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Two-Loop Master Integrals for $\gamma^* \to 3$ Jets: The planar topologies

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Abstract

The calculation of the two-loop corrections to the three jet production rate and to event shapes in electron-positron annihilation requires the computation of a number of up to now unknown two-loop four-point master integrals with one off-shell and three on-shell legs. In this paper, we compute those master integrals which correspond to planar topologies by solving differential equations in the external invariants which are fulfilled by the master integrals. We obtain the master integrals as expansions in $\epsilon = (4-d)/2$, where d is the space-time dimension. The results are expressed in terms of newly introduced two-dimensional harmonic polylogarithms, whose properties are shortly discussed. For all two-dimensional harmonic polylogarithms appearing in the divergent parts of the integrals, expressions in terms of Nielsen's polylogarithms are given.

1 Introduction

The calculation of perturbative next-to-next-to-leading order corrections to $2 \to 2$ scattering or $1 \to 3$ decay processes is a yet outstanding task for many precision applications in particle physics phenomenology. Progress in this field was up to very recently hampered by difficulties in the calculation of the virtual two-loop corrections to the corresponding Feynman amplitudes.

Using dimensional regularisation [1–3] with $d = 4 - 2\epsilon$ dimensions as regulator for ultraviolet and infrared divergencies, the large number of different integrals appearing in the two-loop Feynman amplitudes for $2 \to 2$ scattering or $1 \to 3$ decay processes can be reduced to a small number of master integrals. The techniques used in these reduction are integration-by-parts identities [3–5] and Lorentz invariance [6]. A computer algorithm for the automatic reduction of all two-loop four-point integrals was described in [6].

For two-loop four-point functions with massless internal propagators and all legs on-shell, which are relevant for example in the next-to-next-to-leading order calculation of two jet production at hadron colliders, all master integrals have been calculated over the past year [7–12]. To compute the next-to-next-to-leading order corrections to observables such as the three jet production rate in electron-positron annihilation, two plus one jet production in deep inelastic electron-proton scattering or vector boson plus jet production at hadron colliders, one requires a different class of integrals: two-loop four-point functions with massless internal propagators and one external leg off-shell. Since these functions involve one scale more than their on-shell counterparts, one expects the corresponding master integrals to be more complicated, and also to be more numerous. First progress towards the computation of these master integrals has been made very recently by Smirnov, who computed the planar double box integral with all propagators raised to unit power [13]. Using a Mellin-Barnes contour integral technique, Smirnov obtained an analytic expression for the divergent parts and a one-dimensional integral representation for the finite term. Complementary work on the purely numerical evaluation of this type of master integrals has been presented recently by Binoth and Heinrich [14].

In this paper, we compute all master integrals appearing in the reduction of planar two-loop four-point functions with one external leg off-shell. The method employed here is substantially different from the techniques used in [13, 14]: for all master integrals under consideration, we derived inhomogeneous differential equations [6] in the external invariants. The master integrals are then determined by solving these equations in terms of newly introduced two-dimensional harmonic polylogarithms and subsequent matching of the solution to the boundary condition.

In Section 2, we briefly review the use of the differential equation approach to the computation of master integrals. To solve the differential equations for two-loop four-point master integrals with one off-shell leg, we employ an ansatz which is the product of a rational function of the kinematic invariants times the sum of two-dimensional harmonic polylogarithms (2dHPL). We explain how the unknown parameters in the ansatz are determined from the differential equation and its boundary condition. To illustrate how the method works in practice, we outline the calculation of two master integrals in Section 3.

The master integrals computed using this method are listed in Section 4, which also contains a reference list of simpler master integrals for planar two- and there-point functions, which appear in the reduction of the four-point functions considered here. Section 5 contains concluding remarks and an outlook on potential applications and open problems.

We enclose an Appendix which summarises the properties of harmonic polylogarithms [15] and extends the underlying formalism towards the newly introduced two-dimensional harmonic polylogarithms. In the Appendix, we also tabulate the relation between harmonic polylogarithms and the more commonly known Nielsen's generalised polylogarithms [16, 17]. We find that all (ordinary and two-dimensional) harmonic polylogarithms appearing in the divergent parts of the master integrals computed in this paper can be expressed in terms of generalised polylogarithms. The harmonic polylogarithms appearing in the finite parts of the master integrals can all be represented as one-dimensional integrals over ordinary Nielsen's polylogarithms up to weight three. Their precise numerical evaluation can be obtained by the techniques of [17].

2 Differential Equations for Master Integrals

The use of differential equations for the computation of master integrals was first suggested in [18] as means to relate integrals with massive internal propagators to their massless counterparts. The method was developed in detail in [19], where it was also extended towards differential equations in the external invariants. As a first application of this method, the two-loop sunrise diagram with arbitrary internal masses was studied in [20]. We have developed the differential equation formalism for two-loop four-point functions with massless internal propagators, three external legs on-shell and one external leg off-shell in [6]. We derived an algorithm for the automatic reduction by means of computer algebra (using FORM [21] and Maple [22]) of any two-loop four-point integral to a small set of master integrals, for which we derived differential equations in the external invariants. For all master integrals with up to five internal propagators, we solved the differential equations in a closed form for arbitrary number of space-time dimensions d. In the present work, we elaborate on these results by deriving Laurent expansions around $\epsilon = 0$ for all master integrals derived in [6] as well as all remaining planar master integrals with six or seven propagators.

Four-point functions depend on three linearly independent momenta: p_1, p_2 and p_3. In our calculation, we take all these momenta on-shell ($p_i^2 = 0$), while the fourth momentum $p_{123} = (p_1 + p_2 + p_3)$ is taken off-shell. The kinematics of the four-point function is then fully described by specifying the values of the three Lorentz invariants $s_{12} = (p_1 + p_2)^2$, $s_{13} = (p_1 + p_3)^2$ and $s_{23} = (p_2 + p_3)^2$.

Expressing the system of differential equations obtained in [6] for any master integral in the variables $s_{123} = s_{12} + s_{13} + s_{23}$, $y = s_{13}/s_{123}$ and $z = s_{23}/s_{123}$, we obtain a homogeneous equation in s_{123}, and inhomogeneous equations in y and z. Since s_{123} is the only quantity carrying a mass dimension, the corresponding differential equation is nothing but the rescaling relation obtained by investigating the behaviour of the master integral under a rescaling of all external momenta by a constant factor.

Some of the master integrals under consideration do not depend explicitly on all three invariants, but only on certain combinations of them, thus being one-scale or two-scale integrals. The one-scale integrals can not be determined using the differential equation method, since the only non-trivial differential equation fulfilled by them is the homogeneous rescaling relation. These integrals are relatively simple and can all be computed using Feynman parameters. They were known in the literature for a long time [23, 24]. We list them in Section 4.1 for completeness. The two-scale integrals fulfil, besides the rescaling relation, one inhomogeneous differential equation, which can be employed for their computation. These integrals have all been computed in [6, 25–27], where they are given in a closed form for arbitrary ϵ. We list their ϵ-expansions in Section 4.2.

The solution of differential equations for two-loop four-point functions with one off-shell leg in terms of hypergeometric functions, was discussed in detail in [6]. Although this method yields at first sight very compact results for arbitrary ϵ, it is not very well suited for practical applications, where an expansion around $\epsilon = 0$ is needed. The major problem with this approach is the appearance of generalised hypergeometric functions which can be expressed only in terms of multidimensional integrals. The evaluation of these multiple integrals to the required order of the ϵ-expansion is a problem comparable to the direct evaluation of the master integral from its Feynman parameter representation. To solve the differential equations for the three-scale integrals, we choose therefore an approach different from the one pursued in [6].

The procedure employed here is as follows. In the y and z differential equations for the master integral under consideration, the coefficient of the homogeneous term as well as the full inhomogeneous term (coefficients and subtopologies) are expanded as a series in ϵ. For the master integral, we make an ansatz which has the form

$$\sum_i \mathcal{R}_i(y, z; s_{123}, \epsilon)\mathcal{H}_i(y, z; \epsilon) , \tag{2.1}$$

where the prefactor $\mathcal{R}_i(y, z; s_{123}, \epsilon)$ is a rational function of y and z, which is multiplied with an overall normalisation factor to account for the correct dimension in s_{123}, while $\mathcal{H}_i(y, z; \epsilon)$ is a Laurent series in ϵ. The coefficients of its ϵ-expansion are then written as the sum of 2dHPL up to a weight determined by the order of the series:

$$\mathcal{H}_i(y,z;\epsilon) = \frac{\epsilon^p}{\epsilon^4}\sum_{n=0}^{4}\epsilon^n\left[T_n(z) + \sum_{j=1}^{n}\sum_{\vec{m}_j\in V_j(z)} T_{n,\vec{m}_j}(z)H(\vec{m}_j;y)\right] . \tag{2.2}$$

This form is motivated by the expectation that two-loop four-point functions can diverge up to $1/\epsilon^4$, as obtained by considering soft and collinear limits of internal propagators [28]; p is an – a priori unknown – integer number, accounting for the fact that some of the master integrals might have a lower superficial degree of divergence. $V_j(z)$ is the set of all possible indices for 2dHPL of weight j (j-dimensional vectors made from the components $(0,1,1-z,z)$). The $T_n(z)$ and $T_{n,\vec{m}}(z)$ are unknown coefficients, which can contain in turn ordinary one-dimensional HPL depending on z. With the ansatz (2.2), the leading $1/\epsilon^4$-singularity corresponds to $p=0$ and $n=0$.

The rational functions $\mathcal{R}_i(y,z;s_{123},\epsilon)$ can be determined from the homogeneous part of the differential equations in y and z by inserting only the constant $n=0$ term of $\mathcal{H}_i(y,z;\epsilon)$.

It turns out that for all topologies which contain a single master integral, only one $\mathcal{R}(y,z;s_{123},\epsilon)$, and consequently also only one $\mathcal{H}(y,z;\epsilon)$, are present, such that the sum in (2.1) is not needed. For both topologies containing two master integrals, only one $\mathcal{R}(y,z;s_{123},\epsilon)$ and $\mathcal{H}(y,z;\epsilon)$ per master integral are sufficient, provided that the basis of first and second master integral is chosen appropriately, as explained later in this Section.

Having determined the prefactor $\mathcal{R}(y,z;s_{123},\epsilon)$ for the master integral under consideration, we rewrite the y differential equation for the master integral into a y differential equation for $\mathcal{H}(y,z;\epsilon)$. From the ϵ-expansion of the inhomogeneous term, one can read off the value of p, required to match the order of the Laurent series of the master integral to the Laurent series of the inhomogeneous term.

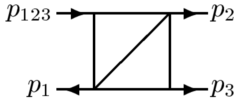
By construction of the 2dHPL, their derivative with respect to y is straightforward (see Appendix A.2):

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}y}H(m;y) &= f(m;y)\,,\\ \frac{\mathrm{d}}{\mathrm{d}y}H(m,\vec{m}_{j-1};y) &= f(m;y)H(\vec{m}_{j-1};y)\,,\end{aligned} \tag{2.3}$$

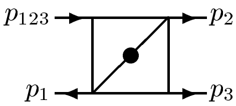
where the $f(m;y)$ are nothing but the y-dependent denominators present in the differential equation. Inserting the right hand side of (2.2) into the y differential equation for $\mathcal{H}(y,z;\epsilon)$ and differentiating according to the above rules, the differential equation becomes a purely algebraic equation. Grouping the different inverse powers of y, $(1-y)$, $(1-y-z)$ and $(y+z)$ and using the linear independence of the $H(\vec{m}_j;y)$, this algebraic equation can be translated into a linear system of equations for the coefficients $T_{n,\vec{m}_j}(z)$. The resulting system of typically several hundreds of equations can be solved for the $T_{n,\vec{m}_j}(z)$ using computer algebra (here we employ the same algorithm as used already in [6] for solving large numbers of integration-by-parts and Lorentz-invariance identities). The $T_n(z)$ do not multiply any y-dependent function in the ansatz and can therefore not be determined in the above way from the differential equation. They correspond to the boundary condition.

