left[\frac{9}{2}\ln^4\left(\frac{s}{M^2}\right) - \frac{449}{6}\ln^3\left(\frac{s}{M^2}\right) + \left(\frac{4855}{18} + \frac{37\pi^2}{3}\right)\ln^2\left(\frac{s}{M^2}\right)\right]\sigma_B\,,\end{aligned} \tag{33}$$

and

$$\begin{aligned}\sigma^{(1)} &= \left[-3\ln^2\left(\frac{s}{M^2}\right) + \frac{26}{3}\ln\left(\frac{s}{M^2}\right) + \left(\frac{218}{9} - 3\pi^2\right)\right]\sigma_B\,,\\ \sigma^{(2)} &= \left[\frac{9}{2}\ln^4\left(\frac{s}{M^2}\right) - \frac{125}{6}\ln^3\left(\frac{s}{M^2}\right) - \left(\frac{799}{9} - \frac{37\pi^2}{3}\right)\ln^2\left(\frac{s}{M^2}\right)\right]\sigma_B\,,\end{aligned} \tag{34}$$

for the initial and final state fermions of the same or opposite isospin, respectively. Here $\sigma_B$ is the Born cross section with the $\overline{MS}$ couplings constant normalized at the scale $\sqrt{s}$.

For the $U(1)$ group we have

$$\begin{aligned}\sigma^{(1)} &= \left[-4\ln^2\left(\frac{s}{M^2}\right) + 12\ln\left(\frac{s}{M^2}\right) - \left(\frac{382}{9} - \frac{4\pi^2}{3}\right)\right]\sigma_B\,,\\ \sigma^{(2)} &= \left[8\ln^4\left(\frac{s}{M^2}\right) - \frac{532}{9}\ln^3\left(\frac{s}{M^2}\right) + \left(\frac{1142}{3} + \frac{16\pi^2}{3}\right)\ln^2\left(\frac{s}{M^2}\right)\right]\sigma_B\,.\end{aligned} \tag{35}$$

Similar to the form factor, we observe a relatively small coefficient of the LL terms and a large coefficient of the NNLL terms.

# 4 NNL logarithms in electroweak processes at high energies

We are interested in the process $f'\bar{f}' \to f\bar{f}$. In the Born approximation, its amplitude is of the following form

$$A_B = \frac{ig^2}{s}\sum_{I,J=L,R}\left(T^3_{f'}T^3_f + t_W^2\frac{Y_{f'}Y_f}{4}\right)A^{f'f}_{IJ}\,, \tag{36}$$

where

$$A^{f'f}_{IJ} = \bar{f}'_I\gamma_\mu f'_I\bar{f}_J\gamma_\mu f_J\,, \tag{37}$$

$t_W = \tan\theta_W$ with $\theta_W$ being the Weinberg angle and $T_f$ $(Y_f)$ is the isospin (hypercharge) of the fermion which depends on the fermion chirality.

To analyze the electroweak correction to the above process we use the approximation with the $W$ and $Z$ bosons of the same mass $M$, the Higgs boson of the mass $M_H \sim M$ and massless quarks and leptons. A fictitious photon mass $\lambda$ has to be introduced to regularize the infrared divergences. We insert the mass into the gauge boson propagators "by hand" to investigate the leading in $s^{-1}$ behavior of the amplitudes, leaving aside the Higgs mechanism of the gauge boson mass generation. This approach is gauge invariant as far as power unsuppressed terms are considered. The NNLL approximation is not sensitive to the fine details of the mass generation because we need only the hard part of the potentially dangerous self-energy insertion to the gauge boson propagator. Indeed, the only effect of the virtual Higgs boson is the modification of the functions $\beta_0$, $\gamma^{(2)}$ and $\tilde{A}_0^{(1)\lambda}$. The function $\gamma^{(2)}$ as well as the running of the coupling constant in $\gamma^{(1)}$ and $\chi^{(1)}$ are determined by the singularities of the hard part of the corrections. On the other hand, the vacuum polarization of the off-shell gauge boson in the Born amplitude contributing to $\tilde{A}_0^{(1)\lambda}$ is infrared safe and can be computed in the massless approximation, i.e. in the leading order in $s^{-1}$ it receives only the contribution of the hard region. The only effect of the Higgs mechanism is that we have two different masses $M_Z$ and $M_W = \cos\theta_W M_Z$. Since $\cos\theta_W \sim 1$ we neglect this difference in our calculation. The correction due to the heavy gauge boson mass splitting will be discussed at the end of the section.

