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Two Loop QCD Vertices and Three Loop MOM β functions

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Abstract

We present numerical results for the two loop QCD corrections to the ghost gluon vertex, the quark gluon vertex, and the triple gluon vertex in the massless limit at the symmetric point. We restrict ourselves to the tensor structures existing in the tree level. The corrections are used to examine different coupling constant definitions in momentum subtraction schemes and the corresponding three-loop β functions.

1 Introduction

There is no unique definition of the QCD coupling constant – α_s: its value depends on the renormalization prescription employed. Within pQCD the definition which is most often used is based on the $\overline{\text{MS}}$-scheme [1, 2]. Such a definition is of great convenience for dealing with inclusive physical observables dominated by short distances (for a review see [3]). On one side the underlying use of dimensional regularization makes advanced calculations possible: e.g. the corresponding $\beta^{\overline{\text{MS}}}$-function is available now with four-loop accuracy [4]. On the other one, this makes the $\overline{\text{MS}}$-scheme virtually inapplicable in cases when one chooses to utilize a different regularization, including, most regrettably, the lattice one. In

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addition, the physical meaning of the $\overline{\text{MS}}$ normalization parameter μ is not transparent and leads to the well-known ambiguities when considering the decoupling of heavy particles.

These shortcomings are absent for a wide class of so-called momentum subtraction (MOM) schemes. The MOM-schemes require the values of properly chosen Green functions with predefined μ-dependent configurations of external momenta to be fixed (usually to their tree values) independently on the considered order. Practical calculations can then be performed with any regularization, including the lattice one.

Unfortunately, even for massless QCD, there are infinitely many possibilities to define a momentum subtraction renormalization scheme. Not only is there an ambiguity which vertex to subtract, and at which exact configuration of external momenta. In addition, there is the freedom to use a certain linear combination of the scalar functions appearing in the gluon and quark vertices, which can be related to fixed polarization states of the external particles [5].

Recently, the momentum subtraction approach has been heavily used to relate lattice results for quark masses [6, 7, 8] and coupling constants [9, 10, 11, 12, 13, 14, 15] to their perturbatively determined $\overline{\text{MS}}$ counterparts.

In one of these papers [13] it has been argued that the knowledge of three-loop coefficients for the corresponding β-functions is necessary and even the four-loop contributions should be taken in to account. This is because the accessible energy ranges in these calculations are just reaching a level where perturbative QCD calculations start to be valid approximations.

The results of these lattice calculations have mainly been analyzed within the so-called $\widetilde{\text{MOM}}$-scheme. The scheme is defined using an asymmetric subtraction at $q_1^2 = q_2^2 = -\mu^2, q_3 = 0$. The perturbative calculations with such a definition are certainly *much* simpler than those for the schemes employing the symmetric subtraction point $q_1^2 = q_2^2 = q_3^2 = -\mu^2$. As a consequence only for that scheme the non-trivial three-loop coefficient of the β-function is known[1]. On the other hand, on general grounds it is rather clear that α_s defined through the symmetric subtraction point should be less prone to all kinds of non-perturbative effects than the one based on the asymmetric subtraction.

In this paper we suggest a new (approximate) way to compute (massless) vertices at the symmetric point by using a large momentum expansion. The method reduces the problem to evaluation of massless propagators. We compute the two loop QCD corrections to the ghost gluon vertex, the quark gluon vertex, and the triple gluon vertex in the massless limit at the symmetric point in the $\overline{\text{MS}}$-scheme. The obtained results are then used to construct different coupling constant definitions in momentum subtraction schemes and the corresponding three-loop β functions.

This letter is organized as follows: In the next Section we discuss the crucial difference concerning possible non-perturbative corrections to the running of the coupling constant defined with subtraction at the symmetric momentum configuration (which is used in MOM-schemes) and the asymmetric one (used in the so-called $\widetilde{\text{MOM}}$-schemes). Section 3 briefly describes our calculational method. In Section 4 the results for the vertex corrections

[1] Very recently even the four-loop term has been computed in [16]

at two loops are given. Then we derive the relations between the coupling constants in the MOM- and the $\overline{\mathrm{MS}}$-scheme (Section 5) and close by a short discussion on the effects of our results on the three loop β function in the MOM-scheme (Section 6).

