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Analytical Result for Dimensionally Regularized Massless Master Non-planar Double Box with One Leg off Shell

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Abstract

The dimensionally regularized massless non-planar double box Feynman diagram with powers of propagators equal to one, one leg off the mass shell, i.e. with $p_1^2 = q^2 \neq 0$, and three legs on shell, $p_i^2 = 0,\ i = 2, 3, 4$, is analytically calculated for general values of q^2 and the Mandelstam variables s, t and u (not necessarily restricted by the physical condition $s + t + u = q^2$). An explicit result is expressed through (generalized) polylogarithms, up to the fourth order, dependent on rational combinations of q^2, s, t and u, and simple finite two- and three fold Mellin–Barnes integrals of products of gamma functions which are easily numerically evaluated for arbitrary non-zero values of the arguments.

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1. Feynman diagrams with four external lines contribute to many important physical quantities. They are rather complicated mathematical objects because depend at least on two independent Mandelstam variables. In the pure massless case with all end-points on shell, i.e., $p_i^2 = 0,\ i = 1, 2, 3, 4$, the problem of the analytical evaluation of two-loop four point diagrams (when the planar and non-planar double boxes shown in Fig. 1 are mostly complicated), in expansion in $\epsilon = (4 - d)/2$ in the framework

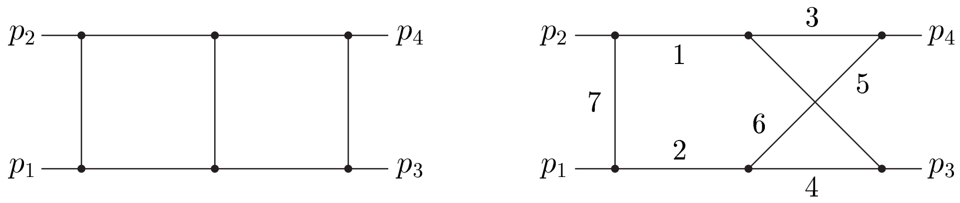

Figure 1: The planar and non-planar double boxes

of dimensional regularization [1] with the space-time dimension d as a regularization parameter, was completely solved during last year in [2, 3, 4, 5]. The corresponding analytical algorithms were quite recently applied to the evaluation of two-loop virtual corrections to $e^+e^- \to \mu^+\mu^-$ and Bhabba scattering [6] and quark scattering [7].

In the massless case with one-leg off shell, $p_1^2 = q^2 \neq 0,\ p_i^2 = 0,\ i = 2, 3, 4$, which is relevant to the process $e^+e^- \to$ 3jets (see, e.g., [8]), the planar double box diagram with powers of propagators equal to one has been analytically calculated in [9] for general values of q^2 and the Mandelstam variables s and t. An explicit result is expressed through (generalized) polylogarithms, up to the fourth order, dependent on rational combinations of q^2, s and t, and a one-dimensional integral with a simple integrand consisting of logarithms and dilogarithms. This result represented through so-called two-dimensional harmonical polylogarithms was confirmed in [10] where the method of differential equations [11, 12] was applied and some other planar master Feynman diagrams with one leg off shell were also evaluated.

There are two different non-planar Feynman diagrams (see Fig. 1) with all powers of propagators equal to one: when $p_1^2 \neq 0$ (or $p_2^2 \neq 0$) and when $p_3^2 \neq 0$ (or $p_4^2 \neq 0$). The purpose of this paper is to analytically evaluate the first of these two master integrals. As in the pure on-shell case (see [4]) it turns out that it is natural to consider non-planar double boxes as functions of $s, t, u = (p_1 + p_4)^2$ and $q^2 = p_1^2$ not necessarily restricted by the physical condition $s + t + u = q^2$ which does not simplify the result. We shall present an explicit result for the considered master non-planar double box in terms of (generalized) polylogarithms, up to the fourth order, dependent on rational combinations of q^2, s, t and u, and simple finite two- and three fold Mellin–Barnes (MB) integrals of products of gamma functions which are easily numerically evaluated for arbitrary non-zero values of the arguments.