Let us first consider the equal mass case $\lambda = M$, where we can work in terms of the fields of unbroken phase. In the massless quark approximation the Higgs boson couples only to the gauge field. Therefore the result of Sects. 2 and 3 for the $SU(2)_L$ gauge group with the coupling $g$ and the $U(1)$ gauge group with the coupling $t_W g$ can be directly applied to the electroweak processes. For the standard electroweak model with one charged Higgs doublet one has to replace in the above expressions $n_f \to 2N_g + 1/4$ for $SU(2)_L$ $\beta$-function, $T_F n_f \to 5N_g/3 + 1/8$ for $U(1)$ $\beta$-function, $n_f \to 2N_g + 2/5$ for $SU(2)_L$ $\gamma_2$ and $\tilde{A}_0^{(1)}$ coefficients, and $T_F n_f \to 5N_g/3 + 1/5$ for $U(1)$ $\gamma_2$ and $\tilde{A}_0^{(1)}$ coefficients, with $N_g = 3$ being the number of generations. For example we have

$$
\begin{aligned}
\gamma^{(2)} &= \frac{10}{3}N_g - \frac{65}{3} + \pi^2\,, \\
\beta_0 &= -\frac{4}{3}N_g + \frac{43}{6}\,,
\end{aligned}
\tag{38}
$$

for $SU(2)_L$ and

$$
\begin{aligned}
\gamma^{(2)} &= \left(\frac{200}{27}N_g + \frac{8}{9}\right) t_W^2 \frac{Y_f^2}{4}\,, \\
\beta_0 &= -\frac{20}{9}N_g - \frac{1}{6}\,,
\end{aligned}
\tag{39}
$$

for $U(1)$.

The result for the amplitudes is obtained by projecting on a relevant initial/final state with the proper assignment of isospin/hypercharge. For example, the projection of the basis (19) on the states corresponding to the neutral current processes reads $\mathcal{A}^{\lambda}_{IJ} \to T^3_f T^3_{f'} \mathcal{A}^{f'f}_{IJ}$, $\mathcal{A}^{d}_{IJ} \to \mathcal{A}^{f'f}_{IJ}$. The only complication in combinatorics is related to the fact that now we are having different gauge groups for the fermions of different chirality. In particular, the double logarithmic approximation is given by the exponential factor

$$\exp\left[-\left(T_f(T_f+1)+t_W^2\frac{Y_f^2}{4}+(f\leftrightarrow f')\right)L(Q^2)\right], \tag{40}$$

where

$$L(Q^2)=\frac{g^2}{16\pi^2}\ln^2\left(\frac{Q^2}{M^2}\right), \tag{41}$$

and $T_f(T_f+1)=C_F$.

The photon is, however, massless and the corresponding infrared divergent contributions should be accompanied by the real soft photon radiation integrated to some resolution energy $\omega_{res}$ to get an infrared safe cross section independent on an auxiliary photon mass. In practice, the resolution energy is much less than the $W$ $(Z)$ boson mass and the massive gauge bosons are supposed to be detected as separate particles. To study the virtual corrections in the limit of the vanishing photon mass we follow a general approach of the infrared evolution equations developed in [6] (see also references therein). It is convenient to use the auxiliary photon mass $\lambda$ as a variable of the infrared evolution equation below the electroweak scale $M$. The dependence of the virtual corrections on $\lambda$ in the limit $\lambda \ll M$ is canceled by the contribution of the real soft photon radiation. For $\omega_{res} \ll M$, the soft photon emission is of the pure QED nature. Therefore, the kernel of the infrared evolution equation which governs the $\lambda$ dependence of the virtual corrections to the amplitudes is Abelian. This dependence is given by the QED factor $\mathcal{U}$ which can be directly obtained from the general formulae given above:

$$\begin{aligned}\mathcal{U} &= U_0(\alpha_e)\exp\Bigg\{-\frac{\alpha_e(\lambda^2)}{4\pi}\Bigg[\left(\left(Q_f^2+Q_{f'}^2-\left(\frac{76}{27}\left(Q_f^2+Q_{f'}^2\right)+\frac{16}{9}Q_fQ_{f'}\right)\frac{\alpha_e}{\pi}N_g\right)\right)\\ &\quad\times\ln^2\left(\frac{Q^2}{\lambda^2}\right)-\left(3\left(Q_f^2+Q_{f'}^2\right)+4Q_fQ_{f'}\ln\left(\frac{x_+}{x_-}\right)\right)\ln\left(\frac{Q^2}{\lambda^2}\right)+\frac{8}{27}\left(Q_f^2+Q_{f'}^2\right)\frac{\alpha_e}{\pi}N_g\\ &\quad\times\ln^3\left(\frac{Q^2}{\lambda^2}\right)\Bigg]+\mathcal{O}(\alpha_e^3)\Bigg\},\end{aligned} \tag{42}$$

where $\alpha_e$ is the $\overline{MS}$ QED coupling constant and we use the following expressions for the QED functions

$$\begin{aligned}\zeta_e^{(1)} &= 3Q_f^2\,,\\ \chi_e^{(1)} &= 4Q_{f'}Q_f\ln\left(\frac{x_+}{x_-}\right),\\ \beta_0^e &= -\frac{32}{9}N_g\,,\\ \gamma_e^{(2)} &= \frac{320}{27}N_gQ_f^2\,.\end{aligned} \tag{43}$$

The expressions for $\beta_0^e$ and $\gamma_e^{(2)}$ can be obtained by substituting $T_F n_f \to 8N_g/3$ to the general formulae. The coefficient $U_0(\alpha_e)$ in eq. (42) is a two-component vector in the chiral basis.