The boundary terms $T_n(z)$ are determined by exploiting the fact that all planar master integrals as well as their derivatives are analytic in the whole kinematical plane with the exception of the two branch points at $y=0$ and $z=0$. As a consequence, any factor $(1-y)$, $(1-y-z)$ or $(y+z)$ appearing in the denominator of the homogeneous term of the differential equations for a master integral can be used to determine the boundary condition in $y=1$, $y=(1-z)$ or $y=-z$ respectively: multiplying the differential equation with this factor and taking the limit where the factor vanishes, one obtains a linear relation between the master integral in this special kinematical point and its subtopologies, no longer involving any derivatives.

Some special care has to be taken in the case of topologies with two master integrals. Our choice of basis for the master integrals in these cases is as follows. For the five-propagator (diagonal box) master integrals, we use the basis already employed in [6]:

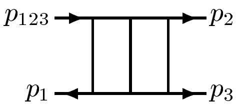


$$= \int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{\mathrm{d}^d l}{(2\pi)^d}\frac{1}{k^2(k-p_{123})^2(k-l-p_2)^2 l^2(l-p_3)^2}, \tag{2.4}$$

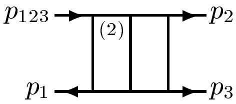


$$= \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{1}{k^2 (k-p_{123})^2 (k-l-p_2)^4 l^2 (l-p_3)^2}. \tag{2.5}$$

For the seven-propagator double box integrals, we use:



$$= \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{1}{k^2 (k-p_{23})^2 (k-p_{123})^2 (k-l)^2 l^2 (l-p_2)^2 (l-p_{23})^2}, \tag{2.6}$$



$$= \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{(k-p_2)^2}{k^2 (k-p_{23})^2 (k-p_{123})^2 (k-l)^2 l^2 (l-p_2)^2 (l-p_{23})^2}. \tag{2.7}$$

This choice of basis for the double box integrals has been first suggested by Anastasiou, Tausk and Tejeda-Yeomans in [12] for the double box integrals with all legs on-shell. We find it to be convenient in the off-shell case as well. Apart from yielding a compact expression for the second master integral, this choice also circumvents problems with subleading terms [29] appearing in the reduction of tensor integrals.

For both topologies, one obtains coupled sets of differential equations for the two master integrals. If only the $n = 0$ term of $\mathcal{H}_i(y, z; \epsilon)$ is retained, the equations decouple, thus enabling the determination of the $\mathcal{R}_i(y, z; s_{123}, \epsilon)$ for each of the two master integrals. The boundary conditions for both master integrals are determined as above by using analyticity properties in the kinematical variables. Boundary conditions in two kinematical points are needed to fully constrain both master integrals. In the case of the seven propagator double box integrals (2.6,2.7), only one of those can be obtained in $y = (1 - z)$, no other y-dependent denominator corresponding to an analytic point is present in the homogeneous term. To find the second boundary condition, we first determine the z-dependence of the $T_n(z)$ by investigating their z differential equations, obtained from the y and z differential equations of the master integrals in $y = (1-z)$. The remaining constant terms in $T_n(z)$ are then obtained by using the fact that the master integrals are regular in $z = 1$. We would like to point out that this procedure relies crucially on the possibility to vary y and z independently, which is not the case for the on-shell double box integral, where the kinematical properties of the master integrals are insufficient [11] to determine both boundary conditions from the differential equation alone.

Following the steps described in this section, we have derived the ϵ-expansions of all planar four-point master integrals with one off-shell leg. It is obvious that this calculation had to follow a bottom-up approach starting from master integrals with few different denominators, which subsequently appeared in the inhomogeneous term of the differential equations for master integrals with more different denominators.

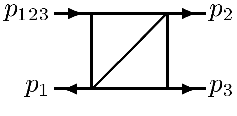
As a check on all three-scale integrals obtained with this method, the results were inserted into the corresponding z differential equation. After transforming the $H(\vec{m}_j \in V_j(z); y)$ into $H(\vec{m}_j \in V_j(y); z)$, the z derivatives can be carried out straightforwardly, thus providing a check on our results.

In the following Section, we illustrate on the example of the master integrals (2.4,2.5) how our method is applied in practice. In Section 4, we list our results for all master integrals appearing in the reduction of any planar two-loop four-point integral with one off-shell leg.

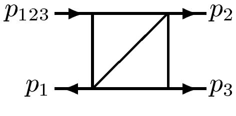
3 Computing a Master Integral from its Differential Equation

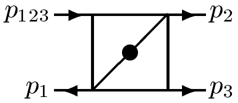
The diagonal box master integrals (2.4,2.5) are symmetric under the interchange of $p_1 \leftrightarrow p_2$, and consequently symmetric under $y \leftrightarrow z$. The coupled set of y differential equations for them reads:

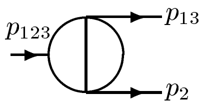
$$\frac{\partial}{\partial y}$$

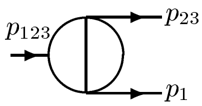


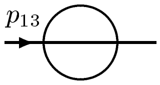
$$= \left[-\frac{1}{y+z} + (d-4) \left(\frac{1}{y} - \frac{1}{4} \frac{1}{1-y-z} - 2 \frac{1}{y+z} \right) \right]$$

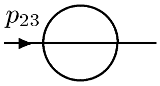


$$+\frac{1}{4}\frac{yz}{(1-y-z)(y+z)}$$


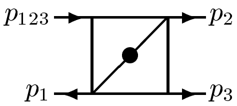
$$+\frac{(d-3)(3d-10)}{4(d-4)}\frac{-4+4z+5y}{y(y+z)(1-y-z)}$$


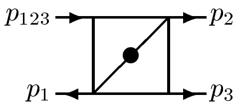
$$-\frac{(d-3)(3d-10)}{4(d-4)}\frac{-4+4z+3y}{y(y+z)(1-y-z)}$$


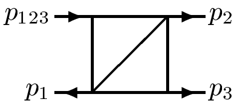
$$-\frac{(d-3)(3d-8)(3d-10)}{2(d-4)^2}\frac{1}{y(y+z)(1-y-z)}$$


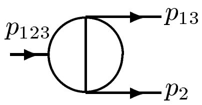
$$-\frac{(d-3)(3d-8)(3d-10)}{2(d-4)^2}\frac{1}{z(y+z)(1-y-z)}$$


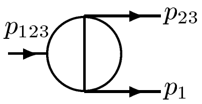
, (3.8)

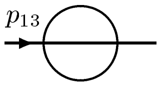
$$\frac{\partial}{\partial y}$$


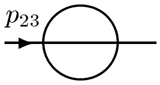
$$= \left[-\frac{1}{y}+(d-4)\left(\frac{1}{y}+\frac{3}{4}\frac{1}{1-y-z}\right)\right]$$


$$-\frac{3(d-4)^2}{4}\frac{y+z}{yz(1-y-z)}$$


$$+\frac{3(d-3)(3d-10)}{4}\frac{-1+y+2z}{y(1-y)z(1-y-z)}$$


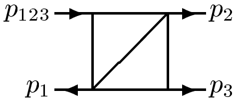
$$+\frac{3(d-3)(3d-10)}{4}\frac{1}{yz(1-y-z)}$$


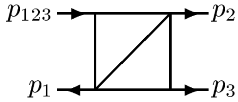
$$-\frac{3(d-3)(3d-8)(3d-10)}{2(d-4)}\frac{1}{y^2(1-y)(1-y-z)}$$


$$-\frac{3(d-3)(3d-8)(3d-10)}{2(d-4)}\frac{1}{yz^2(1-y-z)}$$


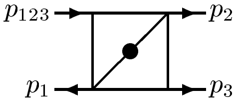
. (3.9)

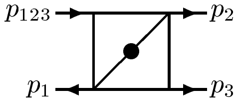
From the powers of $(d-4)$ multiplying the inhomogeneous terms, it can be read off that the second master integral (2.5) is suppressed by one power of $(d-4)$ with respect to the first master integral (2.4). Expanding the homogeneous term of the differential equations and retaining only the leading contribution, the homogeneous differential equations decouple:

$$\frac{\partial}{\partial y}$$


$$= -\frac{1}{y+z}$$


, (3.10)

$$\frac{\partial}{\partial y}$$


$$= -\frac{1}{y}$$


. (3.11)

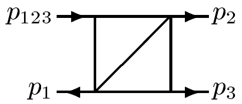
The homogeneous z differential equations follow from the above by $y \leftrightarrow z$, using the $y \leftrightarrow z$ interchange symmetry of the integrals. As a result, one finds the prefactors for the first master integral (2.4):

$\mathcal{R}(y,z;s_{123},\epsilon) \sim 1/(y+z)$ and for the second master integral (2.5): $\mathcal{R}(y,z;s_{123},\epsilon) \sim 1/(yz)$.

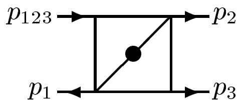
Separating off these prefactors, we can transform (3.8,3.9) into a set of coupled differential equations for the two $\mathcal{H}(y,z;\epsilon)$, corresponding to both master integrals. Inserting the ansatz (2.2) for the $\mathcal{H}(y,z;\epsilon)$, we can then determine by comparison with the ϵ-expansion of the inhomogeneous term that $p=0$ for (2.4) and $p=1$ for (2.5). The coefficients $T_{n,\vec{m}_j}(z)$ appearing in the ansatz are then determined as outlined in the previous Section.

To determine the boundary terms $T_n(z)$ for both master integrals, we first investigate the differential equations (3.8,3.9) in the point $y = 1-z$, where both master integrals are analytic. Multiplying both equations with $(1-y-z)$ and taking $y = 1-z$, the left hand sides of the equations vanish. From (3.8) we obtain,

$$0 = -\frac{d-4}{4}$$



$$\Bigg|_{y=1-z} + \frac{z(1-z)}{4}$$



$$\Bigg|_{y=1-z} + \text{(Subtopologies)}\,, \tag{3.12}$$

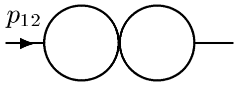
while (3.9) yields a multiple of this. The point $y = (1-z)$ can therefore be used only to determine a linear combination of boundary terms from both master integrals. A second boundary condition is obtained by multiplying (3.8) with $(y+z)$ and subsequently taking $y = -z$, which fixes the boundary terms for the first master integral (2.4). The boundary terms for the second master integral (2.5) follow then from (3.12), thus completing the computation of both master integrals. The results are given in the following section, together with the results for the other master integrals.

