Freiburg-THEP 00/11
TTP00–15
hep-ph/0007088

Three-loop Three-Linear Vertices and Four-Loop $\widetilde{\text{MOM}}$ β functions in massless QCD

K.G. Chetyrkin[*a,b] and A. Rétey[b]

[a] *Fakultät für Physik,*
Albert-Ludwigs-Universität Freiburg, D-79104 Freiburg, Germany
[b] *Institut für Theoretische Teilchenphysik,*
Universität Karlsruhe, D-76128 Karlsruhe, Germany

Abstract

In this paper we present a full set of 2- and 3-point functions for massless QCD at three-loop order in the $\overline{\text{MS}}$ scheme. The vertex functions are evaluated at the asymmetric point with one vanishing momentum. These results are used to relate the $\overline{\text{MS}}$ coupling constant to that of various momentum subtraction (MOM) renormalization schemes at three-loop order. With the help of the known four-loop $\overline{\text{MS}}$ β-function we then determine the four-loop coefficients of the corresponding $\widetilde{\text{MOM}}$ β-functions.

As an application we consider the momentum dependence (running) of the three-gluon asymmetrical vertex recently computed within the lattice approach by Ph. Boucaud et al. in [1]. An account of the four-loop term in the corresponding β-function leads to a significant (around 30%) decrease of the value of the non-perturbative $1/p^2$ correction to the running found in [1].

*Permanent address: Institute for Nuclear Research, Russian Academy of Sciences, 60th October Anniversary Prospect 7a, Moscow 117312, Russia.

1 Introduction

Momentum subtraction schemes provide a possibility to define a renormalization description for QCD in a regularization independent way. The concept of momentum subtraction is very old. It played an important role in the discussion of renormalization description dependence of physical quantities long ago [2, 3, 4, 5]. Recently, the momentum subtraction approach has been heavily used to relate lattice results for quark masses and coupling constants to their perturbatively determined $\overline{\text{MS}}$ counterparts (see, e.g. [6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16]). In [1, 13, 14] it has been argued that the knowledge of three-loop coefficients for the corresponding β-functions is necessary and even the four-loop contributions should be taken in to account. This is because the accessible energy ranges in these calculations are just reaching a level where perturbative QCD calculations start to be valid approximations.

A large subclass (in fact infinitely many) of MOM schemes can be defined by subtracting vertices at the asymmetric point where one external momentum vanishes. We will call this point the zero point (ZP). These schemes are referred to as $\widetilde{\text{MOM}}$ schemes and were introduced and discussed in some detail at two-loop order in [3]. A crucial fact for $\widetilde{\text{MOM}}$ schemes is that setting one external momentum to zero for all three 3-vertices of massless QCD never will produce infrared divergencies.

In this paper we present the perturbative calculation of the gluon, ghost and quark self-energies and all fundamental[1] 3-vertices with one vanishing external momentum for massless QCD in a general covariant gauge. The one-loop triple gluon vertex was obtained in [17] in Feynman gauge, the result for general gauge can be found in [18]. At two loops, the triple gluon and ghost gluon vertices at the ZP have been determined in general gauge in [19]. References to earlier relevant publications and results for various momentum configurations can also be found in this work. The quark-gluon vertex can be found to two-loop order in Feynman gauge in [3].

These three-loop results at hand allow one to relate the coupling constants of any $\widetilde{\text{MOM}}$-like scheme to the $\overline{\text{MS}}$ scheme at three-loop order. Recently, in [20] even the four-loop term of the $\overline{\text{MS}}$ β-function has been computed. Using this result, we can also determine the β-functions of any such $\widetilde{\text{MOM}}$-scheme up to (and including) four loops.

The paper is organized as follows: In Section 2 the definitions of the ghost and quark self-energy and the gluon polarization are introduced and a brief outline of their calculation is given. In Section 3 we introduce the triple gluon, quark gluon and ghost gluon vertices at the ZP and our notation for these along with a description of their calculation. The calculation of the triple gluon vertex is performed directly and additionally in an independent way using the Ward-Slavnov-Taylor (WST) identity relating the triple gluon vertex to the ghost gluon vertex. This is content of Section 4. In Section 5 the $\overline{\text{MS}}$ renormalization procedure is described. In Sections 6 and 7 we use the results of the previous Sections to obtain the coupling constants and β-functions for 4 different $\widetilde{\text{MOM}}$ schemes of particular interest from their $\overline{\text{MS}}$ counterparts. Finally, in Section 8 we discuss the numerical importance of our results on the example of a recent lattice computation [1] of the momentum dependence (running) of the three-gluon asymmetrical vertex.

The complete results are given in the appendix and also will be made available as input file for the algebraic programs FORM and MATHEMATICA at:

`http://www-ttp.physik.uni-karlsruhe.de/Progdata/`

[1] That is appearing in the QCD Lagrangian

2 The Gluon Polarization and the Ghost and Quark Self-Energies

The QCD Lagrangian with n_f massless quark flavors in the covariant gauge is:

$$
\begin{aligned}
\mathcal{L} &= -\frac{1}{4}G^a_{\mu\nu}G^{a\mu\nu} + \mathrm{i}\sum_{f=1}^{n_f}\bar{\psi}^f_i[\not{D}]_{ij}\psi^f_j - \frac{1}{2\xi_L}(\partial^\mu A^a_\mu)^2 + \partial^\mu\bar{\eta}^a(\partial\eta^a - gf^{abc}\eta^b A^c_\mu), \qquad (1)\\
G^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^a_\mu A^b_\nu, \quad [D_\mu]_{ij} = \delta_{ij}\partial_\mu - \mathrm{i}gA^a_\mu T^a_{ij}.
\end{aligned}
$$

The quark fields ψ^f_i transform as the fundamental representation and the gluon fields A^a_μ as the adjoint representation of the gauge group SU(3). T^a_{ij} and f^{abc} are the generators of the fundamental and adjoint representation of the corresponding Lie algebra. The η^a are the ghost fields and ξ_L is the gauge parameter ($\xi_L = 0$ corresponds to Landau gauge).