In a full analogy with the renormalization group all the information on the non-Abelian gauge dynamics above the electroweak scale up to power suppressed contributions is contained in the initial condition for this Abelian infrared evolution equation at the point $\lambda = M$. To fix a relevant initial condition for the evolution in $\lambda$ below the electroweak scale one has to subtract the QED virtual correction (42) computed with the photon of the mass $M$ from the complete result with $\lambda = M$ [6]. This leads, in particular, to the modification of the function $\gamma(\alpha)$ so that the double logarithmic exponential factor becomes

$$\exp\left[-\left(T_f(T_f+1) + t_W^2\frac{Y_f^2}{4} - s_W^2 Q_f^2 + (f \leftrightarrow f')\right) L(Q^2)\right], \tag{44}$$

where $s_W = \sin\theta_W$. In the NNLL approximation after the subtraction we get

$$\begin{aligned} \gamma^{(2)} \quad &= \quad -2\left[\left(-\frac{20}{9}N_g + \frac{130}{9} - \frac{2\pi^2}{3}\right) T_f(T_f+1) - \left(\frac{100}{27}N_g + \frac{4}{9}\right) t_W^2 \frac{Y_f^2}{4}\right. \\ &\quad \left. + \frac{160}{27} N_g s_W^2 Q_f^2\right] . \end{aligned} \tag{45}$$

A similar subtraction should be done for the parameters $\zeta^{(1)}$ and $\chi^{(1)}$ which take the form [9]

$$\zeta^{(1)} = 3\left(T_f(T_f+1) + t_W^2\frac{Y_f^2}{4} - s_W^2 Q_f^2\right), \tag{46}$$

and

$$\begin{aligned} \chi_{\lambda\lambda}^{(1)} &= -4\left(\ln(x_+) + i\pi\right) + \left(t_W^2 Y_{f'} Y_f - 4 s_W^2 Q_{f'} Q_f + 2\right) \ln\left(\frac{x_+}{x_-}\right), \\ \chi_{\lambda d}^{(1)} &= \frac{3}{4}\ln\left(\frac{x_+}{x_-}\right), \\ \chi_{d\lambda}^{(1)} &= 4\ln\left(\frac{x_+}{x_-}\right), \\ \chi_{dd}^{(1)} &= \left(t_W^2 Y_{f'} Y_f - 4 s_W^2 Q_{f'} Q_f\right) \ln\left(\frac{x_+}{x_-}\right). \end{aligned} \tag{47}$$

For $I$ or $J = R$ the matrix $\chi^{(1)}$ is reduced to

$$\chi^{(1)} = \left(t_W^2 Y_{f'} Y_f - 4 s_W^2 Q_{f'} Q_f\right) \ln\left(\frac{x_+}{x_-}\right). \tag{48}$$

At the same time we have some freedom in the definition of the coefficients $F_0(\alpha)$, $\tilde{A}_0(\alpha)$ and $U_0(\alpha_e)$. If we use the one-loop normalization condition $\mathcal{U}|_{Q^2=M^2} = 1$, then $U_0^{(1)} = 0$

and no QED subtraction is necessary for $F_0^{(1)}$ and $\tilde{A}_0^{(1)}$. In this case the QED factor $\mathcal{U}$ is universal and has no matrix structure. To summarize, we have two evolution equations and corresponding initial conditions. The coefficients $F_0(\alpha)$ and $\tilde{A}_0(\alpha)$ give the initial condition for the hard evolution of the amplitudes in $Q$ at $Q = M$ while the above subtraction of the QED contribution gives the initial condition for the infrared evolution of the amplitudes in $\lambda$ at $\lambda = M$.