4 Master Integrals

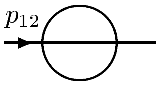
The common normalisation factor of all master integrals is

$$S_\epsilon = \left[(4\pi)^\epsilon \frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\right]\,, \tag{4.1}$$

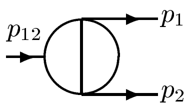
4.1 One-scale Integrals



$$= \left(\frac{S_\epsilon}{16\pi^2}\right)^2 (-s_{12})^{-2\epsilon}\,\frac{-1}{\epsilon^2(1-2\epsilon)^2}\,, \tag{4.2}$$

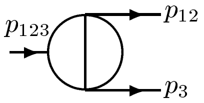


$$= \left(\frac{S_\epsilon}{16\pi^2}\right)^2 (-s_{12})^{1-2\epsilon}\left[-\frac{1}{4\epsilon} - \frac{13}{8} - \frac{115}{16}\epsilon - \left(\frac{865}{32} - \frac{3}{2}\zeta_3\right)\epsilon^2 - \left(\frac{5971}{64} - \frac{39}{4}\zeta_3 - \frac{\pi^4}{40}\right)\epsilon^3 + \mathcal{O}(\epsilon^4)\right]\,, \tag{4.3}$$



$$= \left(\frac{S_\epsilon}{16\pi^2}\right)^2 (-s_{12})^{-2\epsilon}\left[-\frac{1}{2\epsilon^2} - \frac{5}{2\epsilon} - \left(\frac{19}{2} + \frac{\pi^2}{6}\right) - \left(\frac{65}{2} + \frac{5\pi^2}{6} - 2\zeta_3\right)\epsilon - \left(\frac{211}{2} + \frac{19\pi^2}{6} - 10\zeta_3\right)\epsilon^2 + \mathcal{O}(\epsilon^3)\right]. \tag{4.4}$$

4.2 Two-scale Integrals



$$= \left(\frac{S_\epsilon}{16\pi^2}\right)^2 (-s_{123})^{-2\epsilon} \sum_{i=-2}^{2} \frac{g_{4.1,i}\left(\frac{s_{12}}{s_{123}}\right)}{\epsilon^i} + \mathcal{O}(\epsilon^3), \tag{4.5}$$

with:

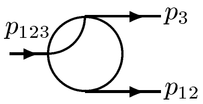
$$g_{4.1,2}(x) = -\frac{1}{2} \tag{4.6}$$

$$g_{4.1,1}(x) = -\frac{5}{2} \tag{4.7}$$

$$g_{4.1,0}(x) = -\frac{19}{2} - H(1,0;x) - \frac{\pi^2}{6} \tag{4.8}$$

$$g_{4.1,-1}(x) = -\frac{65}{2} - 5H(1,0;x) + 2H(1,0,0;x) - H(1,1,0;x) + 2\zeta_3 + \frac{\pi^2}{6}\left[-5 - H(1;x)\right] \tag{4.9}$$

$$\begin{aligned} g_{4.1,-2}(x) &= -\frac{211}{2} - 19H(1,0;x) + 10H(1,0,0;x) - 4H(1,0,0,0;x) - 5H(1,1,0;x) + 2H(1,1,0,0;x) \\ &\quad -H(1,1,1,0;x) + \zeta_3\left[10 - H(1;x)\right] + \frac{\pi^2}{6}\left[-19 - 5H(1;x) - H(1,1;x)\right] \end{aligned} \tag{4.10}$$



$$= \left(\frac{S_\epsilon}{16\pi^2}\right)^2 (-s_{123})^{-2\epsilon} \sum_{i=-2}^{2} \frac{g_{4.2,i}\left(\frac{s_{12}}{s_{123}}\right)}{\epsilon^i} + \mathcal{O}(\epsilon^3), \tag{4.11}$$

with:

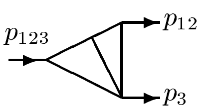
$$g_{4.2,2}(x) = -\frac{1}{2} \tag{4.12}$$

$$g_{4.2,1}(x) = -\frac{5}{2} + H(0;x) \tag{4.13}$$

$$g_{4.2,0}(x) = -\frac{19}{2} + 5H(0;x) - H(0,0;x) + H(1,0;x) + \frac{\pi^2}{6} \tag{4.14}$$

$$\begin{aligned} g_{4.2,-1}(x) &= -\frac{65}{2} + 19H(0;x) - 5H(0,0;x) + H(0,0,0;x) - H(0,1,0;x) + 5H(1,0;x) - H(1,0,0;x) \\ &\quad +H(1,1,0;x) + \zeta_3 + \frac{\pi^2}{6}\left[+5 - H(0;x) + H(1;x)\right] \end{aligned} \tag{4.15}$$

$$\begin{aligned} g_{4.2,-2}(x) &= -\frac{211}{2} + 65H(0;x) - 19H(0,0;x) + 5H(0,0,0;x) - H(0,0,0,0;x) + H(0,0,1,0;x) \\ &\quad -5H(0,1,0;x) + H(0,1,0,0;x) - H(0,1,1,0;x) + 19H(1,0;x) - 5H(1,0,0;x) \\ &\quad +H(1,0,0,0;x) - H(1,0,1,0;x) + 5H(1,1,0;x) - H(1,1,0,0;x) + H(1,1,1,0;x) + \frac{3\pi^4}{40} \\ &\quad +\zeta_3\left[+5 - 4H(0;x) - 2H(1;x)\right] + \frac{\pi^2}{6}\Big[+19 - 5H(0;x) + H(0,0;x) - H(0,1;x) \\ &\quad +5H(1;x) - H(1,0;x) + H(1,1;x)\Big] \end{aligned} \tag{4.16}$$



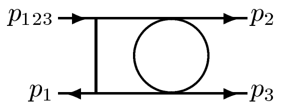
$$= \left(\frac{S_\epsilon}{16\pi^2}\right)^2 \frac{(-s_{123})^{-2\epsilon}}{s_{123} - s_{12}} \sum_{i=0}^{4} \frac{g_{5.1,i}\left(\frac{s_{12}}{s_{123}}\right)}{\epsilon^i} + \mathcal{O}(\epsilon), \tag{4.17}$$

with:

$$g_{5.1,4}(x) = g_{5.1,3}(x) = g_{5.1,2}(x) = g_{5.1,1}(x) = 0 \tag{4.18}$$

$$g_{5.1,0}(x) = -H(0,0,1,0;x) + H(0,1,0,0;x) - \frac{\pi^2}{6}H(0,0;x) + 3\zeta_3 H(0;x) - \frac{\pi^4}{15} \tag{4.19}$$

4.3 Three-scale Integrals



$$= \left(\frac{S_\epsilon}{16\pi^2}\right)^2 \frac{(-s_{123})^{-2\epsilon}}{s_{12}+s_{13}} \sum_{i=-1}^{3} \frac{f_{5.1,i}\left(\frac{s_{13}}{s_{123}}, \frac{s_{23}}{s_{123}}\right)}{\epsilon^i} + \mathcal{O}(\epsilon^2), \tag{4.20}$$

with:

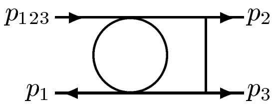
$$f_{5.1,3}(y,z) = 0 \tag{4.21}$$

$$f_{5.1,2}(y,z) = -H(0;z) \tag{4.22}$$

$$f_{5.1,1}(y,z) = +H(0;y)H(0;z) - 2H(0;z) + 2H(0,0;z) + H(1,0;z) + \frac{\pi^2}{6} \tag{4.23}$$

$$\begin{aligned} f_{5.1,0}(y,z) = {} & +2H(0;y)H(0;z) - 2H(0;y)H(1,0;z) - 4H(0;z) - H(0;z)H(1-z,0;y) \\ & -2H(0,0;y)H(0;z) + 4H(0,0;z) - 2H(0,0;z)H(0;y) - 4H(0,0,0;z) - H(0,1,0;y) \\ & -2H(0,1,0;z) + 2H(1,0;z) - H(1,0;z)H(1-z;y) - 2H(1,0,0;z) - 2H(1,1,0;z) \\ & -H(1-z,1,0;y) + \frac{\pi^2}{6}\left[+2 - 2H(0;y) - 3H(0;z) - 2H(1;z) - H(1-z;y)\right] \end{aligned} \tag{4.24}$$

$$\begin{aligned} f_{5.1,-1}(y,z) = {} & +4H(0;y)H(0;z) - 4H(0;y)H(1,0;z) + 4H(0;y)H(1,0,0;z) + 4H(0;y)H(1,1,0;z) \\ & -8H(0;z) - 2H(0;z)H(1-z,0;y) + 2H(0;z)H(1-z,0,0;y) \\ & +H(0;z)H(1-z,1-z,0;y) - 4H(0,0;y)H(0;z) + 4H(0,0;y)H(0,0;z) \\ & +4H(0,0;y)H(1,0;z) + 8H(0,0;z) - 4H(0,0;z)H(0;y) + 2H(0,0;z)H(1-z,0;y) \\ & +4H(0,0,0;y)H(0;z) - 8H(0,0,0;z) + 4H(0,0,0;z)H(0;y) + 8H(0,0,0,0;z) \\ & +2H(0,0,1,0;y) + 4H(0,0,1,0;z) - 2H(0,1,0;y) - 4H(0,1,0;z) \\ & +4H(0,1,0;z)H(0;y) + 2H(0,1,0;z)H(1-z;y) + 2H(0,1,0,0;y) + 4H(0,1,0,0;z) \\ & -H(0,1,1,0;y) + 4H(0,1,1,0;z) + 2H(0,1-z;y)H(1,0;z) \\ & +2H(0,1-z,0;y)H(0;z) + 2H(0,1-z,1,0;y) + 4H(1,0;z) - 2H(1,0;z)H(1-z;y) \\ & +2H(1,0;z)H(1-z,0;y) + H(1,0;z)H(1-z,1-z;y) - 4H(1,0,0;z) \\ & +2H(1,0,0;z)H(1-z;y) + 4H(1,0,0,0;z) + 4H(1,0,1,0;z) - 4H(1,1,0;z) \\ & +2H(1,1,0;z)H(1-z;y) + 4H(1,1,0,0;z) + 4H(1,1,1,0;z) + H(1-z,0,1,0;y) \\ & -2H(1-z,1,0;y) + 2H(1-z,1,0,0;y) - H(1-z,1,1,0;y) + H(1-z,1-z,1,0;y) \\ & +\frac{7\pi^4}{90} + 5\zeta_3 H(0;z) \\ & +\frac{\pi^2}{6}\Big[+4 - 4H(0;y) + 4H(0;y)H(0;z) + 4H(0;y)H(1;z) - 6H(0;z) \\ & +H(0;z)H(1-z;y) + 4H(0,0;y) + 6H(0,0;z) - H(0,1;y) + 4H(0,1;z) \\ & +2H(0,1-z;y) - 4H(1;z) + 2H(1;z)H(1-z;y) + 4H(1,0;z) + 4H(1,1;z) \\ & -2H(1-z;y) + 2H(1-z,0;y) - H(1-z,1;y) + H(1-z,1-z;y)\Big] \end{aligned} \tag{4.25}$$



$$= \left(\frac{S_\epsilon}{16\pi^2}\right)^2 \frac{(-s_{123})^{-2\epsilon}}{s_{23}} \sum_{i=-1}^{3} \frac{f_{5.2,i}\left(\frac{s_{13}}{s_{123}}, \frac{s_{23}}{s_{123}}\right)}{\epsilon^i} + \mathcal{O}(\epsilon^2), \tag{4.26}$$

with:

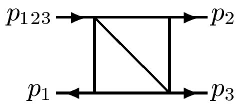
$$f_{5.2,3}(y,z) = -1 \tag{4.27}$$

$$f_{5.2,2}(y,z) = -2 + H(0;y) + H(0;z) \tag{4.28}$$

$$f_{5.2,1}(y,z) = -4 + 2H(0;y) - H(0;y)H(0;z) + 2H(0;z) - 2H(0,0;y) - H(0,0;z) - H(1,0;y) \tag{4.29}$$

$$\begin{aligned} f_{5.2,0}(y,z) = {} & -8 + 4H(0;y) - 2H(0;y)H(0;z) + H(0;y)H(1,0;z) + 4H(0;z) + H(0;z)H(1-z,0;y) \\ & -4H(0,0;y) + 2H(0,0;y)H(0;z) - 2H(0,0;z) + H(0,0;z)H(0;y) + 4H(0,0,0;y) \\ & +H(0,0,0;z) + 2H(0,1,0;y) - 2H(1,0;y) + H(1,0;z)H(1-z;y) + 2H(1,0,0;y) \\ & +H(1,1,0;z) + H(1-z,1,0;y) + 5\zeta_3 + \frac{\pi^2}{6}\left[+H(0;y) + H(1;z) + H(1-z;y)\right] \end{aligned} \tag{4.30}$$

$$\begin{aligned} f_{5.2,-1}(y,z) = {} & -16 + 8H(0;y) - 4H(0;y)H(0;z) + 2H(0;y)H(1,0;z) - H(0;y)H(1,0,0;z) \\ & -H(0;y)H(1,1,0;z) + 8H(0;z) + 2H(0;z)H(1-z,0;y) - 2H(0;z)H(1-z,0,0;y) \\ & -H(0;z)H(1-z,1-z,0;y) - 8H(0,0;y) + 4H(0,0;y)H(0;z) - 2H(0,0;y)H(0,0;z) \\ & -2H(0,0;y)H(1,0;z) - 4H(0,0;z) + 2H(0,0;z)H(0;y) - H(0,0;z)H(1-z,0;y) \\ & +8H(0,0,0;y) - 4H(0,0,0;y)H(0;z) + 2H(0,0,0;z) - H(0,0,0;z)H(0;y) \\ & -8H(0,0,0,0;y) - H(0,0,0,0;z) - 4H(0,0,1,0;y) + 4H(0,1,0;y) \\ & -H(0,1,0;z)H(0;y) - H(0,1,0;z)H(1-z;y) - 4H(0,1,0,0;y) - H(0,1,1,0;z) \\ & -2H(0,1-z;y)H(1,0;z) - 2H(0,1-z,0;y)H(0;z) - 2H(0,1-z,1,0;y) \\ & -4H(1,0;y) + 2H(1,0;z)H(1-z;y) - H(1,0;z)H(1-z,0;y) \\ & -H(1,0;z)H(1-z,1-z;y) + 4H(1,0,0;y) - H(1,0,0;z)H(1-z;y) \\ & -4H(1,0,0,0;y) - H(1,0,1,0;z) + 2H(1,1,0;z) - H(1,1,0,0;z) \\ & -2H(1-z,0,1,0;y) + 2H(1-z,1,0;y) - 2H(1-z,1,0,0;y) \\ & -H(1-z,1-z,1,0;y) + \frac{37\pi^4}{360} + \zeta_3\left[10 - 6H(0;y) - 5H(0;z) - H(1;z) - H(1-z;y)\right] \\ & +\frac{\pi^2}{6}\Big[+ 2H(0;y) - H(0;y)H(0;z) - H(0;y)H(1;z) - H(0;z)H(1-z;y) \\ & -2H(0,0;y) - H(0,1;z) - 2H(0,1-z;y) + 2H(1;z) \\ & -H(1,0;z) + 2H(1-z;y) - H(1-z,0;y) - H(1-z,1-z;y)\Big] \end{aligned} \tag{4.31}$$



$$= \left(\frac{S_\epsilon}{16\pi^2}\right)^2 \frac{(-s_{123})^{-2\epsilon}}{s_{12}} \sum_{i=0}^{4} \frac{f_{5.3,i}\left(\frac{s_{13}}{s_{123}}, \frac{s_{23}}{s_{123}}\right)}{\epsilon^i} + \mathcal{O}(\epsilon), \tag{4.32}$$

with:

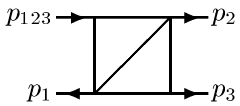
$$f_{5.3,4}(y,z) = 0 \tag{4.33}$$

$$f_{5.3,3}(y,z) = 0 \tag{4.34}$$

$$f_{5.3,2}(y,z) = -H(0;y)H(0;z) - H(1,0;y) - H(1,0;z) - \frac{\pi^2}{6} \tag{4.35}$$

$$\begin{aligned} f_{5.3,1}(y,z) = \; & +2H(0;y)H(1,0;z) + 2H(0;z)H(1-z,0;y) + 2H(0,0;y)H(0;z) + 2H(0,0;z)H(0;y) \\ & +2H(0,1,0;y) + 2H(0,1,0;z) + 2H(1,0;z)H(1-z;y) + 2H(1,0,0;y) + 2H(1,0,0;z) \\ & +2H(1,1,0;z) + 2H(1-z,1,0;y) \\ & +\frac{\pi^2}{6}\left[+2H(0;y) + 2H(0;z) + 2H(1;z) + 2H(1-z;y)\right] \end{aligned} \tag{4.36}$$

$$\begin{aligned} f_{5.3,0}(y,z) = \; & -4H(0;y)H(1,0,0;z) - 4H(0;y)H(1,1,0;z) - 4H(0;z)H(1-z,0,0;y) \\ & -4H(0;z)H(1-z,1-z,0;y) - 4H(0,0;y)H(0,0;z) - 4H(0,0;y)H(1,0;z) \\ & -4H(0,0;z)H(1-z,0;y) - 4H(0,0,0;y)H(0;z) - 4H(0,0,0;z)H(0;y) \\ & -4H(0,0,1,0;y) - 4H(0,0,1,0;z) - 4H(0,1,0;z)H(0;y) - 4H(0,1,0;z)H(1-z;y) \\ & -4H(0,1,0,0;y) - 4H(0,1,0,0;z) - 4H(0,1,1,0;z) - 4H(0,1-z;y)H(1,0;z) \\ & -4H(0,1-z,0;y)H(0;z) - 4H(0,1-z,1,0;y) - 4H(1,0;z)H(1-z,0;y) \\ & -4H(1,0;z)H(1-z,1-z;y) - 4H(1,0,0;z)H(1-z;y) - 4H(1,0,0,0;y) \\ & -4H(1,0,0,0;z) - 4H(1,0,1,0;z) - 4H(1,1,0;z)H(1-z;y) - 4H(1,1,0,0;z) \\ & -4H(1,1,1,0;z) - 4H(1-z,0,1,0;y) - 4H(1-z,1,0,0;y) - 4H(1-z,1-z,1,0;y) \\ & -\frac{7\pi^4}{90} + \frac{\pi^2}{6}\Big[-4H(0;y)H(0;z) - 4H(0;y)H(1;z) - 4H(0;z)H(1-z;y) - 4H(0,0;y) \\ & -4H(0,0;z) - 4H(0,1;z) - 4H(0,1-z;y) - 4H(1;z)H(1-z;y) - 4H(1,0;z) \\ & -4H(1,1;z) - 4H(1-z,0;y) - 4H(1-z,1-z;y)\Big] \end{aligned} \tag{4.37}$$



$$= \left(\frac{S_\epsilon}{16\pi^2}\right)^2 \frac{(-s_{123})^{-2\epsilon}}{s_{13}+s_{23}} \sum_{i=0}^{4} \frac{f_{5.4,i}\left(\frac{s_{13}}{s_{123}}, \frac{s_{23}}{s_{123}}\right)}{\epsilon^i} + \mathcal{O}(\epsilon), \tag{4.38}$$

with:

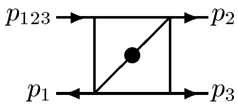
$$f_{5.4,4}(y,z) = 0 \tag{4.39}$$

$$f_{5.4,3}(y,z) = 0 \tag{4.40}$$

$$f_{5.4,2}(y,z) = 0 \tag{4.41}$$

$$\begin{aligned} f_{5.4,1}(y,z) = {} & +H(0;y)H(1,0;z) + H(0;z)H(1-z,0;y) - H(0,1,0;y) - H(0,1,0;z) \\ & +H(1,0;z)H(1-z;y) + H(1,1,0;z) + H(1-z,1,0;y) \\ & +\frac{\pi^2}{6}\left[+H(1;z) + H(1-z;y)\right] \end{aligned} \tag{4.42}$$

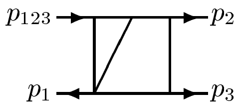
$$\begin{aligned} f_{5.4,0}(y,z) = {} & -2H(0;y)H(1,0,0;z) - H(0;y)H(1,1,0;z) - 2H(0;z)H(1-z,0,0;y) \\ & -H(0;z)H(1-z,1-z,0;y) + 4H(0;z)H(z,1-z,0;y) - 2H(0,0;y)H(1,0;z) \\ & -2H(0,0;z)H(1-z,0;y) + 2H(0,0,1,0;y) - 2H(0,0,1,0;z) + 2H(0,1,0;z)H(0;y) \\ & -2H(0,1,0;z)H(1-z;y) - 4H(0,1,0;z)H(z;y) + 2H(0,1,0,0;y) + 2H(0,1,0,0;z) \\ & -H(0,1,1,0;y) + H(0,1,1,0;z) - 2H(0,1-z;y)H(1,0;z) - 2H(0,1-z,0;y)H(0;z) \\ & -2H(0,1-z,1,0;y) - H(1,0;z)H(1-z,0;y) - H(1,0;z)H(1-z,1-z;y) \\ & +4H(1,0;z)H(z,0;y) + 4H(1,0;z)H(z,1-z;y) - 2H(1,0,0;z)H(1-z;y) \\ & -2H(1,0,1,0;z) + 4H(1,1,0;z)H(z;y) - 2H(1,1,0,0;z) - 2H(1-z,0,1,0;y) \\ & -2H(1-z,1,0,0;y) + H(1-z,1,1,0;y) - H(1-z,1-z,1,0;y) - 4H(z,0,1,0;y) \\ & +4H(z,1-z,1,0;y) + \zeta_3\left[-2H(1;z) - 2H(1-z;y)\right] \\ & +\frac{\pi^2}{6}\Big[-H(0;y)H(1;z) - H(0;z)H(1-z;y) - H(0,1;y) + H(0,1;z) - 2H(0,1-z;y) \\ & +4H(1;z)H(z;y) - H(1,0;z) - H(1-z,0;y) + H(1-z,1;y) - H(1-z,1-z;y) \\ & +4H(z,1-z;y)\Big] \end{aligned} \tag{4.43}$$



$$= \left(\frac{S_\epsilon}{16\pi^2}\right)^2 \frac{(-s_{123})^{-2\epsilon}}{s_{13}s_{23}} \sum_{i=-1}^{3} \frac{f_{5.5,i}\left(\frac{s_{13}}{s_{123}}, \frac{s_{23}}{s_{123}}\right)}{\epsilon^i} + \mathcal{O}(\epsilon^2), \tag{4.44}$$

with:

$$
\begin{aligned}
f_{5.5,3}(y,z) &= 1 &(4.45)\\
f_{5.5,2}(y,z) &= -2H(0;y) - 2H(0;z) &(4.46)\\
f_{5.5,1}(y,z) &= +4H(0;y)H(0;z) + 4H(0,0;y) + 4H(0,0;z) + 3H(1,0;y) + 3H(1,0;z) + \frac{\pi^2}{3} &(4.47)\\
f_{5.5,0}(y,z) &= -6H(0;y)H(1,0;z) - 6H(0;z)H(1-z,0;y) - 8H(0,0;y)H(0;z) - 8H(0,0;z)H(0;y)\\
&\quad -8H(0,0,0;y) - 8H(0,0,0;z) - 6H(0,1,0;y) - 6H(0,1,0;z) - 6H(1,0;z)H(1-z;y)\\
&\quad -6H(1,0,0;y) - 6H(1,0,0;z) + 3H(1,1,0;y) - 3H(1,1,0;z) - 6H(1-z,1,0;y)\\
&\quad -16\zeta_3 + \frac{\pi^2}{6}\left[-4H(0;y) - 4H(0;z) + 3H(1;y) - 3H(1;z) - 6H(1-z;y)\right] &(4.48)\\
f_{5.5,-1}(y,z) &= +12H(0;y)H(1,0,0;z) + 6H(0;y)H(1,1,0;z) + 12H(0;z)H(1-z,0,0;y)\\
&\quad +6H(0;z)H(1-z,1-z,0;y) + 16H(0,0;y)H(0,0;z) + 12H(0,0;y)H(1,0;z)\\
&\quad +12H(0,0;z)H(1-z,0;y) + 16H(0,0,0;y)H(0;z) + 16H(0,0,0;z)H(0;y)\\
&\quad +16H(0,0,0,0;y) + 16H(0,0,0,0;z) + 12H(0,0,1,0;y) + 12H(0,0,1,0;z)\\
&\quad +12H(0,1,0;z)H(0;y) + 12H(0,1,0;z)H(1-z;y) + 12H(0,1,0,0;y)\\
&\quad +12H(0,1,0,0;z) - 6H(0,1,1,0;y) + 6H(0,1,1,0;z) + 12H(0,1-z;y)H(1,0;z)\\
&\quad +12H(0,1-z,0;y)H(0;z) + 12H(0,1-z,1,0;y) + 6H(1,0;z)H(1-z,0;y)\\
&\quad +6H(1,0;z)H(1-z,1-z;y) + 12H(1,0,0;z)H(1-z;y) + 12H(1,0,0,0;y)\\
&\quad +12H(1,0,0,0;z) + 12H(1,0,1,0;z) - 6H(1,1,0,0;y) + 6H(1,1,0,0;z)\\
&\quad +3H(1,1,1,0;y) + 3H(1,1,1,0;z) + 12H(1-z,0,1,0;y) + 12H(1-z,1,0,0;y)\\
&\quad -6H(1-z,1,1,0;y) + 6H(1-z,1-z,1,0;y) - \frac{2\pi^4}{5}\\
&\quad +\zeta_3\left[+32H(0;y) + 32H(0;z) + 3H(1;y) + 15H(1;z) + 12H(1-z;y)\right]\\
&\quad +\frac{\pi^2}{6}\Big[+8H(0;y)H(0;z) + 6H(0;y)H(1;z) + 6H(0;z)H(1-z;y) + 8H(0,0;y)\\
&\quad +8H(0,0;z) - 6H(0,1;y) + 6H(0,1;z) + 12H(0,1-z;y) + 6H(1,0;z) + 3H(1,1;y)\\
&\quad +3H(1,1;z) + 6H(1-z,0;y) - 6H(1-z,1;y) + 6H(1-z,1-z;y)\Big] &(4.49)
\end{aligned}
$$



$$= \left(\frac{S_\epsilon}{16\pi^2}\right)^2 \frac{(-s_{123})^{-2\epsilon}}{(s_{12}+s_{13})s_{23}} \sum_{i=0}^{4} \frac{f_{6.1,i}\left(\frac{s_{13}}{s_{123}}, \frac{s_{23}}{s_{123}}\right)}{\epsilon^i} + \mathcal{O}(\epsilon), \tag{4.50}$$

with:

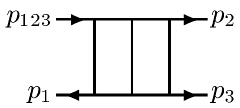
$$f_{6.1,4}(y,z) = 0 \tag{4.51}$$

$$f_{6.1,3}(y,z) = -\frac{1}{2}H(0;y) \tag{4.52}$$

$$f_{6.1,2}(y,z) = +H(0,0;y) - H(1,0;y) - \frac{\pi^2}{6} \tag{4.53}$$

$$\begin{aligned} f_{6.1,1}(y,z) = {} & +H(0;y)H(1,0;z) + H(0;z)H(1-z,0;y) + H(0,0;z)H(0;y) - 2H(0,0,0;y) \\ & -H(0,1,0;y) + H(0,1,0;z) + H(1,0;z)H(1-z;y) + 2H(1,0,0;y) - 2H(1,1,0;y) \\ & +H(1,1,0;z) + H(1-z,1,0;y) - 3\zeta_3 \\ & +\frac{\pi^2}{6}\left[+H(0;z) - 2H(1;y) + H(1;z) + H(1-z;y)\right] \end{aligned} \tag{4.54}$$

$$\begin{aligned} f_{6.1,0}(y,z) = {} & -3H(0;y)H(1,0,0;z) - H(0;y)H(1,1,0;z) + 2H(0;z)H(1,1-z,0;y) \\ & -2H(0;z)H(1-z,0,0;y) - H(0;z)H(1-z,1-z,0;y) - 2H(0,0;y)H(0,0;z) \\ & -2H(0,0;y)H(1,0;z) + 2H(0,0;z)H(1,0;y) - 3H(0,0;z)H(1-z,0;y) \\ & -3H(0,0,0;z)H(0;y) + 4H(0,0,0,0;y) + 2H(0,0,1,0;y) - 3H(0,0,1,0;z) \\ & -H(0,1,0;z)H(0;y) + 2H(0,1,0;z)H(1;y) - 3H(0,1,0;z)H(1-z;y) + 2H(0,1,0,0;y) \\ & -3H(0,1,0,0;z) + H(0,1,1,0;y) - 2H(0,1-z;y)H(1,0;z) - 2H(0,1-z,0;y)H(0;z) \\ & -2H(0,1-z,1,0;y) + 2H(1,0;y)H(1,0;z) + 2H(1,0;z)H(1,1-z;y) \\ & -H(1,0;z)H(1-z,0;y) - H(1,0;z)H(1-z,1-z;y) - 3H(1,0,0;z)H(1-z;y) \\ & -4H(1,0,0,0;y) - 2H(1,0,1,0;y) - 3H(1,0,1,0;z) + 2H(1,1,0;z)H(1;y) \\ & +4H(1,1,0,0;y) - 3H(1,1,0,0;z) - 4H(1,1,1,0;y) + 2H(1,1-z,1,0;y) \\ & -2H(1-z,0,1,0;y) - 2H(1-z,1,0,0;y) + 2H(1-z,1,1,0;y) - H(1-z,1-z,1,0;y) \\ & +\frac{\pi^4}{72} + \zeta_3\left[-2H(0;y) - 3H(0;z) - 6H(1;y) - 3H(1;z) - 3H(1-z;y)\right] \\ & +\frac{\pi^2}{6}\Big[-H(0;y)H(0;z) - H(0;y)H(1;z) + 2H(0;z)H(1;y) - H(0;z)H(1-z;y) \\ & -H(0,0;z) + H(0,1;y) - 2H(0,1-z;y) + 2H(1;y)H(1;z) - H(1,0;z) - 4H(1,1;y) \\ & +2H(1,1-z;y) - H(1-z,0;y) + 2H(1-z,1;y) - H(1-z,1-z;y)\Big] \end{aligned} \tag{4.55}$$



$$= \left(\frac{S_\epsilon}{16\pi^2}\right)^2 \frac{(-s_{123})^{-2\epsilon}}{s_{13}s_{23}^2} \sum_{i=0}^{4} \frac{f_{7.1,i}\left(\frac{s_{13}}{s_{123}}, \frac{s_{23}}{s_{123}}\right)}{\epsilon^i} + \mathcal{O}(\epsilon), \tag{4.56}$$

with:

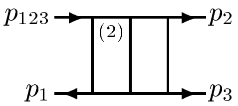
$$f_{7.1,4}(y,z) = -1 \tag{4.57}$$

$$f_{7.1,3}(y,z) = 2\left[H(0;y)+H(0;z)\right] \tag{4.58}$$

$$f_{7.1,2}(y,z) = -4H(0;y)H(0;z)-4H(0,0;y)-4H(0,0;z)-H(1,0;y)-3H(1,0;z)-\frac{7\pi^2}{12} \tag{4.59}$$

$$\begin{aligned} f_{7.1,1}(y,z) = {} & +6H(0;y)H(1,0;z)+2H(0;z)H(1-z,0;y)+8H(0,0;y)H(0;z) \\ & +8H(0,0;z)H(0;y)+8H(0,0,0;y)+8H(0,0,0;z)+5H(0,1,0;y) \\ & +6H(0,1,0;z)+2H(1,0;z)H(1-z;y)+2H(1,0,0;y)+9H(1,0,0;z) \\ & -2H(1,1,0;y)+6H(1,1,0;z)+2H(1-z,1,0;y)-\frac{\zeta_3}{2} \\ & +\frac{\pi^2}{6}\left[+7H(0;y)+7H(0;z)-2H(1;y)+6H(1;z)+2H(1-z;y)\right] \end{aligned} \tag{4.60}$$

$$\begin{aligned} f_{7.1,0}(y,z) = {} & -18H(0;y)H(1,0,0;z)-12H(0;y)H(1,1,0;z)+2H(0;z)H(1,1-z,0;y) \\ & -4H(0;z)H(1-z,0,0;y)-2H(0;z)H(1-z,1-z,0;y)-16H(0,0;y)H(0,0;z) \\ & -12H(0,0;y)H(1,0;z)+2H(0,0;z)H(1,0;y)-6H(0,0;z)H(1-z,0;y) \\ & -16H(0,0,0;y)H(0;z)-16H(0,0,0;z)H(0;y)-16H(0,0,0,0;y) \\ & -16H(0,0,0,0;z)-16H(0,0,1,0;y)-12H(0,0,1,0;z)-12H(0,1,0;z)H(0;y) \\ & +2H(0,1,0;z)H(1;y)-6H(0,1,0;z)H(1-z;y)-10H(0,1,0,0;y) \\ & -18H(0,1,0,0;z)+H(0,1,1,0;y)-12H(0,1,1,0;z)-10H(0,1-z;y)H(1,0;z) \\ & -10H(0,1-z,0;y)H(0;z)-10H(0,1-z,1,0;y)+2H(1,0;y)H(1,0;z) \\ & +2H(1,0;z)H(1,1-z;y)-4H(1,0;z)H(1-z,0;y)-2H(1,0;z)H(1-z,1-z;y) \\ & -6H(1,0,0;z)H(1-z;y)-4H(1,0,0,0;y)-21H(1,0,0,0;z)-2H(1,0,1,0;y) \\ & -15H(1,0,1,0;z)+2H(1,1,0;z)H(1;y)-4H(1,1,0;z)H(1-z;y) \\ & +4H(1,1,0,0;y)-21H(1,1,0,0;z)-4H(1,1,1,0;y)-12H(1,1,1,0;z) \\ & +2H(1,1-z,1,0;y)-2H(1-z,0,1,0;y)-4H(1-z,1,0,0;y) \\ & +4H(1-z,1,1,0;y)-2H(1-z,1-z,1,0;y)-\frac{\pi^4}{4} \\ & +\zeta_3\left[+H(0;y)-2H(1-z;y)-6H(1;y)+3H(1;z)+H(0;z)\right] \\ & +\frac{\pi^2}{6}\Big[-14H(0;y)H(0;z)-12H(0;y)H(1;z)+2H(0;z)H(1;y)-2H(0;z)H(1-z;y) \\ & -14H(0,0;y)-14H(0,0;z)+H(0,1;y)-12H(0,1;z)-10H(0,1-z;y) \\ & +2H(1;y)H(1;z)-4H(1;z)H(1-z;y)-15H(1,0;z)-4H(1,1;y)-12H(1,1;z) \\ & +2H(1,1-z;y)-4H(1-z,0;y)+4H(1-z,1;y)-2H(1-z,1-z;y)\Big] \end{aligned} \tag{4.61}$$