2 MOM vs. $\widetilde{\mathrm{MOM}}$

A well-known way to take into account at least some of non-perturbative physics is the QCD sum rules method [17] based on the use of Operator Product Expansion (OPE) (for a recent review, see [18]). For instance, let us consider a correlator of two local operators

$$G^{AB}(q) = i \int \mathrm{e}^{iqx} \mathrm{d}x \langle 0|T\{A(x)B(0)\}|0\rangle. \tag{1}$$

At large (and Euclidean) external momentum transfer q the correlator (1) can be schematically represented in the form

$$G^{AB}(q) \underset{q\to\infty}{=\!=\!=} G_0^{AB}(q) + \sum_n C_n^{AB}(q)\langle O_n(0)\rangle, \tag{2}$$

where the first term, $G_0^{AB}(q)$, stands for the purely perturbation theory contribution while the sum describes the factorization of the non-perturbative effects due to the large distances (hidden inside of vacuum expectation values (VEV) of various composite operators) and the coefficient functions $C_n^{AB}(q)$. The latter correspond to the short distance (of order $1/|q|$) contributions coming from the integration with respect to x in (1) and, thus, are computable within perturbation theory.

Next, let us consider a three-point correlator

$$G^{ABC}(p,q) = i^2 \int \mathrm{e}^{iqx+ipy} \mathrm{d}x\mathrm{d}y \langle 0|T\{A(x)B(0)C(y)\}|0\rangle \tag{3}$$

in the kinematical regime relevant for the MOM-scheme, that is for both q and p being large. In this case one can straitforwardly apply the OPE for *three* operators and write:

$$G^{ABC}(p,q) \underset{q,p\to\infty}{=\!=\!=} G_0^{ABC}(p,q) + \sum_n C_n^{ABC}(p,q)\langle O_n(0)\rangle. \tag{4}$$

Here the coefficient functions $C_n^{ABC}(p,q)$ are computable as an expansion in the (small) coupling constant $\alpha_s(\mu)$ normalized at (large) momentum scale of order $|p| \approx |q|$. An important feature of both Eqs. (2) and (4) is that the VEV of composite operators are universal and do not depend on the correlator.

The situation is significantly different for the case when only one external momentum, say, q is large and another one is fixed or even set to zero (the latter case corresponds to a $\widetilde{\mathrm{MOM}}$-scheme). Indeed, in this case the representation (4) should be replaced with [19]

$$\begin{aligned} G^{ABC}(p,q) \underset{q\to\infty}{=\!=\!=} \; & G_0^{ABC}(p,q) + \sum_i C_n^{\widetilde{ABC}}(p,q)\langle O_n(0)\rangle \\ & + \sum_n C_n^{AB}(q)\; i \int \mathrm{e}^{ipy}\mathrm{d}y\langle 0|T\{C(y)O_n(0)\}|0\rangle. \end{aligned} \tag{5}$$

In this representation the first sum comes from the integration region where *both* x and y are small and of order $1/|q|$. Note that the functions $C_n^{ABC}(p,q)$ are in general different from $C_n^{\widetilde{ABC}}(p,q)$.

The second sum results from integration regions of small $x \approx 1/|q|$ only (where the OPE of two operators A and B can be employed). As the momentum p is assumed to be small or even zero the two-point correlators in the second sum *can not* be perturbatively computed and should be considered as phenomenological quantities analogous to "condensates".

Thus, we conclude that the three-point correlators employed in $\widetilde{\text{MOM}}$-schemes contain, in in addition to the VEV of composite operators, an extra source of the non-perturbative corrections — VEV of bilocal operators — in comparison to the same correlators with symmetrical pattern of the external momenta.

3 Method of the calculation

The analytic computation of arbitrary two loop three point functions seems to be excluded at the time being. Therefore our computation is based on the method of asymptotic expansions [20] which reduces the complexity of the integrals. We consider three point graphs with incoming momenta q_1, q_2, and $-(q_1+q_2)$ and identify one momentum as large, say q_1. The ordinary large momentum procedure is used to find an expansion with respect to q_2^2/q_1^2 and $q_1.q_2/q_1^2$. The symmetric point further demands that $(q_1+q_2)^2 = q_1^2$, so $q_1.q_2$ can be reexpressed by q_2^2. Setting $q_2^2 = zq_1^2$ yields a series in the variable z.

Using the described method we found very good numerical agreement with the analytical results for the one loop vertex corrections given in [21, 22] and the one and two loop scalar three point integrals given in [23] by computing only 4 to 6 terms of the expansion.

The use of the large momentum procedure results in an asymmetric approach and marks one momentum. Therefore one obtains at least two independent series by choosing different momentum distributions. The numerical values of the series must coincide at the symmetrical point ($z = 1$). This provides a powerful check on our method and can be tested by examining the non-symmetric scalar ladder type topology computed in [23].