The result for the $n$-loop correction to the amplitude (36) can be decomposed as

$$A^{(n)} = A_{LL}^{(n)} + A_{NLL}^{(n)} + A_{NNLL}^{(n)} + \dots \tag{49}$$

Explicit expressions for $A_{LL}^{(1)}$ and $A_{NLL}^{(1)}$ can be found, for example, in [9]. In the NNLL approximation one has to take into account also the one-loop constant contribution corresponding to $F_0^{(1)}$ and $\tilde{\mathcal{A}}_0^{(1)}$ terms of eq. (26). It reads

$$\begin{aligned} a\,A_{NNLL}^{(1)} = {} & \\ & \frac{ig^2}{s}\sum_{I,J=L,R}\left\{-\left(\frac{7}{2}+\frac{2\pi^2}{3}\right)\left[T_f(T_f+1)+t_W^2\frac{Y_f^2}{4}+(f\leftrightarrow f')\right]\left[T_{f'}^3T_f^3+t_W^2\frac{Y_{f'}Y_f}{4}\right]\right. \\ & +\left(2T_{f'}^3T_f^3+t_W^2\frac{Y_{f'}Y_f}{4}\right)t_W^2\frac{Y_{f'}Y_f}{4}\Big[f(x_+,x_-)\left(\delta_{IR}\delta_{JR}+\delta_{IL}\delta_{JL}\right) \\ & -f(x_-,x_+)\left(\delta_{IR}\delta_{JL}+\delta_{IL}\delta_{JR}\right)\Big]-\left[\left(\frac{20}{9}N_g+\frac{4}{9}\right)T_{f'}^3T_f^3+\left(\frac{100}{27}N_g+\frac{4}{9}\right)t_W^2\frac{Y_{f'}Y_f}{4}\right] \\ & \left.+\left[\frac{1}{2}f(x_+,x_-)+\frac{170}{9}+2\pi^2\right]T_{f'}^3T_f^3+\frac{3}{16}f(x_+,x_-)\delta_{IL}\delta_{JL}\right\}aA_{IJ}^{f'f}\,, \end{aligned} \tag{50}$$

where $a = g^2/16\pi^2$ and we keep only the real part of $\tilde{\mathcal{A}}_0^{(1)}$.

Let us consider the two-loop corrections. The two-loop LL corrections to the chiral amplitudes were obtained in [6]

$$\begin{aligned} a^2\,A_{LL}^{(2)} = {} & \frac{ig^2}{s}\sum_{I,J=L,R} \\ & \frac{1}{2}\left(T_f(T_f+1)+t_W^2\frac{Y_f^2}{4}-s_W^2Q_f^2+(f\leftrightarrow f')\right)^2\left[T_{f'}^3T_f^3+t_W^2\frac{Y_{f'}Y_f}{4}\right]L^2(Q^2)A_{IJ}^{f'f}\,, \end{aligned} \tag{51}$$

and the two-loop NLL corrections with the exception of the trivial corrections proportional to $\beta_0$ can be found in [9]

$$\begin{aligned} a^2\,A_{NLL}^{(2)} = {} & -\frac{ig^2}{s}\sum_{I,J=L,R}\left[T_f(T_f+1)+t_W^2\frac{Y_f^2}{4}-s_W^2Q_f^2+(f\leftrightarrow f')\right] \\ & \times\left\{3\left[T_f(T_f+1)+t_W^2\frac{Y_f^2}{4}-s_W^2Q_f^2+(f\leftrightarrow f')\right]\left[T_{f'}^3T_f^3+t_W^2\frac{Y_{f'}Y_f}{4}\right]\right. \\ & +\left[-4\left(\ln(x_+)+i\pi\right)+\ln\left(\frac{x_+}{x_-}\right)\left(2+t_W^2Y_{f'}Y_f\right)\right]T_{f'}^3T_f^3+\frac{3}{4}\ln\left(\frac{x_+}{x_-}\right)\delta_{IL}\delta_{JL} \\ & \left.+\ln\left(\frac{x_+}{x_-}\right)\left[t_W^2Y_{f'}Y_f-4s_W^2Q_{f'}Q_f\right]\left[T_{f'}^3T_f^3+t_W^2\frac{Y_{f'}Y_f}{4}\right]\right\}L(Q^2)l(Q^2)A_{IJ}^{f'f}\,, \end{aligned} \tag{52}$$

where

$$l(Q^2) = \frac{g^2}{16\pi^2} \ln\left(\frac{Q^2}{M^2}\right), \tag{53}$$

with $\delta_{IL} = 1$ for $I = L$ and zero otherwise. The second line of eq. (52) corresponds to the $\zeta^{(1)}$ term of eq. (31) while the third and forth lines correspond to the $\chi^{(1)}$ terms in eq. (31). A part of the $\beta_0$ NLL corrections is absorbed by choosing the normalization point of the coupling constants in eq. (36) to be $Q$. The rest is due to the running of the coupling constant in the double logarithmic integral and corresponds to the $\beta_0$ term in eq. (31). It is of the form

$$\begin{aligned} a^2\, A^{(2)}_{NLL}|_{\beta_0} \;=\; & -\frac{ig^2}{s} \sum_{I,J=L,R} \\ & \frac{1}{3}\left[\left(\frac{4}{3}N_g - \frac{43}{6}\right) T_f(T_f+1) + \left(\frac{20}{9}N_g + \frac{1}{6}\right) t_W^2 \frac{Y_f^2}{4} - \frac{32}{9} N_g s_W^2 Q_f^2 + (f \leftrightarrow f')\right] \\ & \times \left[T^3_{f'} T^3_f + t_W^2 \frac{Y_{f'} Y_f}{4}\right] l(Q^2) L(Q^2) A^{f'f}_{IJ}\,, \end{aligned} \tag{54}$$

provided the normalization point of the coupling constants is $M$ with the exception of the coupling constants entering the Born amplitude (36) normalized at the scale $Q$.