From this Lagrangian one can derive three types of 2-point functions

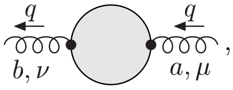
$$
\begin{aligned}
G^{(2)\,ab}_{\mu\nu}(q) &= D^{ab}_{\mu\nu}(q) = \mathrm{i}\int \mathrm{d}x\, \mathrm{e}^{\mathrm{i}qx}\langle T[A^a_\mu(x)A^b_\nu(0)]\rangle,\\
G^{(2)\,ab}(q) &= \Delta^{ab}(q) = \mathrm{i}\int \mathrm{d}x\, \mathrm{e}^{\mathrm{i}qx}\langle T[\eta^a(x)\bar{\eta}^b(0)]\rangle, \qquad (2)\\
G^{(2)}_{ij}(q) &= S_{ij}(q) = \mathrm{i}\int \mathrm{d}x\, \mathrm{e}^{\mathrm{i}qx}\langle T[\psi_i(x)\bar{\psi}_j(0)]\rangle,
\end{aligned}
$$

where as usual $T[AB]$ is the time ordered product of A and B and from now on we will skip the flavor index of the quarks. The propagators in eq. (2) can be expressed in terms of the corresponding self-energies in the following way:

$$
\begin{aligned}
D^{ab}_{\mu\nu}(q) &= \frac{\delta^{ab}}{(-q^2)}\left[(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2})\frac{1}{1+\Pi(q^2)} - \xi_L\frac{q_\mu q_\nu}{q^2}\right],\\
\Delta^{ab}(q) &= \frac{\delta^{ab}}{(-q^2)}\frac{1}{1+\tilde{\Pi}(p^2)}, \qquad (3)\\
S_{ij}(q) &= \frac{\delta^{ij}}{(-q^2)}\frac{\not{q}}{1+\Sigma_V(q^2)}.
\end{aligned}
$$

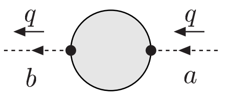
The self-energies $\Pi(q^2)$, $\tilde{\Pi}(q^2)$ and $\Sigma_V(q^2)$ can be calculated by applying the following projections to all one particle irreducible (1PI) diagrams with two external legs of the corresponding type:

$$
\Pi(q^2) = \frac{\delta^{ab}}{N^2-1}\left[\frac{1}{D-1}(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2})\frac{1}{q^2}\right] \times
$$



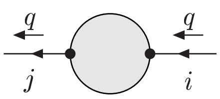
,

$$
\tilde{\Pi}(q^2) = \frac{\delta^{ab}}{N^2-1}\frac{1}{q^2} \times
$$



, (4)

$$
\Sigma_V(q^2) = \frac{\delta^{ab}}{N}\mathrm{Tr}\left[\frac{\not{q}}{4q^2} \times \right.
$$



$\Big]$.

These projectors take into account that we are using dimensional regularization with $D = 4 - 2\epsilon$ and are valid for a general SU(N) gauge group. The trace has to be taken over Dirac matrices. The resulting scalar integrals are of the massless propagator type and can be evaluated with standard methods. For details about the technical setup used to perform these calculations we refer to the appendix.

3 The Triple Gluon, Ghost and Quark Vertex Functions

In massless QCD there are the following 3-point functions:

$$
\begin{aligned}
G^{(3)\,abc}_{\mu\nu\rho}(p,q) &= \mathrm{i}^2 \int \mathrm{d}x\,\mathrm{d}y\,\mathrm{e}^{-\mathrm{i}(px+qy)}\langle T[A^a_\mu(x)A^b_\nu(y)A^c_\rho(0)]\rangle, \\
G^{(3)\,abc}_{\mu}(p,q) &= \mathrm{i}^2 \int \mathrm{d}x\,\mathrm{d}y\,\mathrm{e}^{-\mathrm{i}(px+qy)}\langle T[\eta^b(x)\bar{\eta}^c(y)A^a_\mu(0)]\rangle, \\
G^{(3)\,a}_{\mu\,ij}(p,q) &= \mathrm{i}^2 \int \mathrm{d}x\,\mathrm{d}y\,\mathrm{e}^{-\mathrm{i}(px+qy)}\langle T[\psi_i(x)\bar{\psi}_j(y)A^a_\mu(0)]\rangle.
\end{aligned}
\tag{5}
$$

These are related to the vertex functions by

$$
\begin{aligned}
G^{(3)\,abc}_{\mu\nu\rho}(p,q) &= D^{ad}_{\mu\mu'}(-p)D^{be}_{\nu\nu'}(-q)D^{cf}_{\rho\rho'}(-p-q)\Gamma^{def}_{\mu'\nu'\rho'}(p,q,-p-q), \\
G^{(3)\,abc}_{\mu}(p,q) &= \Delta^{ad}(-p)\tilde{\Gamma}^{def}_{\mu'}(p,q;-p-q)\Delta^{eb}(q)D^{cf}_{\mu'\mu}(q+p), \\
G^{(3)\,a}_{\mu\,ij}(p,q) &= S_{ii'}(-p)\Lambda^{d}_{\mu'\,i'j'}(p,q;-q-p)S_{j'j}(q)D^{ad}_{\mu'\mu}(p+q).
\end{aligned}
\tag{6}
$$

Setting one external momentum to zero leaves only one momentum which can be used to construct tensor decompositions of the vertex functions. The color and Lorentz structure of all vertices at the ZP and our conventions are discussed in the next subsections.

3.1 The Triple Gluon Vertex

For symmetry reasons setting any of the three external gluon momenta of the triple gluon vertex to zero will lead to the same scalar functions. Up to two loops, the triple gluon vertex is known to be proportional to the totally antisymmetric color structure functions f^{abc}. A color structure proportional to the totally symmetric d^{abc} will not have a divergent part. In this work we only consider the f^{abc} part.

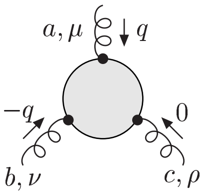
The triple gluon vertex needs to be totally symmetric under exchange of any two of the (bosonic) gluons and we are interested in the part proportional to f^{abc}. So the Lorentz structure for this part in the case of one vanishing external momentum is limited to the following three tensor structures, which are all antisymmetric under exchange of the two gluons with non vanishing momentum:

$$
\begin{aligned}
\Gamma^{abc}_{\mu\nu\rho}(q,-q,0) &= -\mathrm{i}gf^{abc}\Big((2g_{\mu\nu}q_\rho - g_{\mu\rho}q_\nu - g_{\rho\nu}q_\mu)T_1(q^2) \\
&\qquad -(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2})q_\rho T_2(q^2) + q_\mu q_\nu q_\rho T_3(q^2)\Big) \quad .
\end{aligned}
\tag{7}
$$