To arrive at this result we straightforwardly apply the method of refs. [2, 9]: we start from the alpha-representation of the double box and, after expanding some of the involved functions in MB integrals, arrive at a five-fold MB integral representation with gamma functions in the integrand. Then we use a standard procedure of taking residues and shifting contours[2] to resolve the structure of singularities in the parameter of dimensional regularization, ϵ. This leads to the appearance of multiple terms where Laurent expansion in ϵ becomes possible and provides our result.

Finally we discuss the problem of systematical evaluation of general double boxes with one leg off shell and the most general class of functions that can appear in the results.

2. The alpha representation of the non-planar double box with one leg off shell, $p_1^2 = q^2 \neq 0$ differs from its on-shell variant (see, e.g., eqs. (4)–(6) of ref. [4]) by a term depending on q^2 inserted into one of the functions involved:

$$F(s,t,u,q^2;\epsilon) = -\Gamma(3+2\epsilon)\left(i\pi^{d/2}\right)^2 \int_0^\infty \mathrm{d}\alpha_1 \ldots \int_0^\infty \mathrm{d}\alpha_7 \delta\left(\sum \alpha_i - 1\right) D^{1+3\epsilon} A^{-3-2\epsilon} \,, \quad (1)$$

where

$$\begin{aligned} D &= (\alpha_1+\alpha_2+\alpha_7)(\alpha_3+\alpha_4+\alpha_5+\alpha_6) + (\alpha_3+\alpha_5)(\alpha_4+\alpha_6)\,, & (2)\\ A &= [\alpha_1\alpha_2(\alpha_3+\alpha_4+\alpha_5+\alpha_6) + \alpha_1\alpha_4\alpha_5 + \alpha_2\alpha_3\alpha_6](-s) \\ &\quad +\alpha_5\alpha_6\alpha_7(-t) + \alpha_3\alpha_4\alpha_7(-u) + \alpha_7[\alpha_4\alpha_5 + \alpha_2(\alpha_3+\alpha_4+\alpha_5+\alpha_6)](-q^2)\,. & (3)\end{aligned}$$

By introducing six times MB representation

$$\frac{1}{(X+Y)^\nu} = \frac{1}{\Gamma(\nu)} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \mathrm{d}w \frac{Y^w}{X^{\nu+w}} \Gamma(\nu+w)\Gamma(-w)\,, \quad (4)$$

in a suitable way, one can take all the parametrical integrals in gamma functions. Fortunately, one of the MB integrations is then explicitly performed by use of the first Barnes lemma and we arrive at the following 5-fold MB integral:

$$\begin{aligned} F(s,t,q^2;\epsilon) &= -\frac{\left(i\pi^{d/2}\right)^2 \Gamma(-\epsilon)^2}{\Gamma(-1-3\epsilon)\Gamma(-2\epsilon)(-s)^{3+2\epsilon}} \frac{1}{(2\pi i)^5} \int \mathrm{d}v \mathrm{d}w_1 \mathrm{d}w_2 \mathrm{d}z_1 \mathrm{d}z_2 \left(\frac{q^2}{s}\right)^v \\ &\times \left(\frac{t}{s}\right)^{w_1} \left(\frac{u}{s}\right)^{w_2} \frac{\Gamma(3+2\epsilon+v+w_1+w_2+z_1)\Gamma(3+2\epsilon+w_1+w_2+z_1+z_2)}{\Gamma(3+2\epsilon+w_1+w_2+z_1)\Gamma(2+w_1+w_2+z_1+z_2)^2} \\ &\times\Gamma(2+\epsilon+w_1+w_2+z_1+z_2)\Gamma(1+v+w_1+w_2)\Gamma(1+w_1+z_1)\Gamma(1+w_2+z_1) \\ &\times\Gamma(1+w_1+z_2)\Gamma(1+w_2+z_2)\Gamma(-2-2\epsilon-v-w_1-w_2-z_1) \\ &\times\Gamma(-2-2\epsilon-w_1-w_2-z_2)\Gamma(-v)\Gamma(-w_1)\Gamma(-w_2)\Gamma(-z_1)\Gamma(-z_2)\,. \quad (5)\end{aligned}$$

It differs from its analog for $q^2 = 0$ (see eqs. (8)–(10) of ref. [4]) by the additional integration in v. This variable enters only four gamma functions in the integrand.