$$= \left(\frac{S_\epsilon}{16\pi^2}\right)^2 \frac{(-s_{123})^{-2\epsilon}}{(s_{12}+s_{13})s_{23}} \sum_{i=0}^{4} \frac{f_{7.2,i}\left(\frac{s_{13}}{s_{123}}, \frac{s_{23}}{s_{123}}\right)}{\epsilon^i} + \mathcal{O}(\epsilon)\,, \qquad (4.62)$$

with:

$$f_{7.2,4}(y,z) = 0 \qquad (4.63)$$

$$f_{7.2,3}(y,z) = -\frac{3}{2}H(0;z) \qquad (4.64)$$

$$f_{7.2,2}(y,z) = +2H(0;y)H(0;z)+\frac{9}{2}H(0,0;z)+\frac{5}{2}H(1,0;z)+\frac{5\pi^2}{12} \qquad (4.65)$$

$$\begin{aligned}
f_{7.2,1}(y,z) = {} & -4H(0;y)H(1,0;z)-4H(0;z)H(1-z,0;y)-4H(0,0;y)H(0;z)-6H(0,0;z)H(0;y)\\
& -\frac{21}{2}H(0,0,0;z)-4H(0,1,0;y)-\frac{11}{2}H(0,1,0;z)-4H(1,0;z)H(1-z;y)\\
& -\frac{15}{2}H(1,0,0;z)-\frac{7}{2}H(1,1,0;z)-4H(1-z,1,0;y)\\
& +\frac{\pi^2}{6}\left[-4H(0;y)-\frac{15}{2}H(0;z)-\frac{7}{2}H(1;z)-4H(1-z;y)\right] \qquad (4.66)
\end{aligned}$$

$$\begin{aligned}
f_{7.2,0}(y,z) = {} & +12H(0;y)H(1,0,0;z)+8H(0;y)H(1,1,0;z)+8H(0;z)H(1-z,0,0;y)\\
& +10H(0;z)H(1-z,1-z,0;y)+12H(0,0;y)H(0,0;z)+8H(0,0;y)H(1,0;z)\\
& +12H(0,0;z)H(1-z,0;y)+8H(0,0,0;y)H(0;z)+14H(0,0,0;z)H(0;y)\\
& +\frac{45}{2}H(0,0,0,0;z)+10H(0,0,1,0;y)+\frac{23}{2}H(0,0,1,0;z)+12H(0,1,0;z)H(0;y)\\
& +12H(0,1,0;z)H(1-z;y)+8H(0,1,0,0;y)+\frac{27}{2}H(0,1,0,0;z)-2H(0,1,1,0;y)\\
& +\frac{21}{2}H(0,1,1,0;z)+10H(0,1-z;y)H(1,0;z)+10H(0,1-z,0;y)H(0;z)\\
& +10H(0,1-z,1,0;y)+8H(1,0;z)H(1-z,0;y)+10H(1,0;z)H(1-z,1-z;y)\\
& +12H(1,0,0;z)H(1-z;y)+\frac{35}{2}H(1,0,0,0;z)+\frac{25}{2}H(1,0,1,0;z)\\
& +8H(1,1,0;z)H(1-z;y)+\frac{21}{2}H(1,1,0,0;z)+\frac{17}{2}H(1,1,1,0;z)+10H(1-z,0,1,0;y)\\
& +8H(1-z,1,0,0;y)-2H(1-z,1,1,0;y)+10H(1-z,1-z,1,0;y)+\frac{37\pi^4}{240}\\
& +\zeta_3\left[+4H(0;y)+6H(1;z)+4H(1-z;y)\right]\\
& +\frac{\pi^2}{6}\Big[+10H(0;y)H(0;z)+8H(0;y)H(1;z)+10H(0;z)H(1-z;y)+8H(0,0;y)\\
& +\frac{35}{2}H(0,0;z)-2H(0,1;y)+\frac{21}{2}H(0,1;z)+10H(0,1-z;y)+8H(1;z)H(1-z;y)\\
& +\frac{21}{2}H(1,0;z)+\frac{17}{2}H(1,1;z)+8H(1-z,0;y)-2H(1-z,1;y)+10H(1-z,1-z;y)\Big]
\end{aligned}$$

(4.67)

4.4 Comparison with results in the literature

The one-scale integrals of Section 4.1 were known in the literature [23, 24] for a long time and are listed only for completeness. The two-scale integrals of Section 4.2 existed in a closed form for arbitrary ϵ in terms of ordinary hypergeometric functions [6, 25–27]. We checked that our results, obtained from solving the differential equations for these integrals, agree with the ϵ-expansions of the hypergeometric functions quoted in [6, 25–27].

For the three-scale double box integral (4.56), we confirm the recent result of Smirnov [13], which was obtained using a Mellin-Barnes representation. In [13] analytic forms containing Nielsen's polylogarithms are given for all ϵ-divergent terms of (4.56); these terms can be compared directly. For the finite part of (4.56), Smirnov provides in [13] a result involving a one-dimensional integral plus other terms containing only Nielsen's polylogarithms of non-simple arguments. The integrand of the one-dimensional integral involves products of logarithms with dilogarithms. To compare with our result, we rewrote the finite part quoted in [13] in terms of 2dHPL. The procedure employed was as follows. In the limit $y \to 0$, we determined the z-dependent coefficients of $\ln^i y$ $(i = 1 \ldots 4)$ as well as the finite z-dependent term. The remaining terms, which are regular in $y \to 0$ were determined from the y-derivative of the result. Applying this derivative, one obtains $(1/y, 1/(1-y), 1/(1-y-z))$ times Nielsen's polylogarithms of level 3. These polylogarithms are differentiated with respect to y again, resulting in $(1/y, 1/(1-y), 1/(1-y-z))$ times Nielsen's polylogarithms of level 2, which are translated into 2dHPL of weight 2. The two y derivatives are undone by integration (keeping correct account of boundary terms by verifying the limit $y \to 0$ at each stage). As a result, we obtain 2dHPL of weight 4, as well as products of 2dHPL with ordinary HPL of z. After converting the overall normalisation factor, we find agreement of [13] with our result (4.56).

5 Conclusion

Two-loop four-point functions with one off-shell leg are an important ingredient to the calculation of next-to-next-to-leading order corrections to three jet production and related observables in electron-positron annihilation. By exploiting integration-by-parts [3–5] and Lorentz invariance [6] identities, one can express the loop integrals appearing in these functions as a linear combination of a small set of master integrals. These master integrals are scalar functions of the external invariants, whose determination was up to now a major obstacle to further progress in next-to-next-to-leading order calculations.

In the present paper, we compute all master integrals appearing in the reduction of planar two-loop four-point functions with massless internal propagators and one off-shell leg. The method employed here relies on the fact that all master integrals fulfil inhomogeneous differential equations in their external invariants. For the master integrals under consideration, these differential equations were derived in [6]. We determined the master integrals by solving the corresponding differential equations, starting from a product ansatz involving a rational function of the external invariants and a sum of newly introduced two-dimensional harmonic polylogarithms. The two-dimensional harmonic polylogarithms are an extension of the harmonic polylogarithms of [15], built *à la carte* to suit the needs of our calculation. We describe the underlying formalism in the Appendix and tabulate the relation of two-dimensional harmonic polylogarithms to the better known generalised polylogarithms of Nielsen [16]. For the divergent parts of the master integrals considered here, we find analytic expressions in terms of generalised polylogarithms, the finite part can be expressed as a one-dimensional integral over a combination of generalised polylogarithms.

A yet outstanding task is the computation of the master integrals appearing in the reduction of amplitudes with crossed topologies. The derivation of these from the corresponding differential equations should follow the procedure outlined in Section 2. The ansatz might however contain more than one rational function $\mathcal{R}_i$, as already observed in the on-shell case [9, 10]. Moreover, the determination of the boundary conditions might be more involved since the non-planar topologies have in general three branch points: in $y = 0$, $z = 0$ and $y = 1 - z$. This work is presently in progress.

A Harmonic Polylogarithms

Harmonic Polylogarithms (HPL) were introduced in [15] as an extension of the generalised polylogarithms of Nielsen [16]. They are constructed in such a way that they form a closed, linearly independent set under a certain class of integrations. We observe that the class of allowed integrations on this set can be extended *à la carte* by enlarging the definition of harmonic polylogarithms in order to suit the needs of a particular calculation. We will make use of this feature by generalising the one-dimensional HPL of [15] to two-dimensional harmonic polylogarithms (2dHPL), which appear in the solution of the differential equations for the three-scale master integrals discussed in this paper.

In the following subsection, we will briefly review the properties of the ordinary HPL, and tabulate expressions for HPL up to weight 4 in terms of generalised polylogarithms. In another subsection, we will then extend this framework to 2dHPL, which can in general be expressed up to weight 3 in generalised polylogarithms. We tabulate all 2dHPL up to weight 3 which are relevant to the calculation presented in this paper. The 2dHPL of weight 4 are expressed as one-dimensional integrals over combinations of generalised polylogarithms and therefore can be evaluated numerically in a straightforward manner.

A.1 One-dimensional Harmonic Polylogarithms

The one-dimensional HPL $H(\vec{m}_w;x)$ is described by a w-dimensional vector $\vec{m}_w$ of parameters and by its argument x. w is called the weight of H. We briefly recall the HPL formalism:

1. Definition of the HPL at $w=1$:

$$\begin{aligned} H(1;x) &\equiv -\ln(1-x)\ , \\ H(0;x) &\equiv \ln x\ , \\ H(-1;x) &\equiv \ln(1+x) \end{aligned} \tag{A.1}$$

   and the rational fractions in x

$$\begin{aligned} f(1;x) &\equiv \frac{1}{1-x}\ , \\ f(0;x) &\equiv \frac{1}{x}\ , \\ f(-1;x) &\equiv \frac{1}{1+x}\ , \end{aligned} \tag{A.2}$$

   such that

$$\frac{\partial}{\partial x}H(a;x) = f(a;x) \qquad \text{with} \quad a=+1,0,-1\ . \tag{A.3}$$

2. For $w>1$:

$$H(0,\ldots,0;x) \equiv \frac{1}{w!}\ln^w x\ , \tag{A.4}$$

$$H(a,\vec{b};x) \equiv \int_0^x \mathrm{d}x f(a;x)H(\vec{b};x)\ , \tag{A.5}$$

   which results in

$$\frac{\partial}{\partial x}H(a,\vec{b};x) = f(a;x)H(\vec{b};x)\ . \tag{A.6}$$

   This last relation is the main tool for verifying identities among different HPL. Such identities can be verified by first checking a special point (typically $x=0$) and subsequently checking the derivatives. If agreement in the derivatives is not obvious, this procedure can be repeated until one arrives at relations involving only HPL with $w=1$.