Let us consider the NNLL contribution. For convenience we split it in four parts

$$A^{(2)}_{NNLL} \;=\; \Delta_1 A^{(2)}_{NNLL} + \Delta_2 A^{(2)}_{NNLL} + \Delta_3 A^{(2)}_{NNLL} + \Delta_4 A^{(2)}_{NNLL}\,, \tag{55}$$

where the trivial $\beta_0^2$ renormalization group logarithms which can be absorbed into the running of the coupling constants in eq. (36) are not included. The correction $\Delta_1 A^{(2)}_{NNLL}$ corresponding to the $\gamma^{(2)}$, ${\zeta^{(1)}}^2$, $\beta_0\zeta^{(1)}$ and $F_0^{(1)}\gamma^{(1)}$ terms of eq. (32) is

$$\begin{aligned} a^2\, \Delta_1 A^{(2)}_{NNLL} \;=\; & \frac{ig^2}{s} \sum_{I,J=L,R} \\ & \Bigg\{ -\Bigg[\left(-\frac{20}{9}N_g + \frac{130}{9} - \frac{2\pi^2}{3}\right) T_f(T_f+1) - \left(\frac{100}{27}N_g + \frac{4}{9}\right) t_W^2 \frac{Y_f^2}{4} + \frac{160}{27} N_g s_W^2 Q_f^2 \\ & + (f \leftrightarrow f')\Bigg] + \frac{9}{2}\left[T_f(T_f+1) + t_W^2 \frac{Y_f^2}{4} - s_W^2 Q_f^2 + (f \leftrightarrow f')\right]^2 \\ & + \frac{3}{2}\left[\left(\frac{4}{3}N_g - \frac{43}{6}\right) T_f(T_f+1) + \left(\frac{20}{9}N_g + \frac{1}{6}\right) t_W^2 \frac{Y_f^2}{4} - \frac{32}{9} N_g s_W^2 Q_f^2 + (f \leftrightarrow f')\right] \\ & + \left(\frac{7}{2} + \frac{2\pi^2}{3}\right)\left[T_f(T_f+1) + t_W^2 \frac{Y_f^2}{4} - s_W^2 Q_f^2 + (f \leftrightarrow f')\right]\Bigg[T_f(T_f+1) + t_W^2 \frac{Y_f^2}{4} \\ & + (f \leftrightarrow f')\Bigg]\Bigg\} \left[T^3_{f'} T^3_f + t_W^2 \frac{Y_{f'} Y_f}{4}\right] l^2(Q^2) A^{f'f}_{IJ}\,. \end{aligned} \tag{56}$$

The correction $\Delta_2 A^{(2)}_{NNLL}$ corresponding to the $\zeta^{(1)}\chi^{(1)}$ and $\beta_0\chi^{(1)}$ terms of eq. (32) reads

$$
\begin{aligned}
a^2\,\Delta_2 A^{(2)}_{NNLL} \;=\; & \frac{ig^2}{s}\sum_{I,J=L,R} \qquad (57)\\
& \left\{\left[\left(-4\left(\ln\left(x_+\right)+i\pi\right)+\ln\left(\frac{x_+}{x_-}\right)\left(2+t_W^2 Y_{f'}Y_f\right)\right)T^3_{f'}T^3_f+\frac{3}{4}\ln\left(\frac{x_+}{x_-}\right)\delta_{IL}\delta_{JL}\right]\right.\\
& \times\left[\left(\frac{2}{3}N_g-\frac{43}{12}\right)+3\left(T_f(T_f+1)+t_W^2\frac{Y_f^2}{4}-s_W^2Q_f^2+(f\leftrightarrow f')\right)\right]\\
& +4\ln\left(\frac{x_+}{x_-}\right)\left[\left(\left(\frac{10}{9}N_g+\frac{1}{12}\right)t_W^2\frac{Y_{f'}Y_f}{4}-\frac{16}{9}N_g s_W^2 Q_{f'}Q_f\right)+3\left(t_W^2\frac{Y_{f'}Y_f}{4}-s_W^2Q_{f'}Q_f\right)\right.\\
& \left.\left.\times\left(T_f(T_f+1)+t_W^2\frac{Y_f^2}{4}-s_W^2Q_f^2+(f\leftrightarrow f')\right)\right]\left[T^3_{f'}T^3_f+t_W^2\frac{Y_{f'}Y_f}{4}\right]\right\}l^2(Q^2)A^{f'f}_{IJ}\,.
\end{aligned}
$$