[2] The procedures presented in [2] and [4] are a little bit different. In the latter variant, one systematically uses integration in MB integrals in straight lines along imaginary axes. We prefer to use the former variant.

When taking minus residue of the integrand in v one reproduces the corresponding 4-fold MB integral in [4]. Integral (5) is evaluated in expansion in ϵ, up to a finite part, by resolving singularities in ϵ in the same style as in [9]. We obtain and confirm as a by-product the on-shell result of ref. [4].

To present the final result let us turn to the variables $x = s/q^2$, $y = t/q^2$ and $z = u/q^2$:

$$F(s,t,u,q^2;\epsilon) = \frac{\left(i\pi^{d/2} \mathrm{e}^{-\gamma_{\mathrm{E}}\epsilon}\right)^2}{s^2 tu(-q^2)^{2\epsilon-1}} \sum_{i=0}^{4} \frac{f_i(x,y,z)}{\epsilon^i} + O(\epsilon) \ . \tag{6}$$

We obtain

$$\begin{aligned}
f_4(x,y,z) &= \frac{1}{4}(1-x-yz) - \frac{3}{8}(z+y) \ , &(7)\\
f_3(x,y,z) &= -\frac{1}{4}\left[3(1+x+y+z) + (2-2x-3y-3z-2yz)l_x\right.\\
&\quad \left. +(2-2x+y-3z+2yz)l_y + (2-2x-3y+z+2yz)l_z\right] \ , &(8)\\
f_2(x,y,z) &= 3(x-1)\mathrm{Li}_2\,(x) + 2(1-y)z\mathrm{Li}_2\,(y) + 2y(1-z)\mathrm{Li}_2\,(z)\\
&+\frac{1}{4}\Big[(2-2x-3y-3z-2yz)l_x^2 + (2-2x+y-3z+2yz)l_y^2\\
&+(2-2x-3y+z+2yz)l_z^2\Big] - 3(1-x)\bar{l}_x l_x + 2z(1-y)\bar{l}_y l_y + 2y(1-z)\bar{l}_z l_z\\
&+\frac{1}{2}\left[(2-2x+y-3z+2yz)l_x l_y + (2-2x+z-3y+2yz)l_x l_z\right.\\
&\left.+(2-2x+y+z-2yz)l_y l_z\right] + \frac{3}{2}\left[(1-x+y+z)l_x + (1+x-y+z)l_y\right.\\
&\left.+(1+x+y-z)l_z\right] + \frac{\pi^2}{48}(30(1-x)+11(y+z)+26yz) + 3(1+x+y+z) \ , &(9)\\
f_1(x,y,z) &= 3(1-x)(2\mathrm{Li}_3\,(x) + 5\mathrm{Li}_3\,(1-x)) + 2(2-2x+4y-z-3yz)\mathrm{Li}_3\,(y)\\
&-3(1-y)z\mathrm{Li}_3\,(1-y) + 2(2-2x-y+4z-3yz)\mathrm{Li}_3\,(z) - 3y(1-z)\mathrm{Li}_3\,(1-z)\\
&-3(1-x)(5\bar{l}_x - 2(1+l_y+l_z))\mathrm{Li}_2\,(x) - 15(1-x)\bar{l}_x\mathrm{Li}_2\,(1-x)\\
&+((1-y)(6-4zl_x+3z\bar{l}_y+4zl_z) - (2-2x+4y+z-5yz)l_y)\mathrm{Li}_2\,(y)\\
&+((1-z)(6-4yl_x+3y\bar{l}_z+4yl_y) - (2-2x+y+4z-5yz)l_z)\mathrm{Li}_2\,(z)\\
&+3(1-y)z\bar{l}_y\mathrm{Li}_2\,(1-y) + 3y(1-z)\bar{l}_z\mathrm{Li}_2\,(1-z)\\
&-\frac{1}{6}(2-2x-3y-3z-2yz)l_x^3 + 3(1-x)\bar{l}_x l_x^2 - 15(1-x)\bar{l}_x^2 l_x\\
&-\frac{1}{6}(2-2x+y-3z+2yz)l_y^3 - 2(1-y)z\bar{l}_y l_y^2 + 3(1-y)z\bar{l}_y^2 l_y\\
&-\frac{1}{6}(2-2x-3y+z+2yz)l_z^3 - 2y(1-z)\bar{l}_z l_z^2 + 3y(1-z)\bar{l}_z^2 l_z\\
&-\frac{1}{2}\left[(2-2x+y-3z+2yz)l_x l_y(l_x+l_y) + (2-2x-3y+z+2yz)l_x l_z(l_x+l_z)\right.\\
&\left.+(2-2x+y+z-2yz)l_y l_z(l_y+l_z)\right] - (2-2x+y+z-2yz)l_x l_y l_z
\end{aligned}$$