3. The HPL fulfil an algebra (see Section 3 of [15]), such that a product of two HPL (with weights w_1 and w_2) of the same argument x is a combination of HPL of argument x with weight $w = w_1 + w_2$, for example at $w_1 = 1, w_2 = 3$, this relation reads:

$$\begin{aligned} H(m_1;x)H(m_2,m_3,m_4;x) &= +H(m_1,m_2,m_3,m_4;x) + H(m_2,m_1,m_3,m_4;x) \\ &\quad +H(m_2,m_3,m_1,m_4;x) + H(m_2,m_3,m_4,m_1;x) \, . \end{aligned} \tag{A.7}$$

4. The HPL are linearly independent.

It can be seen from (A.5), that the HPL of parameters $(+1, 0, -1)$ form a closed set under the class of integrations

$$\int_0^x \mathrm{d}x \left(\frac{1}{x}, \frac{1}{1-x}, \frac{1}{1+x} \right) H(\vec{b};x) \, . \tag{A.8}$$

In the context of the present calculation, it turns out that integrals involving denominators $1/(1+x)$ do never occur, the HPL of parameters $(0, 1)$ are therefore sufficient.

The HPL of parameters $(0, 1)$ can all be expressed in terms of logarithms and Nielsen's polylogarithms. Up to level $m = 4$, they read:

$$\begin{aligned}
H(0;x) &= \ln x \, , \\
H(1;x) &= -\ln(1-x) \, , \\
H(0,0;x) &= \frac{1}{2!} \ln^2 x \, , \\
H(0,1;x) &= \mathrm{Li}_2(x) \, , \\
H(1,0;x) &= -\ln x \ln(1-x) - \mathrm{Li}_2(x) \, , \\
H(1,1;x) &= \frac{1}{2!} \ln^2(1-x) \, , \\
H(0,0,0;x) &= \frac{1}{3!} \ln^3 x \, , \\
H(0,0,1;x) &= \mathrm{Li}_3(x) \, , \\
H(0,1,0;x) &= -2\,\mathrm{Li}_3(x) + \ln x \,\mathrm{Li}_2(x) \, , \\
H(0,1,1;x) &= \mathrm{S}_{1,2}(x) \, , \\
H(1,0,0;x) &= -\frac{1}{2} \ln(1-x) \ln^2 x - \ln x \,\mathrm{Li}_2(x) + \mathrm{Li}_3(x) \, , \\
H(1,0,1;x) &= -2\,\mathrm{S}_{1,2}(x) - \ln(1-x)\,\mathrm{Li}_2(x) \, , \\
H(1,1,0;x) &= S_{1,2}(x) + \ln(1-x)\,\mathrm{Li}_2(x) + \frac{1}{2} \ln x \ln^2(1-x) \, , \\
H(1,1,1;x) &= -\frac{1}{3!} \ln^3(1-x) \, , \\
H(0,0,0,0;x) &= \frac{1}{4!} \ln^4 x \, , \\
H(0,0,0,1;x) &= \mathrm{Li}_4(x) \, , \\
H(0,0,1,0;x) &= \ln x \,\mathrm{Li}_3(x) - 3\,\mathrm{Li}_4(x) \, , \\
H(0,0,1,1;x) &= \mathrm{S}_{2,2}(x) \, , \\
H(0,1,0,0;x) &= \frac{1}{2} \ln^2 x \,\mathrm{Li}_2(x) - 2 \ln x \,\mathrm{Li}_3(x) + 3\,\mathrm{Li}_4(x) \, ,
\end{aligned}$$

$$
\begin{aligned}
H(0,1,0,1;x) &= -2\,\mathrm{S}_{2,2}(x)+\frac{1}{2}[\,\mathrm{Li}_2(x)]^2 \ , \\
H(0,1,1,0;x) &= \ln x S_{1,2}(x)-\frac{1}{2}[\,\mathrm{Li}_2(x)]^2 \ , \\
H(0,1,1,1;x) &= \mathrm{S}_{1,3}(x) \ , \\
H(1,0,0,0;x) &= -\frac{1}{6}\ln^3 x\ln(1-x)-\frac{1}{2}\ln^2 x\,\mathrm{Li}_2(x)+\ln(x)\,\mathrm{Li}_3(x)-\,\mathrm{Li}_4(x) \ , \\
H(1,0,0,1;x) &= -\frac{1}{2}[\,\mathrm{Li}_2(x)]^2-\ln(1-x)\,\mathrm{Li}_3(x) \ , \\
H(1,0,1,0;x) &= 2\ln(1-x)\,\mathrm{Li}_3(x)-\ln x\ln(1-x)\,\mathrm{Li}_2(x)-2\ln x\,\mathrm{S}_{1,2}(x)+\frac{1}{2}[\,\mathrm{Li}_2(x)]^2+2\,\mathrm{S}_{2,2}(x) \ , \\
H(1,0,1,1;x) &= -\ln(1-x)\,\mathrm{S}_{1,2}(x)-3\,\mathrm{S}_{1,3}(x) \ , \\
H(1,1,0,0;x) &= \frac{1}{4}\ln^2 x\ln^2(1-x)-\ln(1-x)\,\mathrm{Li}_3(x)+\ln x\ln(1-x)\,\mathrm{Li}_2(x)+\ln x\,\mathrm{S}_{1,2}(x)-\,\mathrm{S}_{2,2}(x) \ , \\
H(1,1,0,1;x) &= \frac{1}{2}\ln^2(1-x)\,\mathrm{Li}_2(x)+2\ln(1-x)\,\mathrm{S}_{1,2}(x)+3\,\mathrm{S}_{1,3}(x) \ , \\
H(1,1,1,0;x) &= -\frac{1}{6}\ln x\ln^3(1-x)-\frac{1}{2}\ln^2(1-x)\,\mathrm{Li}_2(x)-\ln(1-x)\,\mathrm{S}_{1,2}(x)-S_{1,3}(x) \ , \\
H(1,1,1,1;x) &= \frac{1}{4!}\ln^4(1-x) \ .
\end{aligned} \tag{A.9}
$$

A.2 Two-dimensional Harmonic Polylogarithms

The generalisation from one-dimensional to two-dimensional HPL starts from (A.8), which defines the class of integrations under which the HPL form a closed set. By inspection of the various inhomogeneous terms of the y differential equations for the three-scale master integrals discussed in this paper, we find that, besides the denominators $1/y$ and $1/(1-y)$ also $1/(1-y-z)$ and $1/(y+z)$ appear. It is therefore appropriate to introduce an extension of the HPL, which forms a closed set under the class of integrations

$$
\int_0^y \mathrm{d}y \left(\frac{1}{y},\frac{1}{1-y},\frac{1}{1-y-z},\frac{1}{y+z}\right) H(\vec{b};y) \ . \tag{A.10}
$$

Such an extension of the HPL formalism can be made by extending the set of fractions by

$$
\begin{aligned}
f(1-z;y) &\equiv \frac{1}{1-y-z} \ , \\
f(z;y) &\equiv \frac{1}{y+z} \ ,
\end{aligned} \tag{A.11}
$$

and correspondingly the set of HPL at weight 1 by

$$
\begin{aligned}
H(1-z;y) &= -\ln\left(1-\frac{y}{1-z}\right) \ , \\
H(z;y) &= \ln\left(\frac{y+z}{z}\right) .
\end{aligned} \tag{A.12}
$$

Allowing $(z,1-z)$ as components of the vector $\vec{m}_w$ of parameters, (A.5) does then define the extended set of HPL, which we call two-dimensional Harmonic Polylogarithms (2dHPL). They form a closed set under integrations of the form (A.10), as required. This closure was achieved by construction, showing that the HPL formalism of [15] can be extended to suit the needs of a particular calculation. We do not consider -1 among the components of $\vec{m}_w$, since they are not needed in the present context. The 2dHPL fulfil the same algebra as the HPL (an example of the algebra is given in (A.7)), and they are linearly independent. The feature of linear independence distinguishes the 2dHPL from Nielsen's polylogarithms of non-trivial two-variable arguments (in the literature, see for example [30], there are large numbers of relations between Nielsen's polylogarithms with arguments which are rational functions of two variables).

Two-dimensional Harmonic Polylogarithms can be expressed in terms of generalised polylogarithms only up to weight 3. At weight 4, only some special cases relate to generalised polylogarithms. The relations at weight 2 read:

$$\begin{aligned}
H(0,1-z;y) &= \mathrm{Li}_2\left(\frac{y}{1-z}\right) , \\
H(0,z;y) &= -\mathrm{Li}_2\left(-\frac{y}{z}\right) , \\
H(1,1-z;y) &= \frac{1}{2}\ln^2(1-y) - \ln(1-y)\ln(1-z) + \mathrm{Li}_2\left(\frac{z}{1-y}\right) - \mathrm{Li}_2(z) , \\
H(1,z;y) &= -\ln\left(\frac{1-y}{1+z}\right)\ln\left(\frac{y+z}{z}\right) + \mathrm{Li}_2\left(\frac{z}{1+z}\right) - \mathrm{Li}_2\left(\frac{y+z}{1+z}\right) , \\
H(1-z,0;y) &= -\ln y \ln\left(1-\frac{y}{1-z}\right) - \mathrm{Li}_2\left(\frac{y}{1-z}\right) , \\
H(1-z,1;y) &= -\frac{1}{2}\ln^2(1-y) + \ln(1-y-z)\ln(1-y) - \mathrm{Li}_2\left(\frac{z}{1-y}\right) + \mathrm{Li}_2(z) , \\
H(z,0;y) &= \ln y \ln\left(\frac{y+z}{z}\right) + \mathrm{Li}_2\left(-\frac{y}{z}\right) , \\
H(z,1;y) &= -\ln(1+z)\ln\left(\frac{y+z}{z}\right) - \mathrm{Li}_2\left(\frac{z}{1+z}\right) + \mathrm{Li}_2\left(\frac{y+z}{1+z}\right) , \\
H(1-z,z;y) &= -\ln(1-y-z)\ln\left(\frac{y+z}{z}\right) + \mathrm{Li}_2(z) - \mathrm{Li}_2(y+z) , \\
H(z,1-z;y) &= \ln(1-z)\ln\left(\frac{y+z}{z}\right) - \mathrm{Li}_2(z) + \mathrm{Li}_2(y+z) , \\
H(1-z,1-z;y) &= \frac{1}{2!}\ln^2\left(1-\frac{y}{1-z}\right) , \\
H(z,z;y) &= \frac{1}{2!}\ln^2\left(\frac{y+z}{z}\right) .
\end{aligned} \tag{A.13}$$