We use the series with the best convergence to estimate the mean value and the variation of independent expansions as an indication for the size of the uncertainty. This description is in most cases equivalent to an error estimation based on the size of the highest expansion term available.

Throughout the whole computation we made use of several computer programs. The diagrams were generated using `QGRAF` [24] and the large momentum procedure was applied by `EXP` [25]. The actual evaluation of the integrals was done using the `MINCER` [26] package written in `FORM` [27].

4 Two loop vertex corrections

We parametrize the QCD vertices at the symmetrical point ($p_1^2 = p_2^2 = p_3^2 = p^2$) as follows:

	C_F	C_A	Tn_f
$\tilde{\Gamma}^{(1)}(-\mu^2)$	—	$\frac{3}{32}+\frac{1}{96}I$	—
$\Lambda^{(1)}(-\mu^2)$	$-\frac{1}{2}+\frac{1}{12}I$	$\frac{13}{16}-\frac{13}{96}I$	—
$\Gamma^{(1)}(-\mu^2)$	—	$-\frac{3}{32}+\frac{23}{288}I$	$\frac{1}{2}-\frac{2}{9}I$

Table 1: One loop results — by SU(N) color factors.

	C_F^2	C_FC_A	C_A^2	C_ATn_f	C_FTn_f
$\tilde{\Gamma}^{(2)}(-\mu^2)$	—	—	0.197(1)	−0.151(2)	—
$\Lambda^{(2)}(-\mu^2)$	0.206(4)	−0.20(4)	0.679(1)	−0.4968(4)	−0.0211(4)
$\Gamma^{(2)}(-\mu^2)$	—	—	−0.22(4)	0.65(7)	−0.408(10)

Table 2: Two loop results — by SU(N) color factors.

- ghost gluon vertex (p_1 is the incoming momentum of the out-ghost)

$$\tilde{\Gamma}^{abc}_{\mu}(p_1,p_2,p_3) = ig_s f^{abc} p_1^{\nu}\left(g_{\mu\nu}\tilde{\Gamma}(p^2)+\cdots\right), \tag{6}$$

- quark gluon vertex

$$\Lambda^{a}_{\mu,ij}(p_1,p_2,p_3) = g_s T^a_{ij}\left(\gamma_\mu \Lambda(p^2)+\cdots\right), \tag{7}$$

- triple gluon vertex

$$\begin{aligned}\Gamma^{abc}_{\mu\nu\lambda}(p_1,p_2,p_3) = {} & -ig_s f^{abc}\times \\ & \left[\left(g_{\mu\nu}\left(p_1-p_2\right)_\lambda + g_{\nu\lambda}\left(p_2-p_3\right)_\mu + g_{\lambda\mu}\left(p_3-p_1\right)_\nu\right)\Gamma(p^2)+\cdots\right]\end{aligned} \tag{8}$$

with the strong coupling constant g_s, the structure constants of the SU(N) Lie algebra f^{abc}, and the matrices of the fundamental representation T^a_{ij}. The terms contained in the ellipsis are defined in such a way that they do not interfere with the displayed form factor. This means that e.g. for Eq. (7) the dots could contain only the structures $p_i^\mu, \not{p}_i p_j^\mu$ and so on. Note, that our choice of the kinematical structures (6-8) is, probably, the simplest one. However, our approach could be applied, if necessary, to any other choice, should the latter be more convenient and/or natural for a specific physical problem.

With Eqs. (7) and (8) we follow the prescription of [21]. Concerning the ghost gluon vertex our definition partly avoids the ambiguity connected with the identification of the vertex form factors and differs from the one used in [21].

Each form factor receives quantum corrections:

$$\Gamma = 1 + \frac{\alpha_s}{\pi}\Gamma^{(1)} + \left(\frac{\alpha_s}{\pi}\right)^2 \Gamma^{(2)} + \cdots \tag{9}$$

and similarly for $\tilde{\Gamma}$ and Λ.

In the following we restrict ourselves to Landau gauge so the gluon propagator is taken to be purely transversal. A further simplification can be achieved by observing that the MOM-scheme is usually defined at $p^2 = -\mu^2$ with the 't Hooft mass μ. Therefore all logarithms $\log(-p^2/\mu^2)$ drop out. All quantities displayed in the following are computed using the $\overline{\mathrm{MS}}$-scheme for the renormalization.