Due to the WST identity for the triple gluon vertex(for details see Section 4) the third function needs to vanish and finding $T_3(q^2) = 0$ is a check for the correctness of the result. The functions $T_i(q^2)$ can directly be calculated by applying the following projectors to the 1PI diagrams with 3 external gluon legs of which one carries zero momentum:

$$
\begin{aligned}
P^{abc}_{1\,\mu\nu\rho}(q) &= \frac{\mathrm{i}f^{abc}}{N(N^2-1)}\left[\frac{1}{D-1}\left(\frac{q_\mu q_\nu q_\rho}{q^4} - \frac{g_{\nu\rho}q_\mu}{2q^2} - \frac{g_{\mu\rho}q_\nu}{2q^2}\right)\right], \\
P^{abc}_{2\,\mu\nu\rho}(q) &= \frac{\mathrm{i}f^{abc}}{N(N^2-1)}\left[\frac{1}{D-1}\left(\frac{3q_\mu q_\nu q_\rho}{q^4} - g_{\mu\nu}\frac{q_\rho}{q^2} - g_{\mu\rho}\frac{q_\nu}{q^2} - g_{\nu\rho}\frac{q_\mu}{q^2}\right)\right], \\
P^{abc}_{3\,\mu\nu\rho}(q) &= \frac{\mathrm{i}f^{abc}}{N(N^2-1)}\left[\frac{q_\mu q_\nu q_\rho}{q^6}\right],
\end{aligned}
\tag{8}
$$

$$T_i(q^2) = P^{abc}_{i\,\mu\nu\rho}(q) \times$$



$$. \tag{9}$$

The explicit calculation shows that indeed T_3 vanishes.

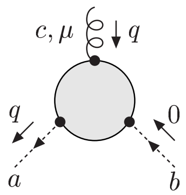
3.2 The Ghost Gluon Vertex

The ghost gluon vertex is at tree level proportional to the momentum of the outgoing ghost. Since this vertex is the only interaction of the ghost field, it is clear that the vertex is also proportional to this momentum at any order in perturbation theory. This leaves two possibilities of one zero external momentum and due to the simple Lorentz structure also only two scalar functions to be determined (again we only consider only the f^{abc} color structure):

$$\begin{aligned}\tilde{\Gamma}^{abc}_{\mu}(-q,0;q) &= -\mathrm{i}gf^{abc}q_\mu\tilde{\Gamma}_{\mathrm{h}}(q^2),\\ \tilde{\Gamma}^{abc}_{\mu}(-q,q;0) &= -\mathrm{i}gf^{abc}q_\mu\tilde{\Gamma}_{\mathrm{g}}(q^2).\end{aligned} \tag{10}$$

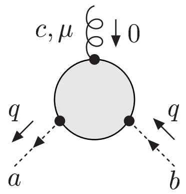
The subscripts g and h stand for the external line that carries zero momentum: g for the gluon and h for the ghost. Again the $\tilde{\Gamma}_i(q^2)$ can be directly computed from the 1PI diagrams with two external ghosts and one external gluon:

$$\tilde{\Gamma}_{\mathrm{h}}(q^2) \quad = \quad +\frac{\mathrm{i}f^{abc}}{N(N^2-1)}\frac{q_\mu}{q^2} \times$$



$$, \tag{11}$$

$$\tilde{\Gamma}_{\mathrm{g}}(q^2) \quad = \quad +\frac{\mathrm{i}f^{abc}}{N(N^2-1)}\frac{q_\mu}{q^2} \times$$



$$. \tag{12}$$

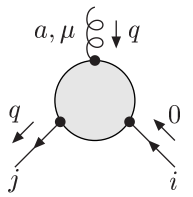
3.3 The Quark Gluon Vertex

Finally the quark gluon vertex is proportional to the color structure T^a_{ij} and for one vanishing external momentum there are two different possibilities (here setting either of the quark momenta to zero gives the same scalar functions up to three loops). A useful tensor decomposition is:

$$\begin{aligned}\Lambda^a_{\mu\,ij}(-q,0;q) &= gT^a_{ij}\left[\gamma_\mu\Lambda_{\mathrm{q}}(q^2) + \gamma_\nu\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right)\Lambda^T_{\mathrm{q}}(q^2)\right],\\ \Lambda^a_{\mu\,ij}(-q,q;0) &= gT^a_{ij}\left[\gamma_\mu\Lambda_{\mathrm{g}}(q^2) + \gamma_\nu\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right)\Lambda^T_{\mathrm{g}}(q^2)\right],\end{aligned} \tag{13}$$

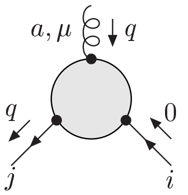
where the additional subscript q corresponds to the case of a vanishing external quark momentum and T marks the transversal part. The Λ_i can directly be calculated in the following way:

$$\Lambda_{\mathrm{q}}(q^2) \;=\; +\frac{T^a_{ij}}{NC_F} Tr \left[\frac{\not{q} q_\mu}{4q^2} \times \right.$$



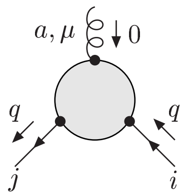
$$\left. \right] ,$$

$$\Lambda^T_{\mathrm{q}}(q^2) \;=\; \frac{T^a_{ij}}{NC_F} Tr \left[\frac{1}{4(D-1)} \left(\gamma_\mu - D \frac{\not{q} q_\mu}{q^2} \right) \times \right.$$



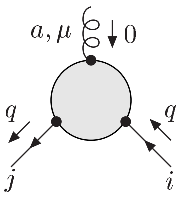
$$\left. \right] , \tag{14}$$

$$\Lambda_{\mathrm{g}}(q^2) \;=\; +\frac{T^a_{ij}}{NC_F} Tr \left[\frac{\not{q} q_\mu}{4q^2} \times \right.$$



$$\left. \right] ,$$

$$\Lambda^T_{\mathrm{g}}(q^2) \;=\; \frac{T^a_{ij}}{NC_F} Tr \left[\frac{1}{4(D-1)} \left(\gamma_\mu - D \frac{\not{q} q_\mu}{q^2} \right) \times \right.$$



$$\left. \right] . \tag{15}$$

4 The Ward-Slavnov-Taylor Identity

By using the WST identity for the triple gluon vertex, one can determine the scalar functions T_1 and T_2 from the gluon and ghost self-energies and functions related to the ghost gluon vertex. This check has also been performed in other determinations of the gluon vertex [17] [18] [19]. The general WST identity was determined in [21] [22] and can be found in the following form for the triple gluon vertex in [17] [18]:

$$\begin{aligned} k^\rho \Gamma_{\mu\nu\rho}(p,q,k) \;=\; & -J(p^2)G(k^2)(g_\mu^\rho p^2 - p^\rho p_\mu)\tilde{\Gamma}_{\rho\nu}(p,k;q) \\ & +J(q^2)G(k^2)(g_\nu^\rho q^2 - q^\rho q_\nu)\tilde{\Gamma}_{\rho\mu}(q,k;p). \end{aligned} \tag{16}$$