The correction $\Delta_3 A^{(2)}_{NNLL}$ corresponding to the ${\chi^{(1)}}^2$ terms of eq. (32) reads

$$
\begin{aligned}
a^2\,\Delta_3 A^{(2)}_{NNLL} \;=\; & \frac{ig^2}{s}\sum_{I,J=L,R} \qquad (58)\\
& \frac{1}{2}\left\{\left(-4\left(\ln\left(x_+\right)+i\pi\right)+\ln\left(\frac{x_+}{x_-}\right)\left(2+t_W^2Y_{f'}Y_f-4s_W^2Q_{f'}Q_f\right)\right)^2T^3_{f'}T^3_f+\ln\left(\frac{x_+}{x_-}\right)\right.\\
& \times\left[-4\left(\ln\left(x_+\right)+i\pi\right)+2\ln\left(\frac{x_+}{x_-}\right)\left(1+t_W^2Y_{f'}Y_f-4s_W^2Q_{f'}Q_f\right)\right]\\
& \times\left[\frac{3}{4}\delta_{IL}\delta_{JL}+t_W^2Y_{f'}Y_fT^3_{f'}T^3_f\right]+\ln^2\left(\frac{x_+}{x_-}\right)\left[3\left(T^3_{f'}T^3_f+t_W^2\frac{Y_{f'}Y_f}{4}\delta_{IL}\delta_{JL}\right)\right.\\
& \left.\left.+\left(t_W^2Y_{f'}Y_f-4s_W^2Q_{f'}Q_f\right)^2t_W^2\frac{Y_{f'}Y_f}{4}\right]\right\}l^2(Q^2)A^{f'f}_{IJ}\,.
\end{aligned}
$$

The correction $\Delta_4 A^{(2)}_{NNLL}$ corresponding to the real part of the $\gamma^{(1)}\tilde{\mathcal{A}}^{(1)}_0$ term of eq. (32) reads

$$
\begin{aligned}
a^2\,\Delta_4 A^{(2)}_{NNLL} \;=\; & -\frac{ig^2}{s}\sum_{I,J=L,R} \qquad (59)\\
& \left(T_f(T_f+1)+t_W^2\frac{Y_f^2}{4}-s_W^2Q_f^2+(f\leftrightarrow f')\right)\left\{\left(2T^3_{f'}T^3_f+t_W^2\frac{Y_{f'}Y_f}{4}\right)t_W^2\frac{Y_{f'}Y_f}{4}\right.\\
& \times\Big[f(x_+,x_-)\left(\delta_{IR}\delta_{JR}+\delta_{IL}\delta_{JL}\right)-f(x_-,x_+)\left(\delta_{IR}\delta_{JL}+\delta_{IL}\delta_{JR}\right)\Big]\\
& -\left[\left(\frac{20}{9}N_g+\frac{4}{9}\right)T^3_{f'}T^3_f+\left(\frac{100}{27}N_g+\frac{4}{9}\right)t_W^2\frac{Y_{f'}Y_f}{4}\right]\\
& \left.+\left[\frac{1}{2}f(x_+,x_-)+\frac{170}{9}+2\pi^2\right]T^3_{f'}T^3_f+\frac{3}{16}f(x_+,x_-)\delta_{IL}\delta_{JL}\right\}l^2(Q^2)A^{f'f}_{IJ}\,.
\end{aligned}
$$

With the expression for the chiral amplitudes at hand, we can compute the leading and subleading logarithmic corrections to the basic observables for $e^+e^-\to f\bar{f}$.

In the NNLL approximation one has to take into account also the effect of analytical continuation to the physical positive real value of the invariant $s$. For the annihilation processes it is more natural to normalize the QED factor at the Minkowskian point $s = M^2$ to $\mathcal{U}|_{s=M^2} = 1$ so that after the expansion in $\alpha_e$ it reads

$$\begin{aligned}\mathcal{U} \;&=\; \left\{1 - \frac{\alpha_e(\lambda^2)}{4\pi}\left[\left(Q_f^2 + Q_{f'}^2\right)\ln^2\left(\frac{s}{\lambda^2}\right) - \left((3+2i\pi)\left(Q_f^2+Q_{f'}^2\right) + 4Q_fQ_{f'}\ln\left(\frac{x_+}{x_-}\right)\right)\right.\right.\\ &\quad\left.\left.\times \ln\left(\frac{s}{\lambda^2}\right)\right] + \mathcal{O}(\alpha_e^2)\right\}, \qquad (60)\end{aligned}$$