The one loop corrections to the ghost gluon vertex, the quark gluon vertex, and the triple gluon vertex can be found in [21]. There the integral

$$I = -2\int_0^1 \frac{\log(x)}{x^2 - x + 1} = 2.3439072\ldots \tag{10}$$

was defined. The results are displayed in table 1 where we have distinguished between contributions proportional to different color factors of the SU(N) (for SU(3) $C_F = 4/3$, $C_A = 3$, and $T = 1/2$, n_f represents the number of fermions).

Our numerical results for the two loop contributions are summarized in table 2. The number in the brackets denotes our error estimation on the last digit.

Inserting the values of the color factors for SU(3) and linearly adding the errors one would surely overestimate the uncertainty. Therefore we first computed the series with color factors already inserted and then extracted the results which read

$$\begin{aligned}
\tilde{\Gamma}^{(2)}(-\mu^2) &= 1.770(9) - 0.1727(3)\, n_f, \\
\Lambda^{(2)}(-\mu^2) &= 5.590(6) - 0.7594(2)\, n_f, \\
\Gamma^{(2)}(-\mu^2) &= -2.0(4) + 0.72(9)\, n_f.
\end{aligned} \tag{11}$$

5 Relations between coupling constants

The MOM renormalization condition implies that the value of one of the above defined functions should be equal to 1 when the corresponding MOM-scheme is used. This allows for the computation of the coupling constant in that MOM-scheme (α_s^{MOM}) as a perturbative series in the coupling constant of the $\overline{\mathrm{MS}}$-scheme ($\alpha_s^{\overline{\mathrm{MS}}}$) which is of the form

$$\frac{\alpha_s^{\mathrm{MOM}}}{\alpha_s^{\overline{\mathrm{MS}}}} = \left\{1 + \frac{\alpha_s^{\overline{\mathrm{MS}}}}{\pi}(d_{10} + d_{11}n_f) + \left(\frac{\alpha_s^{\overline{\mathrm{MS}}}}{\pi}\right)^2 \left(d_{20} + d_{21}n_f + d_{22}n_f^2\right) + \cdots\right\}. \tag{12}$$

We will refer to MOM quantites computed on the basis of the triple gluon vertex, the quark gluon vertex, and the ghost gluon vertex as MOMggg, MOMq, and MOMh, respectively.

Using the well known results for the self energies of the quark, the gluon, and the ghost at one and two loops in the $\overline{\mathrm{MS}}$-scheme we find the coefficients shown in table 3.

	d_{10}	d_{11}	d_{20}	d_{21}	d_{22}
MOMh	$\frac{49}{12}+\frac{1}{16}I$	$-\frac{5}{18}$	35.88(2)	−5.0707(6)	$\frac{25}{324}$
MOMq	$\frac{89}{16}-\frac{85}{144}I$	$-\frac{5}{18}$	29.53(1)	−5.1961(4)	$\frac{25}{324}$
MOMggg	$\frac{11}{2}+\frac{23}{48}I$	$-\frac{1}{3}-\frac{2}{9}I$	59.8(8)	−12.6(2)	$\frac{47}{432}+\frac{7}{54}I+\frac{1}{81}I^2$

Table 3: Coefficients of the relation between the coupling constants in the MOM- and the $\overline{\mathrm{MS}}$-scheme.

6 Beta function

We define the perturbative series of a generic β function by

$$\mu^2 \frac{\mathrm{d}}{\mathrm{d}\mu^2} \frac{\alpha_s(\mu^2)}{\pi} = \beta(\alpha_s) = -\left(\frac{\alpha_s}{\pi}\right)^2 \sum_{i\geq 0} \beta_i \left(\frac{\alpha_s}{\pi}\right)^i . \tag{13}$$

The β function in the MOM-scheme (β^{MOM}) can be computed directly from the results presented in the last section using the relation

$$\beta^{\mathrm{MOM}}(\alpha_s^{\mathrm{MOM}}) = \mu^2 \frac{\mathrm{d}}{\mathrm{d}\mu^2} \frac{\alpha_s^{\mathrm{MOM}}(\mu^2)}{\pi} = \beta^{\overline{\mathrm{MS}}} \frac{\mathrm{d}\alpha_s^{\mathrm{MOM}}}{\mathrm{d}\alpha_s^{\overline{\mathrm{MS}}}}\bigg|_{\alpha_s^{\overline{\mathrm{MS}}}\to\alpha_s^{\mathrm{MOM}}} . \tag{14}$$

In our framework the first two terms in the perturbative series of the β function are scheme-independent. To compute β_2^{MOM} the knowledge of the two loop vertex corrections is sufficient if $\beta_2^{\overline{\mathrm{MS}}}$ is given.