Here $\tilde{\Gamma}_{\nu\mu}(p,q;k)$ is related to the proper ghost gluon vertex $\tilde{\Gamma}^{abc}_\mu(p,q;k)$ by:

$$\tilde{\Gamma}^{abc}_\mu(p,q;k) = -\mathrm{i}gf^{abc}p^\nu \tilde{\Gamma}_{\nu\mu}(p,q;k). \tag{17}$$

and we have introduced the functions

$$J(p^2) = 1 + \Pi(p^2) \quad \text{and} \quad G(p^2) = \frac{1}{1+\tilde{\Pi}(p^2)} .$$

In ref. [17] the following tensor decomposition for $\tilde{\Gamma}_{\mu\nu}(p,q;k)$ is given:

$$
\begin{aligned}
\tilde{\Gamma}_{\mu\nu}(p,k;q) &= +g_{\mu\nu}a(q,k,p) - q_\mu k_\nu b(q,k,p) + p_\mu q_\nu c(q,k,p) \\
&\quad +q_\mu p_\nu d(q,k,p) + p_\mu p_\nu e(q,k,p). \qquad (18)
\end{aligned}
$$

The analytic determination of the full momentum dependence of the generalized ghost gluon vertex $\tilde{\Gamma}_{\nu\mu}$ at three loops is impossible with the current calculation techniques. But of course we do not need to know the full momentum dependence for checking the WST identity at the ZP. Still, even the determination of all possible tensor structures for the case of one vanishing momentum and the corresponding expansions to first order in the vanishing momentum would be a very demanding project. To avoid the calculation of unnecessary parts of the ghost gluon vertex we follow closely the approach of [19].

For the vertex function there are two independent momenta. They can be chosen in such a way that at the ZP one of them vanishes. Contracting the vertex in the WST identity (16) with the momentum that does not vanish at the ZP is relatively simple. In this case one can safely set the other momentum to zero on the right hand side and is left with functions of just one momentum. Inserting the tensor decompositions (7) and (18) gives:

$$
T_1(p^2) = a_3(p^2)G(p^2)J(p^2) \quad , \quad a_3(p^2) = a(0,p,-p). \qquad (19)
$$

There is also the possibility to contract the vertex with the momentum that vanishes in the limit of the ZP. This case gives a differential WST identity at the ZP. If the right hand side is expanded to first order in k_ρ the constant lowest order term cancels. Dividing by k_ρ, setting k to 0 on both sides and taking into account that for *massless* quarks $G(0) = 1$[2] results in the following representation of this differential identity:

$$
\begin{aligned}
\Gamma_{\mu\nu\rho}(p,-p,0) &= +a_2(p^2)J(p^2)\left(2g_{\mu\nu}p_\rho - g_{\mu\rho}p_\nu - g_{\nu\rho}p_\mu\right) \\
&\quad +a_2(p^2)p^2\frac{\mathrm{d}J(p^2)}{\mathrm{d}p^2}\left(-2\frac{p_\mu p_\nu p_\rho}{p^2} + 2g_{\mu\nu}p_\rho\right) \\
&\quad +d_2(p^2)p^2J(p^2)\left(-2\frac{p_\mu p_\nu p_\rho}{p^2} + g_{\mu\rho}p_\nu + g_{\nu\rho}p_\mu\right) \\
&\quad +J(p^2)\left(a'_{23}(p^2) - a'_{21}(p^2)\right)\left(\frac{p_\mu p_\nu p_\rho}{p^2} - g_{\mu\nu}p_\rho\right). \qquad (20)
\end{aligned}
$$

where we have expanded $a(p,k,q)$ to first order in small k and use the following shortcuts:

$$
\begin{aligned}
d(p,0,-p) = d_2(p^2) \quad &, \quad a(p,0,-p) = a(-p,0,p) = a_2(p^2), \\
a(-p-k,k,p) &= a_2(p^2) + \frac{k\cdot p}{p^2}a'_{23}(p^2) + \mathcal{O}(k^2)\ , \qquad (21) \\
a(p,k,-p-k) &= a_2(p^2) + \frac{k\cdot p}{p^2}a'_{21}(p^2) + \mathcal{O}(k^2)\ .
\end{aligned}
$$

Contracting eq. (20) with p^μ once more and projecting out the structure $T_1(p^2)$ gives another possibility to reproduce $T_1(p^2)$. It can be used to produce the simple relation

$$
a_2(p^2) - p^2 d_2(p^2) = G(p^2)a_3(p^2). \qquad (22)
$$

Using this relation and projecting out the structure proportional to $T_2(p^2)$ we can also obtain $T_2(p^2)$ in a completely independent way:

$$
\begin{aligned}
T_2(p^2) &= 2T_1(p^2) - 2a_(p^2)\frac{\mathrm{d}}{\mathrm{d}p^2}\left(p^2J(p^2)\right) \\
&\quad +J(p^2)\left(a'_{23}(p^2) - a'_{21}(p^2)\right). \qquad (23)
\end{aligned}
$$

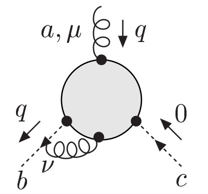
[2] $G(0) = 1$ since all contributing diagrams are massless tadpoles which vanish in dimensional regularization.

Finally, applying the projector P_3 of eq. (8) to the right hand side of eq. (20) should give an alternative representation of $T_3(p^2)$ and it is indeed found to be zero from this identity, as has been mentioned in the last Section.