At weight 3, a total of 56 2dHPL exist. All of them can be expressed as combinations of generalised polylogarithms up to weight 3. We quote only the six 2dHPL of weight 3 which are relevant to the results presented in this paper:

$$\begin{aligned}
H(0,1-z,0;y) &= \ln y \,\mathrm{Li}_2\left(\frac{y}{1-z}\right) - 2\,\mathrm{Li}_3\left(\frac{y}{1-z}\right) , \\
H(1,1-z,0;y) &= \mathrm{S}_{1,2}(y) - \mathrm{Li}_3\left(\frac{yz}{(1-y)(1-z)}\right) + \mathrm{Li}_3\left(\frac{y}{1-z}\right) + \mathrm{Li}_3\left(\frac{z}{1-y}\right) \\
&\quad - \mathrm{Li}_3(y) - \mathrm{Li}_3(z) + \ln y \,\mathrm{Li}_2\left(\frac{z}{1-y}\right) + \ln(1-y)\,\mathrm{Li}_2(y) - \ln y \,\mathrm{Li}_2(z) \\
&\quad + \ln(1-y)\,\mathrm{Li}_2(z) - \ln y \ln(1-y)\ln(1-z) + \frac{1}{2}\ln y \ln^2(1-y) \\
&\quad + \frac{1}{2}\ln(1-z)\ln^2(1-y) ,
\end{aligned}$$

$$
\begin{aligned}
H(1-z,0,0;y) &= -\frac{1}{2}\ln^2 y \ln\left(1-\frac{y}{1-z}\right) - \ln y \,\mathrm{Li}_2\left(\frac{y}{1-z}\right) + \mathrm{Li}_3\left(\frac{y}{1-z}\right) , \\
H(1-z,1,0;y) &= \mathrm{Li}_3\left(\frac{-y}{1-y-z}\right) + \mathrm{Li}_3\left(\frac{-z}{1-y-z}\right) - \mathrm{Li}_3\left(\frac{-yz}{1-y-z}\right) + \mathrm{Li}_3(y) \\
&\quad - \mathrm{Li}_3\left(\frac{-z}{1-z}\right) - \ln y \,\mathrm{Li}_2\left(\frac{z}{1-y}\right) - \ln\left(1-\frac{y}{1-z}\right) \mathrm{Li}_2(z) + \ln y \,\mathrm{Li}_2(z) \\
&\quad -\frac{1}{6}\ln^3\left(1-\frac{y}{1-z}\right) + \ln y \ln(1-y) \ln\left(1-\frac{y}{1-z}\right) - \frac{1}{2}\ln y \ln^2(1-y) \\
&\quad + \ln y \ln(1-y)\ln(1-z) - \frac{1}{2}\ln(1-z)\ln^2\left(1-\frac{y}{1-z}\right) \\
&\quad -\frac{1}{2}\ln^2(1-z)\ln\left(1-\frac{y}{1-z}\right) , \\
H(1-z,1-z,0;y) &= \frac{1}{2}\ln y \ln^2\left(1-\frac{y}{1-z}\right) + \ln\left(1-\frac{y}{1-z}\right) \mathrm{Li}_2\left(\frac{y}{1-z}\right) + \mathrm{S}_{1,2}\left(\frac{y}{1-z}\right) , \\
H(z,1-z,0;y) &= \mathrm{Li}_3\left(\frac{y}{(1-z)(y+z)}\right) - \mathrm{Li}_3\left(\frac{y}{1-z}\right) - \mathrm{Li}_3\left(\frac{y}{y+z}\right) - \mathrm{Li}_3(y+z) + \mathrm{Li}_3(z) \\
&\quad - \ln y \,\mathrm{Li}_2(z) - \ln z \,\mathrm{Li}_2(z) + \ln(y+z)\,\mathrm{Li}_2(z) + \ln y \,\mathrm{Li}_2(y+z) \\
&\quad - \ln y \ln z \ln(1-z) + \ln y \ln(1-z)\ln(y+z) + \ln z \ln(1-z) \ln(y+z) \\
&\quad -\frac{1}{2}\ln(1-z)\ln^2(y+z) - \frac{1}{2}\ln^2 z \ln(1-z) . \qquad \text{(A.14)}
\end{aligned}
$$

The 2dHPL at weight 4 can be written as one-dimensional integrals:

$$
H(n,\vec{m}_3;y) = \int_0^y \mathrm{d}y f(n,y) H(\vec{m}_3,y) . \qquad \text{(A.15)}
$$

A.2.1 Limiting cases

If y or z are fixed to particular numerical values, or if they are related to each other, 2dHPL reduce to ordinary HPL. We have to derive reduction formulae for several cases, which are relevant to match the boundary conditions of the differential equations we study in this paper.

In $z=0$ and $z=1$, the reduction from 2dHPL to HPL is a mere substitution on the components of $\vec{m}_w$, combined with an adjustment of the overall sign of the HPL under consideration. The corresponding rules can be derived by writing out the 2dHPL as a multiple integral by repeated application of (A.5):

$$
H(\vec{m}_w;y) = \int_0^y \mathrm{d}t_1 f(m_1;t_1) \int_0^{t_1} \mathrm{d}t_2 f(m_2;t_2) \ldots \int_0^{t_{w-1}} \mathrm{d}t_w f(m_w;t_w) , \qquad \text{(A.16)}
$$

valid for $m_w \neq 0$, and

$$
H(\vec{m}_w;y) = \int_0^y \mathrm{d}t_1 f(m_1;t_1) \int_0^{t_1} \mathrm{d}t_2 f(m_2;t_2) \ldots \int_0^{t_{v-1}} \mathrm{d}t_v f(m_v;t_v) H(\vec{0}_{w-v};t_v) , \qquad \text{(A.17)}
$$

valid for $\vec{m}_w = (\vec{m}_v, \vec{0}_{w-v})$ with $m_v \neq 0$.

If $m_w \neq z$ (respectively $m_v \neq z$ in the second case), $H(\vec{m}_w;y)$ remains finite in the limit $z=0$. This limit is obtained by replacing $f(z;t_i) \to f(0;t_i)$ and $f(1-z;t_i) \to f(1;t_i)$ in (A.16,A.17) and subsequent integration. For $m_w = z$ (respectively $m_v = z$), $H(\vec{m}_w;y)$ diverges proportional to $\ln z$ in $z \to 0$. This divergence can be made explicit by the use of the HPL-algebra, such that $H(\vec{m}_w;y)$ can be written as linear combination of logarithmically divergent and finite terms in $z \to 0$.

If $m_w \neq 1-z$, (respectively $m_v \neq 1-z$), $H(\vec{m}_w;y)$ remains finite in the limit $z=1$, it develops divergent terms proportional to $\ln(1-z)$ otherwise. The limit $z=1$ is obtained by replacing $f(z;t_i) \to f(-1;t_i)$ and $f(1-z;t_i) \to -f(0;t_i)$ in (A.16,A.17) and subsequent integration.

To reduce 2dHPL in $y=1-z$, $y=-z$ or $y=1$ into HPL of argument z, one uses

$$H(\vec{m}(z);y(z)) = H(\vec{m}(z=0);y(z=0)) + \int_0^z \mathrm{d}z \frac{\mathrm{d}}{\mathrm{d}z} H(\vec{m}(z);y(z)) \,, \tag{A.18}$$

where the boundary $z=0$ can be replaced by $z=1$ if $H(\vec{m}(z=0);y(z=0))$ is divergent. The derivative $\frac{\mathrm{d}}{\mathrm{d}z}H(\vec{m}(z);y(z))$ is then carried out on the multiple integral representation of $H(\vec{m}(z);y(z))$ (A.16,A.17). The resulting $[f(m_i;t_i)]^2$ are reduced to single powers by repeated integration by parts and partial fractioning. The resulting multiple integral can then be identified as multiple integral representation of an ordinary HPL $H(\vec{m};z)$. As an example, consider

$$\begin{aligned}
H(1,1-z;1-z) &= H(1,0;0) + \int_1^z \mathrm{d}z \frac{\mathrm{d}}{\mathrm{d}z} H(1,1-z;1-z) \\
&= \int_1^z \mathrm{d}z \frac{\mathrm{d}}{\mathrm{d}z} \int_0^{1-z} \frac{\mathrm{d}t_1}{1-t_1} \int_0^{t_1} \frac{\mathrm{d}t_2}{1-t_2-z} \\
&= \int_1^z \mathrm{d}z \left[-\frac{1}{z} \int_0^{1-z} \frac{\mathrm{d}t_2}{1-t_2-z} + \int_0^{1-z} \frac{\mathrm{d}t_1}{1-t_1} \int_0^{t_1} \frac{\mathrm{d}t_2}{(1-t_2-z)^2} \right] \\
&= \int_1^z \mathrm{d}z \left(-\frac{1}{1-z} - \frac{1}{z} \right) \int_0^{1-z} \frac{\mathrm{d}t_1}{1-t_1} \\
&= \int_1^z \mathrm{d}z \left(-\frac{1}{1-z} - \frac{1}{z} \right) H(1;1-z) \\
&= \int_1^z \mathrm{d}z \left(+\frac{1}{1-z} + \frac{1}{z} \right) H(0;z) \\
&= H(0,0;z) + H(1,0;z) - H(1,0;1) \,.
\end{aligned} \tag{A.19}$$

To derive the transformation formulae in $y=1-z$, $y=-z$ or $y=1$ for 2dHPL up to weight 4, we have programmed the underlying algorithm in FORM [21]. It is evident from the above example that the transformation is carried out in a recursive manner, i.e. by using the results for 2dHPL of weight $w-1$ in the transformation of 2dHPL of weight w.

A.2.2 Interchange of arguments

To check our results for the master integrals, which were obtained in terms of $H(\vec{m}(z);y)$ by solving the y differential equations, we inserted them into the corresponding z differential equations. To carry out this check, we have to convert the 2dHPL $H(\vec{m}(z);y)$ into 2dHPL $H(\vec{m}(y);z)$ and ordinary HPL $H(\vec{m};y)$. This conversion is made using

$$H(\vec{m}(z);y) = H(\vec{m}(z=0);y) + \int_0^z \mathrm{d}z \frac{\mathrm{d}}{\mathrm{d}z} H(\vec{m}(z);y) \,, \tag{A.20}$$

where the boundary $z=0$ can be replaced by $z=1$ if $H(\vec{m}(z=0);y)$ is divergent. The z-derivative is again carried out on the multiple integral representation of $H(\vec{m}(z);y)$ (A.16,A.17). Repeated application of integration by parts and partial fractioning does then generate a form that can be identified as a bilinear combination of 2dHPL $H(\vec{m}(y);z)$ and ordinary HPL $H(\vec{m};y)$. As an example, consider

$$\begin{aligned}
H(1,1-z;y) &= H(1,1;y) + \int_0^z \mathrm{d}z \frac{\mathrm{d}}{\mathrm{d}z} H(1,1-z;y) \\
&= H(1,1;y) + \int_0^z \mathrm{d}z \frac{\mathrm{d}}{\mathrm{d}z} \int_0^y \frac{\mathrm{d}t_1}{1-t_1} \int_0^{t_1} \frac{\mathrm{d}t_2}{1-t_2-z}
\end{aligned}$$

$$
\begin{aligned}
&= H(1,1;y) + \int_0^z \mathrm{d}z \int_0^y \frac{\mathrm{d}t_1}{1-t_1} \int_0^{t_1} \frac{\mathrm{d}t_2}{(1-t_2-z)^2} \\
&= H(1,1;y) + \int_0^z \mathrm{d}z \int_0^y \mathrm{d}t_1 \left[\left(-\frac{1}{1-z} - \frac{1}{z} \right) \frac{1}{1-t_1} + \frac{1}{z} \frac{1}{1-t_1-z} \right] \\
&= H(1,1;y) + \int_0^z \mathrm{d}z \left[\left(-\frac{1}{1-z} - \frac{1}{z} \right) H(1;y) + \frac{1}{z} H(1-z;y) \right] \\
&= H(1,1;y) - H(1;y) \left[H(0;z) + H(1;z) \right] + \int_0^z \frac{\mathrm{d}z}{z} \left[H(1-y;z) + H(1;y) - H(1;z) \right] \\
&= H(1,1;y) + H(0,1-y;z) - H(0,1;z) - H(1;y)H(1;z) \, . \qquad \text{(A.21)}
\end{aligned}
$$

Interchange formulae have been derived for 2dHPL up to weight 4 by programming the underlying algorithm in FORM [21]. Like the transformation algorithm presented above, the interchange algorithm also works recursively by using interchange formulae obtained for lower weights.

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