$\beta^{\overline{\mathrm{MS}}}$ is known up to four loops [28, 29, 30, 4] and the first three terms read:

$$\begin{aligned}
\beta_0^{\overline{\mathrm{MS}}} &= \frac{1}{4}\left(\frac{11}{3}C_A - \frac{4}{3}Tn_f\right), \\
\beta_1^{\overline{\mathrm{MS}}} &= \frac{1}{16}\left(\frac{34}{3}C_A^2 - 4C_FTn_f - \frac{20}{3}C_ATn_f\right), \\
\beta_2^{\overline{\mathrm{MS}}} &= \frac{1}{64}\bigg(\frac{2857}{54}C_A^3 + 2C_F^2Tn_f - \frac{205}{9}C_FC_ATn_f - \frac{1415}{27}C_A^2Tn_f \\
&\qquad + \frac{44}{9}C_FT^2n_f^2 + \frac{158}{27}C_AT^2n_f^2\bigg).
\end{aligned} \tag{15}$$

The vertex corrections given above lead to the following values for the three loop β function in the MOM-scheme:

$$\begin{aligned}
\beta_2^{\mathrm{MOMh}} &= 44.82(5) - 9.730(5)\,n_f + 0.3276(1)\,n_f^2, \\
\beta_2^{\mathrm{MOMq}} &= 28.86(3) - 9.206(3)\,n_f + 0.35322(7)\,n_f^2, \\
\beta_2^{\mathrm{MOMggg}} &= 24(2) + 0.04(63)\,n_f - 1.05(3)\,n_f^2 + 0.0415330\,n_f^3.
\end{aligned} \tag{16}$$

Note, that a large uncertainty of the n_f-coefficient for the MOMggg-scheme is the result of an accidental cancellation between $\beta_2^{\overline{\rm MS}}$ and the corrections induced by the one and two loop vertex corrections in this case.

It is instructive to compare the MOMggg-scheme with the (standard) $\overline{\rm MS}$-scheme as well as with the $\widetilde{\rm MOM}$gg one. The latter is defined with the help of the asymmetric subtraction of the triple gluon vertex (more details in [11, 16]). The three-loop contributions to the corresponding β-functions read

$$\begin{aligned} \beta_2^{\overline{\rm MS}} &= 22.3203 - 4.36892\, n_f + 0.0940394\, n_f^2, \\ \beta_2^{\widetilde{\rm MOM}\rm gg} &= 37.6899 - 5.57013\, n_f - 0.223177\, n_f^2 + 0.0138889\, n_f^3. \end{aligned} \tag{17}$$

Thus, one observes that (at three loops) the MOMggg-scheme is numerically significantly closer to the $\overline{\rm MS}$-scheme that to the $\widetilde{\rm MOM}$gg one.

Very recently, in [31] an attempt has been made to "predict" the value of $\beta_2^{\rm MOMggg}|_{n_f=0}$ with the help of an OPE analysis of the flavourless non-perturbative gluon propagator and the symmetric triple gluon vertex in the Landau gauge. They have found

$$\beta_2^{\rm MOMggg}|_{lattice} = 1.5(3)\, \beta_2^{\widetilde{\rm MOM}\rm gg}|_{n_f=0} \tag{18}$$

while our result is

$$\beta_2^{\rm MOMggg}|_{n_f=0} = 0.64(5)\, \beta_2^{\widetilde{\rm MOM}\rm gg}|_{n_f=0}. \tag{19}$$

We believe that the rough (that is in the sign and overall magnitude) agreement between Eqs. (18) and (19) could be improved by taking into account anomalous dimensions of power corrections as discussed in [31].

Conclusion

We have presented a new approach to approximately compute three-point Green functions in massless theories. The approach is based on the use of asymptotic expansion methods. It reduces the problem of computing of a three-point N-loop correlator (with Euclidean external momenta) to the calculation of the N-loop massless propagators. We have checked the method by recomputing a few analytically known results. In all cases the resulting series proved to be accurate approximations to the exact results.

We have applied the approach to compute the two loop QCD corrections to the ghost gluon vertex, the quark gluon vertex, and the triple gluon vertex in the massless limit at the symmetric momentum point. The results have been used to construct the corresponding three-loop β functions. In principle, the calculation could be extended to include one more loop (as the three-loop massless propagators are still accessible with `MINCER`). Unfortunately, for the time being the hardware constraints look insurmountable.

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