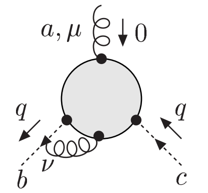
The scalar functions a_2, a_3, a'_{23} and a'_{21} can be calculated from 1PI diagrams with the following operations:

$$P^{abc}_{\mu\nu}(q) = \frac{f^{abc}}{N(N^2-1)} \frac{1}{D-1} \left(q^2 g_{\mu\nu} - q_\mu q_\nu\right) , \tag{24}$$

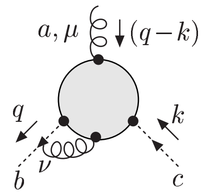
$$a_2(q^2) = 1 + P^{abc}_{\mu\nu}(q) \times \left[\quad \right] , \tag{25}$$



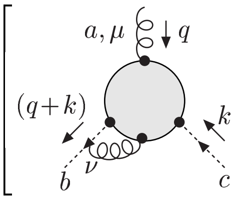
$$a_3(q^2) = 1 + P^{abc}_{\mu\nu}(q) \times \left[\quad \right] , \tag{26}$$



$$a'_{23}(q^2) = \Box_k \left(q\cdot k\, P^{abc}_{\mu\nu}(q) \mathcal{T}^{(1)}_k \times \left[\quad \right] \right) , \tag{27}$$



$$a'_{21}(q^2) = \Box_k \left(q\cdot k\, P^{abc}_{\mu\nu}(q) \mathcal{T}^{(1)}_k \times \left[\quad \right] \right) . \tag{28}$$



The additional gluon line is to visualize that at the outgoing ghost vertex we do not contract with the external momentum, which leaves an additional open index ν as shown in eq. (17). The shortcut $\mathcal{T}^{(1)}_k$ stands for a first order Taylor expansion in k and $\Box_k = g_{\alpha\beta} \frac{\partial}{\partial k_\alpha} \frac{\partial}{\partial k_\beta}$. Both are to be applied to the integrand **before** any of the integrations are performed. These manipulations are necessary because the MINCER package can only handle scalar products of internal and one external momentum for more than one loop. Since there are only the two independent external momenta p and k, the first order terms in k (of a scalar function) have to be proportional to $k \cdot p$. This guarantees that the above manipulations will extract the scalars a'_{2i}.

Using this strategy we found complete agreement for the bare expression with the ones from the direct computation of the triple gluon vertex.

5 Renormalization

For a generic renormalization scheme[3] R, the following relations between bare and renormalized quantities hold in dimensional regularization:

$$
\begin{aligned}
(A^{\mathrm{B}})^a_\nu = \sqrt{Z_3^{\mathrm{R}}(h^{\mathrm{R}},\mu,\epsilon)}\,(A^{\mathrm{R}})^a_\nu(\mu)\ , \qquad \eta^{\mathrm{B}} &= \sqrt{\tilde{Z}_3^{\mathrm{R}}(h^{\mathrm{R}},\mu,\epsilon)}\,\eta^{\mathrm{R}}(\mu)\ ,\\
\psi_i^{\mathrm{B}} = \sqrt{Z_2^{\mathrm{R}}(h^{\mathrm{R}},\mu,\epsilon)}\,\psi_i^{\mathrm{R}}(\mu), \qquad h^{\mathrm{B}} &= \mu^{2\epsilon}\,Z_h^{\mathrm{R}}(h^{\mathrm{R}},\mu,\epsilon)\,h^{\mathrm{R}}(\mu).
\end{aligned}
\tag{29}
$$

Here $h = \alpha_s/(4\pi) = g_s^2/(16\pi^2)$, μ is the 't Hooft unit of mass which is the renormalization point in the MS scheme.

The self-energies and vertex functions are renormalized as follows

$$
\begin{aligned}
1+\Pi^{\mathrm{R}}(q^2,h^R,\mu,\epsilon) &= Z_3^{\mathrm{R}}(h^{\mathrm{R}},\mu,\epsilon)\left[1+\Pi^{\mathrm{B}}(q^2,h^{\mathrm{B}},\epsilon)\right]_{h^{\mathrm{B}}=\mu^{2\epsilon}Z_h h,\ \xi_L^{\mathrm{B}}=Z_3\xi_L}\ ,\\
1+\tilde{\Pi}^{\mathrm{R}}(q^2,h^R,\mu,\epsilon) &= \tilde{Z}_3^{\mathrm{R}}(h^{\mathrm{R}},\mu,\epsilon)\left[1+\tilde{\Pi}^{\mathrm{B}}(q^2,h^{\mathrm{B}},\epsilon)\right]_{h^{\mathrm{B}}=\mu^{2\epsilon}Z_h h,\ \xi_L^{\mathrm{B}}=Z_3\xi_L}\ ,\\
1+\Sigma_V^{\mathrm{R}}(q^2,h^R,\mu,\epsilon) &= Z_2^{\mathrm{R}}(h^{\mathrm{R}},\mu,\epsilon)\left[1+\Sigma_V^{\mathrm{B}}(q^2,h^{\mathrm{B}},\epsilon)\right]_{h^{\mathrm{B}}=\mu^{2\epsilon}Z_h h,\ \xi_L^{\mathrm{B}}=Z_3\xi_L}\ ,\\
T_{1,2}^{\mathrm{R}}(q^2,h^R,\mu,\epsilon) &= Z_1^{\mathrm{R}}(h^{\mathrm{R}},\mu,\epsilon)T_{1,2}^{\mathrm{B}}(q^2,h^{\mathrm{B}},\epsilon)\big|_{h^{\mathrm{B}}=\mu^{2\epsilon}Z_h h,\ \xi_L^{\mathrm{B}}=Z_3\xi_L}\ ,\\
\tilde{\Gamma}_{h,g}^{\mathrm{R}}(q^2,h^R,\mu,\epsilon) &= \tilde{Z}_1^{\mathrm{R}}(h^{\mathrm{R}},\mu,\epsilon)\tilde{\Gamma}_{h,g}^{\mathrm{B}}(q^2,h^{\mathrm{B}},\epsilon)\Big|_{h^{\mathrm{B}}=\mu^{2\epsilon}Z_h h,\ \xi_L^{\mathrm{B}}=Z_3\xi_L}\ ,\\
\Lambda_{g,q}^{\mathrm{R}}(q^2,h^R,\mu,\epsilon) &= \bar{Z}_1^{\mathrm{R}}(h^{\mathrm{R}},\mu,\epsilon)\Lambda_{g,q}^{\mathrm{B}}(q^2,h^{\mathrm{B}},\epsilon)\big|_{h^{\mathrm{B}}=\mu^{2\epsilon}Z_h h,\ \xi_L^{\mathrm{B}}=Z_3\xi_L}\ .
\end{aligned}
\tag{30}
$$

The renormalization constants for the self-energies can be read of eqs.(2,3). Likewise, from eqs. (5,6) and the QCD Lagrangian the renormalization factors for the trilinear operators can be related to the renormalization of the coupling constant and the field renormalizations

$$
\sqrt{Z_a^R Z_3^R} = \frac{Z_1^R}{Z_3^R} = \frac{\tilde{Z}_1^R}{\tilde{Z}_3^R} = \frac{\bar{Z}_1^R}{Z_2^R}\,. \tag{31}
$$

QCD is known to be a renormalizable theory. This means that Z_x^R can be found which make the renormalized Greens functions eqs. (2,5) finite when taking the limit $\epsilon \to 0$ and at the same time fulfill the WST identities, respectively eq. (31), to any order in perturbation theory.