$$
\begin{aligned}
&+6(1-x)\bar{l}_x l_x(l_y+l_z)+4(1-y)z\bar{l}_y l_y(l_z-l_x)-4y(1-z)\bar{l}_z l_z(l_x-l_y)\\
&-\frac{3}{2}\Big[(1-x+y+z)l_x^2-4(1-x)\bar{l}_x l_x+(1+x-y+z)l_y^2-4(1-y)\bar{l}_y l_y\\
&+(1+x+y-z)l_z^2-4(1-z)\bar{l}_z l_z\Big]\\
&-3((1-x-y+z)l_x l_y+(1+x-y-z)l_y l_z+(1-x+y-z)l_x l_z)\\
&-\frac{\pi^2}{24}\left[(30-30x+11y+11z+26yz)l_x+(30-30x+3y+11z-14yz)l_y\right.\\
&\left.-60(1-x)\bar{l}_x+12(1-y)z\bar{l}_y+(30-30x+11y+3z-14yz)l_z+12y(1-z)\bar{l}_z\right]\\
&-6\left[(1-x+y+z)l_x+(1+x-y+z)l_y+(1+x+y-z)l_z\right]-12(1+x+y+z)\\
&-\frac{\zeta(3)}{6}\left[52(1-x)-63(y+z)-40yz\right]-\frac{\pi^2}{8}(11-5(x+y+z))\;, \qquad (10)
\end{aligned}
$$

where $l_a=\ln a$ and $\bar{l}_a=\ln(1-a)$ for $a=x,y,z$. Moreover $\mathrm{Li}_a(z)$ is the polylogarithm [13] and (in the next formula)

$$
S_{a,b}(z)=\frac{(-1)^{a+b-1}}{(a-1)!b!}\int_0^1\frac{\ln^{a-1}(t)\ln^b(1-zt)}{t}\mathrm{d}t \qquad (11)
$$

the generalized polylogarithm [14].

The finite part $f_0=\bar{f}_0+\tilde{f}_0$ consists of two pieces:

$$
\begin{aligned}
\bar{f}_0(x,y,z) \;=\;& 30(1-x)S_{2,2}(x)-2(1-y)(6-7z)S_{2,2}(y)-2(1-z)(6-7y)S_{2,2}(z)\\
&+6(x-yz)\mathrm{Li}_4(yz/x)-12(1-x)\mathrm{Li}_4(x)\\
&-2(3-6x-3y-z+4yz)\mathrm{Li}_4(y)-2(3-3x-y-3z+4yz)\mathrm{Li}_4(z)\\
&+69(1-x)\mathrm{Li}_4(1-x)-3(1-y)z\mathrm{Li}_4(1-y)-3y(1-z)\mathrm{Li}_4(1-z)\\
&-\frac{1}{2}\Big[(14+10y-7z+yz)\mathrm{Li}_2(y)^2+(14-2x-7y+10z+yz)\mathrm{Li}_2(z)^2\Big]\\
&-6(1-x)\Big[(2-5\bar{l}_x+2l_y+2l_z)\mathrm{Li}_3(x)+(4+5l_y+5l_z)\mathrm{Li}_3(1-x)\Big]\\
&-\big[4(2+x+4y-z-3yz)l_x+2(1+5y-5yz)l_y+4(2-x+4y+z+3yz)l_z\\
&+2(1-y)(6-7z)\bar{l}_y+12(1-y)\big]\mathrm{Li}_3(y)\\
&-(6(1-y)(4-l_x z+l_z z)+(20+4y-17z+11yz)l_y)\mathrm{Li}_3(1-y)\\
&-\big[4(2+x-y+4z-3yz)l_x+4(2-2x+y+4z+3yz)l_y+2(1-x+5z-5yz)l_z\\
&+2(6-7y)(1-z)\bar{l}_z+12(1-z)\big]\mathrm{Li}_3(z)-\big[6(4-l_x y+l_y y)(1-z)\\
&+(20-2x-17y+4z+11yz)l_z\big]\mathrm{Li}_3(1-z)+6(l_x-l_y-l_z)(x-yz)\mathrm{Li}_3(yz/x)\\
&-\frac{3}{2}(1-x)\Big[4(l_y+l_z+1)^2+\pi^2+12\Big]\mathrm{Li}_2(x)\\
&+\Big[4(1-y)z(l_x-l_z)^2+(2-2x+4y+z-5yz)l_y^2+2(2+x+4y+z-5yz)l_x l_y\\
&+2(2-x+4y-z+5yz)l_y l_z-\bar{l}_y l_y(14+10y-7z+yz)-12(1-y)(l_x+l_z)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\pi^2}{6}(22y+15z-21yz)-24(1-y)+\frac{\pi^2}{3}\Bigg]\,\mathrm{Li}_2\,(y)\\
&+\Big[4y(1-z)(l_x-l_y)^2+(2-2x+y+4z-5yz)l_z^2+2(2+x+y+4z-5yz)l_xl_z\\
&+2(2-2x-y+4z+5yz)l_yl_z-(14-2x-7y+10z+yz)\bar{l}_zl_z-12(1-z)(l_x+l_y)\\
&-\frac{\pi^2}{6}(2x-15y-22z+21yz)-24(1-z)+\frac{\pi^2}{3}\Bigg]\,\mathrm{Li}_2\,(z)\\
&+(x-yz)(3(l_x-l_y-l_z)^2+4\pi^2)\mathrm{Li}_2\,(yz/x)\\
&+\frac{1}{12}(2-2x-3y-3z+yz)l_x^4-2(1-x)l_x^3\bar{l}_x+\frac{15}{2}(1-x)l_x^2\bar{l}_x^2\\