Let us consider the total cross sections of the quark-antiquark/$\mu^+\mu^-$ production in the $e^+e^-$ annihilation. The LL, NLL and NNLL corrections to the cross sections to one and two loops read

$$\begin{aligned}R_{Q\bar{Q}} &= 1 - 1.66\,L(s) + 5.31\,l(s) - 15.86\,a + 1.93\,L^2(s) - 9.43\,L(s)l(s) + 29.73\,l^2(s)\,,\\ R_{q\bar{q}} &= 1 - 2.18\,L(s) + 20.58\,l(s) - 36.34\,a + 2.79\,L^2(s) - 50.06\,L(s)l(s) + 295.12\,l^2(s)\,,\\ R_{\mu^+\mu^-} &= 1 - 1.39\,L(s) + 10.12\,l(s) - 31.33\,a + 1.42\,L^2(s) - 18.43\,L(s)l(s) + 99.89\,l^2(s)\,,\end{aligned} \qquad (61)$$

where $Q = u, c, t$, $q = d, s, b$, $R_{Q\bar{Q}} = \sigma/\sigma_B(e^+e^- \to Q\bar{Q})$ and so on. The $\overline{MS}$ couplings in the Born cross section are normalized at $\sqrt{s}$. Numerically, we have $L(s) = 0.07$ $(0.11)$ and $l(s) = 0.014$ $(0.017)$ for $\sqrt{s} = 1$ TeV and 2 TeV respectively. Here $M = M_W$ has been chosen for the infrared cutoff and $a = 2.69 \cdot 10^{-3}$, $s_W^2 = 0.231$ for the $\overline{MS}$ couplings normalized at the gauge boson mass. The small difference between the two-loop NLL coefficients in eq. (61) and the result of [9] is due to the $\beta_0$ contribution (54).

To get the infrared safe result for the semi-inclusive cross sections one has to add to the expressions given above the standard QED corrections due to the soft photon emission and the *pure* QED virtual correction which is determined for massless or light fermions of the mass $m_f \ll \lambda \ll M$ by eqs. (42), (60). To derive the QED factor for $\lambda$ far less than the fermion mass $\lambda \ll m_f \ll M$ one has to change the kernel of the infrared evolution equation and match the new solution to eq. (42) at the point $\lambda = m_f$. The sum of the real and virtual QED corrections depends on $s$, $\omega_{res}$ and on the initial/final fermion masses but not on $M_{Z,W}$. Note that our analysis implies the resolution energy for the real photon emission to be smaller than the heavy boson mass. If the resolution energy exceeds $M_{Z,W}$ the analysis is more complicated due to the fact that the radiation of real photons is not of Poisson type because of its non-Abelian $SU(2)_L$ component [6]. In the case of the quark-antiquark final state the strong interaction also produces the logarithmically growing terms. They can be read off the results of Section 1 for the form factor. For massless quarks the complete $\mathcal{O}(\alpha_s^2)$ corrections including the bremsstrahlung effects can be found in [28].

For completeness we give a numerical estimate of corrections to the cross section asymmetries. In the case of the forward-backward asymmetry $A^{FB}$ (the difference of the cross section averaged over forward and backward semispheres with respect to the electron beam direction divided by the total cross section) we get

$$
\begin{aligned}
R^{FB}_{Q\bar{Q}} &= 1 - 0.09\,L(s) - 1.23\,l(s) + \ \ 3.58\,a + 0.12\,L^2(s) + 0.65\,L(s)l(s) + \ \ 2.57\,l^2(s)\,, \\
R^{FB}_{q\bar{q}} &= 1 - 0.14\,L(s) + 7.15\,l(s) - \ \ 7.93\,a + 0.02\,L^2(s) - 1.27\,L(s)l(s) - 36.53\,l^2(s)\,, \\
R^{FB}_{\mu^+\mu^-} &= 1 - 0.04\,L(s) + 5.49\,l(s) - 10.77\,a + 0.27\,L^2(s) - 6.29\,L(s)l(s) + 6.65\,l^2(s)\,,
\end{aligned}
\tag{62}
$$

where $R^{FB} = A^{FB}/A^{FB}_B$. For the left-right asymmetry $A^{LR}$ (the difference of the cross sections of the left and right particles production divided by the total cross section) we obtain in the same notation

$$
\begin{aligned}
R^{LR}_{Q\bar{Q}} &= 1 - \ \ 2.34\,L(s) + \ \ \ 8.98\,l(s) + 17.08\,a - 0.46\,L^2(s) + \ \ 8.37\,L(s)l(s) - \ \ \ \ 5.43\,l^2(s)\,, \\
R^{LR}_{q\bar{q}} &= 1 - \ \ 1.12\,L(s) + \ \ 11.86\,l(s) - \ \ 7.28\,a - 0.81\,L^2(s) + 18.05\,L(s)l(s) - 120.48\,l^2(s)\,, \\
R^{LR}_{\mu^+\mu^-} &= 1 - 13.24\,L(s) + 113.77\,l(s) - 60.36\,a - 0.79\,L^2(s) + 32.40\,L(s)l(s) - 291.25\,l^2(s)\,.
\end{aligned}
\tag{63}
$$