Using all these formulas, renormalization in the $\overline{\mathrm{MS}}$-scheme is straightforward. It requires the $Z_x = Z_x^{\overline{\mathrm{MS}}}(h,\epsilon)$ to only contain poles in ϵ and thus to be of the following form:

$$
Z_x(h,\epsilon) = 1 + \sum_{i>0}\frac{1}{\epsilon^i}Z_x^{(i)}(h) \quad , \quad Z_x^{(i)}(h) = 1 + \sum_{j\geq i} h^j Z_x^{(i,j)} \tag{32}
$$

The μ-dependence of the fields and the coupling h is then described in the usual way by the renormalization group equations:

$$
\begin{aligned}
\mu^2\frac{\mathrm{d}}{\mathrm{d}\mu^2}h(\mu) = -\epsilon h + \beta(h) \quad &, \quad \beta(h) = -\sum_{i=0}^{\infty} h^{(i+2)}\beta_i = h^2\frac{\partial}{\partial h}Z_h^{(1)},\\
2\mu^2\frac{\mathrm{d}}{\mathrm{d}\mu^2}A_\nu^a(\mu) = \gamma_3(h)A_\nu^a(\mu) \quad &, \quad \gamma_3(h) = -\sum_{i=0}^{\infty} h^{(i+1)}\gamma_{3\,i} = h\frac{\partial}{\partial h}Z_3^{(1)},\\
2\mu^2\frac{\mathrm{d}}{\mathrm{d}\mu^2}\eta(\mu) = \tilde{\gamma}_3(h)\eta(\mu) \quad &, \quad \tilde{\gamma}_3(h) = -\sum_{i=0}^{\infty} h^{(i+1)}\tilde{\gamma}_{3\,i} = h\frac{\partial}{\partial h}\tilde{Z}_3^{(1)},\\
2\mu^2\frac{\mathrm{d}}{\mathrm{d}\mu^2}\psi_i(\mu) = \gamma_2(h)\psi_i(\mu) \quad &, \quad \gamma_2(h) = -\sum_{i=0}^{\infty} h^{(i+1)}\gamma_{2\,i} = h\frac{\partial}{\partial h}Z_2^{(1)},
\end{aligned}
\tag{33}
$$

[3] In the following all quantities without explicit superscript refer to the $\overline{\mathrm{MS}}$ scheme, the superscript B marks the bare quantities

where the last equalities can be derived from eqs. (29), the D dimensional β-function $[-\epsilon h+\beta(h)]$ and the fact that the bare quantities must be independent of μ. As the functions in eq. (30) β and the anomalous dimensions are here defined in D dimensions but are finite for $\epsilon\to 0$.

The $\overline{\text{MS}}$ renormalization constants and anomalous dimensions are well known up to three loops [23, 24] and for the β-function [20] and the quark field anomalous dimension γ_2 [7] even the four-loop terms are known. Of course up to three loops the Z_i can also be determined independently from the bare results for the self-energies and vertex functions. The corresponding anomalous dimensions are given in the appendix.

Performing this standard renormalization procedure one arrives at the $\overline{\text{MS}}$ renormalized expressions for all self-energies and vertex functions. The expressions for these in the limit $\epsilon\to 0$ at the point $p^2=-\mu^2$ are given in the appendix for generic color factors and a generic gauge parameter. The momentum dependence can be obtained from the anomalous dimensions. The fact that indeed all poles in ϵ cancel and eq. (31) holds is another check for the two-loop finite part and at least for the pole terms of any of the three-loop results.

6 Four Particular $\widetilde{\text{MOM}}$ like Scheme Definitions

Momentum subtraction schemes are defined by setting some of the 2 and 3-point functions to their tree values for a fixed configuration of the states of the external particles (that is momenta and polarization state) in a certain gauge. This fixes the field renormalization constants at the renormalization point μ:

$$\begin{aligned}
1+\Pi^{\widetilde{\text{MOM}}}(-\mu^2)=1 &= Z_3^{\widetilde{\text{MOM}}}(\mu^2,\epsilon)\left[1+\Pi^{\text{B}}(-\mu^2,\epsilon)\right],\\
\left[1+\tilde{\Pi}^{\widetilde{\text{MOM}}}(-\mu^2)\right]=1 &= \tilde{Z}_3^{\widetilde{\text{MOM}}}(\mu^2,\epsilon)\left[1+\tilde{\Pi}^{\text{B}}(-\mu^2,\epsilon)\right],\\
\left[1+\Sigma_V^{\widetilde{\text{MOM}}}(-\mu^2,\epsilon)\right]=1 &= Z_2^{\widetilde{\text{MOM}}}(\mu^2,\epsilon)\left[1+\Sigma_V^{\text{B}}(-\mu^2,\epsilon)\right].
\end{aligned}\tag{34}$$

For the renormalization of the coupling constant there are infinitely many possibilities to define a momentum subtraction renormalization scheme, even when considering only the ZP. Not only is there an ambiguity in which vertex to subtract, but also there is the freedom to use a certain linear combination of the scalar functions appearing in the gluon and quark vertices, which can be related to fixed polarization states of the external particles [3]. If one considers the generalized ghost gluon vertex as defined in eq. (17) there are even more possibilities.

Each of these choices defines a different renormalization scheme just as a different choice of the arbitrary renormalization scale. In general it is not possible to fix more than one vertex to it's tree value at the ZP with $p^2=-\mu^2$, since they all are related by Ward identities.

"The" $\widetilde{\text{MOM}}$ scheme originally was defined in [3] as a scheme in which all three triple-vertices are subtracted at the ZP without violating the Ward identities. This can be achieved by choosing a special set of the appearing scalar functions at one-loop order. Of course this universality does not hold anymore when considering the two- and three-loop order. This is why even in the original publication a generalization of this scheme for two and more loops is given. It subtracts the ghost vertex and corresponds to the original definition at one-loop order. Here we also consider the other two schemes that coincide with the $\widetilde{\text{MOM}}$ scheme at one-loop order. They are defined by subtracting just the scalar functions that appear at tree level of the triple gluon (we will refer to it as the $\widetilde{\text{MOM}}$g scheme) and quark gluon vertex (quark momentum set to zero, $\widetilde{\text{MOM}}$q), respectively.