&+\frac{1}{12}(2-2x+y-3z+5yz)l_y^4+\frac{4}{3}(1-y)zl_y^3\bar{l}_y-(10+2y-7z+4yz)l_y^2\bar{l}_y^2\\
&+\frac{1}{12}(2-2x-3y+z+5yz)l_z^4+\frac{4}{3}y(1-z)l_z^3\bar{l}_z-(10-x-7y+2z+4yz)l_z^2\bar{l}_z^2\\
&+\frac{1}{3}(2-2x+y-3z-yz)l_xl_y(l_x^2+l_y^2)+\frac{1}{2}(2-2x+y-3z+5yz)l_x^2l_y^2\\
&+\frac{1}{3}(2-2x-3y+z-yz)l_xl_z(l_x^2+l_z^2)+\frac{1}{2}(2-2x-3y+z+5yz)l_x^2l_z^2\\
&+\frac{1}{3}(2-2x+y+z+yz)l_yl_z(l_y^2+l_z^2)+\frac{1}{2}(2-2x+y+z+9yz)l_y^2l_z^2\\
&-6(1-x)\bar{l}_xl_x(l_y+l_z)(l_x+l_y+l_z)\\
&+4(1-y)z\bar{l}_yl_y(l_x-l_z)(l_x+l_y-l_z)+4y(1-z)\bar{l}_z(l_x-l_y)l_z(l_x-l_y+l_z)\\
&+\left[(2-2x+y+z+yz)l_x+(2-2x+y+z-5yz)(l_y+l_z)\right]l_xl_yl_z\\
&-(x-yz)(l_x-l_y-l_z)^3\ln(1-yz/x)\\
&+(1-x+y+z)l_x^3-6(1-x)l_x^2\bar{l}_x+(1+x-y+z)l_y^3-6(1-y)l_y^2\bar{l}_y\\
&+(1+x+y-z)l_z^3-6(1-z)l_z^2\bar{l}_z+3((1-x-y+z)l_xl_y(l_x+l_y)\\
&+(1-x+y-z)l_xl_z(l_x+l_z)+(1+x-y-z)l_yl_z(l_y+l_z))+6(1-x-y-z)l_xl_yl_z\\
&-12\left[l_x\bar{l}_x(l_y+l_z)(1-x)+l_y\bar{l}_y(l_x+l_z)(1-y)+(1-z)(l_x+l_y)l_z\bar{l}_z\right]\\
&+\pi^2\Bigg\{\frac{1}{24}(30-30x+11z+y(11+74z))l_x^2-\frac{13}{2}(1-x)l_x\bar{l}_x\\
&+\frac{1}{24}(30-30x+3y+11z+98yz)l_y^2+\frac{1}{6}(14+y(10-7z)+z)l_y\bar{l}_y\\
&+\frac{1}{24}(30-30x+11y+3z+98yz)l_z^2+\frac{1}{6}(14-2x+y+10z-7yz)l_z\bar{l}_z\\
&+\frac{1}{12}\left[(30(1-x)+3y+11z-62yz)l_xl_y+(30(1-x)+11y+3z-62yz)l_xl_z\right.\\
&\left.+(30(1-x)+3y+3z+50yz)l_yl_z\right]-4(l_x-l_y-l_z)(x-yz)\ln(1-yz/x)\Bigg\}\\
&+6\Big[(1-x+y+z)l_x^2-4(1-x)l_x\bar{l}_x+(1+x-y+z)l_y^2-4(1-y)l_y\bar{l}_y\\
&+(1+x+y-z)l_z^2-4(1-z)l_z\bar{l}_z\Big]
\end{aligned}
$$