Finally, for the left-right asymmetry $\tilde{A}^{LR}$ (the difference of the cross sections for the left and right initial state particles divided by the total cross section) which differs from $A^{LR}$ for the quark-antiquark final state we have

$$
\begin{aligned}
\tilde{R}^{LR}_{Q\bar{Q}} &= 1 - \ \ 2.75\,L(s) + \ \ 10.07\,l(s) + \ \ 8.13\,a - 0.91\,L^2(s) + 12.19\,L(s)l(s) - \ \ 30.40\,l^2(s)\,, \\
\tilde{R}^{LR}_{q\bar{q}} &= 1 - \ \ 1.07\,L(s) + \ \ 11.56\,l(s) - \ \ 7.04\,a - 0.77\,L^2(s) + 17.17\,L(s)l(s) - 116.80\,l^2(s)\,.
\end{aligned}
\tag{64}
$$

In the $1-2$ TeV region the two-loop LL, NLL and NNLL corrections to the cross sections can be as large as $1-4\%$, $5-10\%$, and $5-9\%$ respectively. However, we observe a *significant cancellation* between different terms and the sum of the known two-loop corrections amounts of approximately $1-2\%$. The sum of the two-loop correction to the asymmetries is even smaller and does not exceed 1% level with the exception of the $R^{LR}_{\mu^+\mu^-}$. For this quantity the relatively large corrections are the consequence of the numerically small Born approximation.

Let us discuss the accuracy of our result. At TeV energies the LL, NLL and NNLL corrections of eqs. (61)–(64) provide asymptotic expressions for the cross section and the asymmetries to one and two loops. The complete one-loop corrections are known exactly (see [32] for the most general result) and we have included the dominant one-loop terms in eqs. (61)–(64) to demonstrate the structure of the expansion rather than for precise numerical estimates. For physical applications, the mass difference between the $W$ and the $Z$ gauge boson, power suppressed terms and also top quark mass effects can be important. The effect of the $W$ and $Z$ gauge boson mass difference on the coefficients of the NLL and NNLL terms is suppressed as $(M_Z - M_W)/M \sim 0.1$ while the leading power corrections can be as large as $M^2/s < 0.01$. Thus, except for the production of third generation quarks, the above expressions approximate the exact one-loop result with 1% accuracy in the TeV region. At the same time both effects can be neglected in two-loop approximation. Therefore, the only essential deviation of the complete two-loop NLL and NNLL result from eqs. (61)–(64) for the production of the third generation quarks is due to the large top quark Yukawa coupling. Numerically, the corresponding corrections can be as important as the generic non-Yukawa ones.

Finally, let us emphasize that the angular dependent NLL and NNLL terms are quite important for the cross section and dominate in particular the forward-backward asymmetry.

# 5 Summary

In the present paper we employed the evolution equation approach to analyze the high energy asymptotic behavior of the four-fermion amplitudes in the non-Abelian gauge models. The results were used to compute the NNLL electroweak corrections to the neutral current four-fermion processes at high energy in the massless quark approximation to all orders in the coupling constants. We have shown the NNLL approximation to be insensitive to the details of the gauge boson mass generation as well as to the Higgs boson mass and self-coupling.

We have calculated the explicit expressions for the one- and two-loop terms which saturate the NNLL corrections to the basic observables in the TeV region. In general, the two-loop NLL and NNLL corrections exceed the LL contribution in the TeV region due to the numerically small coefficient in front of the double logarithmic terms. Hence the truly asymptotic behavior sets in only at an significantly higher energy. At the same time the two-loop NNLL corrections are numerically of the same magnitude but slightly smaller than the NLL ones, both being in the range of 1%-10%. This could be considered as a signal of convergence of the logarithmic expansion at TeV energies. Indeed, the two-loop coefficients in front of $\ln^2(s/M^2)$ is (a few units)$\times\alpha^2$. This is not an unusually large value in a non-Euclidean regime where the expansion parameter is $\alpha$, rather then $\alpha/(4\pi)$ as can be seen in eqs. (33)–(35) (see also [22, 29, 31]). Moreover, we have observed a significant cancellation between the two-loop LL, NLL and NNLL terms. As a result of this cancellation the sum of these two-loop corrections to the cross sections is of order $1-2\%$ for all the processes.

Thus, if we assume no further growth of the coefficient for the single logarithmic and non-logarithmic two-loop terms and the observed pattern of cancellation to hold, we would argue that our NNLL result approximates the exact cross sections with 1% accuracy. The accuracy is less for the production of third generation quarks where we cannot neglect the large Yukawa coupling to the Higgs boson that modifies the NLL and NNLL terms in our formulae.

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