$$
\begin{aligned}
&+12((1-x-y+z)l_xl_y+(1-x+y-z)l_xl_z+(1+x-y-z)l_yl_z)\\
&+\zeta(3)\left[-\frac{1}{3}(38(1-x)+63(y+z)+40yz)l_x-30(1-x)\bar{l}_x\right.\\
&+\frac{4}{3}(28-13x+14y-24z+10yz)l_y+2(1-y)(6-7z)\bar{l}_y\\
&\left.+\frac{2}{3}(56-29x-48y+28z+20yz)l_z+2(6-7y)(1-z)\bar{l}_z\right]\\
&+\frac{\pi^2}{4}\left[(11(1-x)-5(y+z))l_x+(11(1-y)-5(x+z))l_y+(11(1-z)-5(x+y))l_z\right]\\
&+24((1-x+y+z)l_x+(1+x-y+z)l_y+(1+x+y-z)l_z)\\
&+48(1+x+y+z)+\frac{\pi^2}{2}(11-5(x+y+z))+4(17-x-y-z)\zeta(3)\\
&+\frac{\pi^4}{2880}(894(1-x)+395(y+z)+10034yz) \qquad (12)
\end{aligned}
$$

and the following two- and three-fold MB integrals:

$$
\begin{aligned}
\tilde{f}_0(x,y,z) \;=\; &-\frac{6}{(2\pi i)^3}\int \mathrm{d}v\mathrm{d}w_1\mathrm{d}w_2\; x^{-1-v-w_1-w_2}y^{1+w_1}z^{1+w_2}\Gamma(3+v+w_1+w_2)\\
&\times\Gamma(1+v+w_1+w_2)\Gamma(1+w_1)\Gamma(-w_1)\Gamma(-1-w_1)\Gamma(-1-v-w_1)\\
&\times\Gamma(1+w_2)\Gamma(-w_2)\Gamma(-1-w_2)\Gamma(-1-v-w_2)\\
&+\frac{6}{(2\pi i)^2}\int \mathrm{d}v\mathrm{d}w\; x^{-w}(y^{1+w}z^{-v}+z^{1+w}y^{-v})\\
&\times\Gamma(1+v)\Gamma(v)\Gamma(-v)\Gamma(1+w)^2\Gamma(w)\Gamma(-w)^2\Gamma(-1-v-w)\\
+\frac{6}{(2\pi i)^2}&\int \mathrm{d}v\mathrm{d}w\; x^{-v-w}(y^{1+w}+z^{1+w})\frac{\Gamma(-v)}{v}\Gamma(1+w)\Gamma(-1-w)^2\Gamma(1+v+w)\\
\times\Gamma(v+w)&\Gamma(-v-w)\left[1+v+w+v(1+w)(\psi(1-v)-\psi(1+w))\right]\\
+\frac{2}{(2\pi i)^2}&\int \mathrm{d}v\mathrm{d}w\; y^{1+w}z^{-1-v-w}\left[(1+v+w)w((1-v)(v+w)x+vz)-5(1-v)vz\right]\\
\times\frac{\Gamma(-v)}{(1-v)v}&\Gamma(w)\Gamma(-w)\Gamma(-1-w)\Gamma(1+v+w)\Gamma(v+w)\Gamma(-1-v-w)\;. \qquad (13)
\end{aligned}
$$

The integration contours in the above 3-fold MB integral are in straight lines along imaginary axes with $-1 <\mathrm{Re}\, v$, $\mathrm{Re}\, w_{1,2}$, $\mathrm{Re}\,(v+w_{1,2})$, $\mathrm{Re}\,(v+w_1+w_2) < 0$ and, in 2-fold MB integrals, with $-1 <\mathrm{Re}\, v$, $\mathrm{Re}\, w$, $\mathrm{Re}\,(v+w) < 0$.

3. Our result (7)–(13) is in agreement with the leading power behaviour when $q^2 \to 0$ which is obtained by use of the strategy of regions of [15]. It also agrees with results based on numerical integration in the space of alpha parameters [16] (where the 1% accuracy for the $1/\epsilon$ and ϵ^0 parts is guaranteed). The result has been also checked with the value of the double box at $s = q^2$ which was evaluated by the same technique based on MB integrals. Note that $F(q^2, q^2, t, u)$ is expressed not only through (generalized) polylogarithms, as the on-shell dobule box in [4], but also through a two-fold MB integral that leads to a one-parametrical integral of dilogarithms and logarithms.

The two-fold MB integrals in (13) can be converted into a form similar to the two-fold MB integrals present in the planar double box [9], or, expressed through so-called two-dimensional harmonic polylogarithms [10] which generalize harmonic polylogarithms introduced in [17]. The three-fold MB integral in (13) can be written, up to some two-fold MB integrals, through the following three-fold series

$$\sum_{n,n_1,n_1=0}^{\infty} \frac{(n+n_1+1)!(n+n_2+1)!}{n!(n+2)!(n_1+1)^2(n_2+1)^2 n_1! n_2!} A_{n,n_1,n_2} x^n y^{n_1} z^{n_2} \,, \tag{14}$$

where A_{n,n_1,n_2} is a linear combination of $(\psi(n+n_{1,2}+1)^2 + \psi'(n+n_{1,2}+2))$, $\psi(n+n_{1,2}+2)$, and $\psi(n_{1,2}+1)$. If we replace A_{n,n_1,n_2} by one the corresponding series can be summed up with a result in terms of polylogarithms, up to Li_3, depending on certain rational combinations of x, y, z. The presence of A_{n,n_1,n_2} makes the situation much more complicated. The series can be summed up in the form of a two-parametrical integral of a cumbersome expression with polylogarithms. It is yet unclear where one of these parametrical integrations can be explicitly performed. We therefore prefer to leave the 3-fold MB integral in the result. This form can be easily used for numerical evaluation for arbitrary non-zero values of x, y and z.

Keeping in mind the characteristic feature of the non-planar diagrams mentioned above one can conclude that the presence of four scales (or, three dimensionless variables) in the problem shows that the two-dimensional harmonic polylogarithms (introduced in [10] just for the problem of the evaluation of the double boxes) could be insufficient to express all the results for the double boxes with one leg off shell, and one can imagine the necessity of introducing three-dimensional harmonic polylogarithms.

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References

[1] G. 't Hooft and M. Veltman, *Nucl. Phys.* B44 (1972) 189; C.G. Bollini and J.J. Giambiagi, *Nuovo Cim.* 12B (1972) 20.

[2] V.A. Smirnov, *Phys. Lett.* B460 (1999) 397.

[3] V.A. Smirnov and O.L. Veretin, *Phys. Lett.* B566 (2000) 469.

[4] J.B. Tausk, *Phys. Lett.* B469 (1999) 225.

[5] C. Anastasiou, E.W.N. Glover and C. Oleari, *Nucl. Phys.* B575 (2000) 416; C. Anastasiou, T. Gehrmann, C. Oleari, E. Remiddi and J.B. Tausk, *Nucl. Phys.* B580 (2000) 577; T. Gehrmann and E. Remiddi, *Nucl. Phys. Proc. Suppl.* 89 (2000) 251; C. Anastasiou, J.B. Tausk and M.E. Tejeda-Yeomans, *Nucl. Phys. Proc. Suppl.* 89 (2000) 262.

[6] Z. Bern, L. Dixon and A. Ghinculov, hep-ph/0010075.

[7] C. Anastasiou, E.W.N. Glover, C. Oleari and M.E. Tejeda-Yeomans, hep-ph/0010212.

[8] R.K. Ellis, D.A. Ross and A.E. Terrano, *Nucl. Phys.* B178 (1981) 421.

[9] V.A. Smirnov, *Phys. Lett.* B491 (2000) 130.

[10] T. Gehrmann and E. Remiddi, hep-ph/0008287.

[11] A.V. Kotikov, *Phys. Lett.* B254 (1991) 158; E. Remiddi, *Nuovo Cim.* 110A (1997) 1435.

[12] T. Gehrmann and E. Remiddi, *Nucl. Phys.* B580 (2000) 485.

[13] L. Lewin, *Polylogarithms and associated functions* (North Holland, 1981).

[14] K.S. Kolbig, J.A. Mignaco and E. Remiddi, *B.I.T* 10 (1970) 38; K.S. Kolbig, *Math. Comp.* 39 (1982) 647; A. Devoto and D.W. Duke, *Riv. Nuovo Cim.* 7 (1984) 1.

[15] M. Beneke and V.A. Smirnov, *Nucl. Phys.* B522 (1998) 321; V.A. Smirnov and E.R. Rakhmetov, *Teor. Mat. Fiz.* 120 (1999) 64; V.A. Smirnov, *Phys. Lett.* B465 (1999) 226.

[16] T. Binoth and G. Heinrich, *Nucl. Phys.* B585 (2000) 741.

[17] E. Remiddi and J.A.M. Vermaseren, *Int. J. Mod. Phys.* A15 (2000) 725.