Another $\widetilde{\text{MOM}}$ like renormalization scheme definition ($\widetilde{\text{MOM}}$gg) that uses the gluon vertex lately has been used to relate lattice results of the triple gluon vertex [12] and the gluon propagator [14] to perturbative calculations. The $\widetilde{\text{MOM}}$q scheme that subtracts the quark gluon vertex has also been used in lattice calculations [16], but is not used beyond one-loop order in this work.

6.1 Subtracting the Ghost Gluon Vertex

A generalization of the original $\widetilde{\text{MOM}}$ scheme to more than one loop is given by subtracting the ghost gluon vertex with the moment of the incoming ghost set to zero and a longitudinal polarization of the external gluon (the case of a transversally polarized gluon will vanish at the ZP). The renormalization condition is:

$$\tilde{\Gamma}_h^{\widetilde{\mathrm{MOM}}}(-\mu^2) = 1 = \tilde{Z}_1^{\widetilde{\mathrm{MOM}}}(\mu^2)\tilde{\Gamma}^{\mathrm{B}}(-\mu^2). \tag{35}$$

Using eqs. (29-31,34), we can connect the coupling constant in this scheme to quantities calculable in the $\overline{\mathrm{MS}}$ scheme through the following chain of equations:

$$\begin{aligned} g_s^{\widetilde{\mathrm{MOM}}}(\mu) &= \mu^{-\epsilon} g_s^b \frac{1}{\tilde{Z}_g^{\widetilde{\mathrm{MOM}}}(\mu)} = \mu^{-\epsilon} g_s^b \frac{\tilde{Z}_3^{\widetilde{\mathrm{MOM}}}(\mu)\sqrt{Z_3^{\widetilde{\mathrm{MOM}}}(\mu)}}{\tilde{Z}_1^{\widetilde{\mathrm{MOM}}}(\mu)} \\ &= g_s^{\overline{\mathrm{MS}}}(\mu) Z_g^{\overline{\mathrm{MS}}}(\mu) \left[\frac{\tilde{\Gamma}_h^{\mathrm{B}}(p^2)}{(1+\tilde{\Pi}^{\mathrm{B}}(p^2))\sqrt{1+\Pi^{\mathrm{B}}(p^2)}}\right]_{p^2=-\mu^2} \\ &= g_s^{\overline{\mathrm{MS}}}(\mu) \left[\frac{\tilde{\Gamma}_h^{\overline{\mathrm{MS}}}(p^2)}{(1+\tilde{\Pi}^{\overline{\mathrm{MS}}}(p^2))\sqrt{1+\Pi^{\overline{\mathrm{MS}}}(p^2)}}\right]_{p^2=-\mu^2} . \end{aligned} \tag{36}$$

Squaring this and inserting the $\overline{\mathrm{MS}}$ expressions for Π, $\tilde{\Pi}$ and $\tilde{\Gamma}_h$ we arrive at the following relation between the coupling in this scheme and the $\overline{\mathrm{MS}}$ coupling, expressed as an expansion in $h = h^{\overline{\mathrm{MS}}}$ and[4] $\xi_L = \xi_L^{\overline{\mathrm{MS}}}$:

$$\begin{aligned} h^{\widetilde{\mathrm{MOM}}} &= h + h^2 \left[+\frac{169}{12} - \frac{10}{9}n_f + \frac{9}{2}\xi_L + \frac{3}{4}\xi_L^2\right] + h^3 \left[+\frac{76063}{144} - \frac{4}{3}n_f\zeta_3 - 5n_f\xi_L \right. \\ &\quad \left. -\frac{1913}{27}n_f + \frac{100}{81}n_f^2 + \frac{117}{8}\zeta_3\xi_L - \frac{351}{8}\zeta_3 + \frac{1719}{16}\xi_L + \frac{549}{16}\xi_L^2 + \frac{81}{16}\xi_L^3\right] \\ &\quad + h^4 \left[+\frac{42074947}{1728} - 159 n_f\zeta_3\xi_L + \frac{3}{4}n_f\zeta_3\xi_L^2 + \frac{8362}{27}n_f\zeta_3 + \frac{2320}{9}n_f\zeta_5 - \frac{6931}{16}n_f\xi_L \right. \\ &\quad -\frac{757}{16}n_f\xi_L^2 - \frac{769387}{162}n_f + \frac{16}{3}n_f^2\zeta_3\xi_L + \frac{28}{9}n_f^2\zeta_3 + \frac{38}{9}n_f^2\xi_L + \frac{199903}{972}n_f^2 \\ &\quad -\frac{1000}{729}n_f^3 + \frac{4893}{4}\zeta_3\xi_L + \frac{1485}{16}\zeta_3\xi_L^2 - \frac{1341}{32}\zeta_3\xi_L^3 - \frac{117}{32}\zeta_3\xi_L^4 - \frac{60675}{16}\zeta_3 \\ &\quad -\frac{8505}{16}\zeta_5\xi_L - \frac{4635}{32}\zeta_5\xi_L^2 + \frac{405}{16}\zeta_5\xi_L^3 + \frac{315}{64}\zeta_5\xi_L^4 - \frac{70245}{64}\zeta_5 + \frac{290371}{64}\xi_L \\ &\quad \left. +\frac{22287}{16}\xi_L^2 + \frac{21141}{64}\xi_L^3 + \frac{2547}{64}\xi_L^4\right] . \end{aligned} \tag{37}$$

6.2 Subtracting the Quark Gluon Vertex

Using again the renormalization condition that at one-loop level is identical to the original $\widetilde{\mathrm{MOM}}$ scheme we have:

$$\Lambda_q^{\widetilde{\mathrm{MOM}}\mathrm{q}}(-\mu^2) = 1. \tag{38}$$

This subtracts just the structure proportional to γ_μ which is present at the tree level and corresponds to a longitudinally polarized gluon. Similar operations as above then lead to

$$h^{\widetilde{\mathrm{MOM}}\mathrm{q}}(\mu) = h^{\overline{\mathrm{MS}}}(\mu) \left[\frac{\Lambda_q^{\overline{\mathrm{MS}}}(-\mu^2)}{(1+\Sigma_V^{\overline{\mathrm{MS}}}(-\mu^2))\sqrt{1+\Pi^{\overline{\mathrm{MS}}}(-\mu^2)}}\right]^2 \tag{39}$$

and inserting the $\overline{\mathrm{MS}}$ results gives:

[4] see Section 7 for details

$$
\begin{aligned}
h^{\widetilde{\mathrm{MOM}}\mathrm{q}} \;&=\; h \;+\; h^2 \;\left[+\frac{169}{12}-\frac{10}{9}n_f+\frac{9}{2}\xi_L+\frac{3}{4}\xi_L^2\right] +\; h^3 \;\left[+\frac{77035}{144}-\frac{4}{3}n_f\zeta_3-5n_f\xi_L\right.\\
&\qquad \left.-\frac{1913}{27}n_f+\frac{100}{81}n_f^2+18\zeta_3\xi_L-54\zeta_3+\frac{1647}{16}\xi_L+\frac{549}{16}\xi_L^2+\frac{81}{16}\xi_L^3\right]\\
&\quad +h^4 \;\left[+\frac{42735097}{1728}-\frac{333}{2}n_f\zeta_3\xi_L+\frac{3}{4}n_f\zeta_3\xi_L^2+\frac{37579}{108}n_f\zeta_3+\frac{2320}{9}n_f\zeta_5-\frac{6759}{16}n_f\xi_L\right.\\
&\qquad -\frac{757}{16}n_f\xi_L^2-\frac{386759}{81}n_f+\frac{16}{3}n_f^2\zeta_3\xi_L+\frac{28}{9}n_f^2\zeta_3+\frac{38}{9}n_f^2\xi_L+\frac{199903}{972}n_f^2\\
&\qquad -\frac{1000}{729}n_f^3+\frac{20067}{16}\zeta_3\xi_L+\frac{2511}{32}\zeta_3\xi_L^2-\frac{225}{8}\zeta_3\xi_L^3-\frac{117}{32}\zeta_3\xi_L^4-\frac{35385}{8}\zeta_3\\
&\qquad -\frac{8325}{16}\zeta_5\xi_L-\frac{3195}{32}\zeta_5\xi_L^2+\frac{255}{16}\zeta_5\xi_L^3+\frac{315}{64}\zeta_5\xi_L^4-\frac{66765}{64}\zeta_5+\frac{281059}{64}\xi_L\\
&\qquad \left.+\frac{43311}{32}\xi_L^2+\frac{20925}{64}\xi_L^3+\frac{2547}{64}\xi_L^4\right]. \qquad (40)
\end{aligned}
$$

6.3 Subtracting the Triple Gluon Vertex I

Subtracting just the Lorentz structure that is present at the tree level the gluon vertex gives the following renormalization condition

$$
T_1^{\widetilde{\mathrm{MOM}}\mathrm{g}}(-\mu^2) = 1. \qquad (41)
$$

This subtracts the triple gluon vertex with the zero momentum gluon and one of the others being polarized longitudinal (in direction of their momenta) and the last being polarized transversal and parallel to the plane defined by the gluon momenta (for details see [3]). A similar derivation as for the ghost gluon vertex leads then to the following relation of the coupling in the $\widetilde{\mathrm{MOM}}$g scheme to the $\overline{\mathrm{MS}}$ coupling:

$$
h^{\widetilde{\mathrm{MOM}}\mathrm{g}}(\mu) = h^{\overline{\mathrm{MS}}}(\mu)\frac{\left(T_1^{\overline{\mathrm{MS}}}(-\mu^2)\right)^2}{\left(1+\Pi^{\overline{\mathrm{MS}}}(-\mu^2)\right)^3}. \qquad (42)
$$

This gives the following relations between the coupling constants of the two schemes:

$$
\begin{aligned}
h^{\widetilde{\mathrm{MOM}}\mathrm{g}} \;&=\; h \;+\; h^2 \;\left[+\frac{169}{12}-\frac{10}{9}n_f+\frac{9}{2}\xi_L+\frac{3}{4}\xi_L^2\right] +\; h^3 \;\left[+\frac{38261}{72}-\frac{4}{3}n_f\zeta_3-5n_f\xi_L\right.\\
&\qquad \left.-\frac{7571}{108}n_f+\frac{100}{81}n_f^2+\frac{81}{4}\zeta_3\xi_L-\frac{207}{4}\zeta_3+\frac{795}{8}\xi_L+\frac{549}{16}\xi_L^2+\frac{81}{16}\xi_L^3\right]\\
&\quad +h^4 \;\left[+\frac{84464417}{3456}-\frac{1375}{8}n_f\zeta_3\xi_L+\frac{3}{4}n_f\zeta_3\xi_L^2+\frac{72161}{216}n_f\zeta_3+\frac{2320}{9}n_f\zeta_5\right.\\
&\qquad -\frac{39197}{96}n_f\xi_L-\frac{187}{4}n_f\xi_L^2-\frac{6098639}{1296}n_f+\frac{16}{3}n_f^2\zeta_3\xi_L+\frac{28}{9}n_f^2\zeta_3+\frac{38}{9}n_f^2\xi_L\\
&\qquad +\frac{197149}{972}n_f^2-\frac{1000}{729}n_f^3+\frac{87729}{64}\zeta_3\xi_L+\frac{6261}{64}\zeta_3\xi_L^2-\frac{1371}{64}\zeta_3\xi_L^3-\frac{117}{32}\zeta_3\xi_L^4\\
&\qquad -\frac{279861}{64}\zeta_3-\frac{26805}{64}\zeta_5\xi_L-\frac{4155}{64}\zeta_5\xi_L^2+\frac{885}{64}\zeta_5\xi_L^3+\frac{315}{64}\zeta_5\xi_L^4-\frac{3285}{4}\zeta_5\\
&\qquad \left.+\frac{267071}{64}\xi_L+\frac{83511}{64}\xi_L^2+\frac{41445}{128}\xi_L^3+\frac{2547}{64}\xi_L^4\right]. \qquad (43)
\end{aligned}
$$

6.4 Subtracting the Triple Gluon Vertex II

The $\widetilde{\text{MOM}}$gg scheme used in [12] and [14] is defined by the following renormalization conditions:

$$T_1^{\widetilde{\text{MOM}}\text{gg}}(-\mu^2) - \frac{1}{2} T_2^{\widetilde{\text{MOM}}\text{gg}}(-\mu^2) = 1. \tag{44}$$

This subtracts the triple gluon vertex with the zero momentum gluon being polarized transversal and parallel to the plane defined by the gluon momenta. The other two gluon are also polarized transversal but perpendicular to the vertex plane. Another possibility to understand this linear combination is to realize that it corresponds to subtracting the transversal part of the 3-point function, which is the only one to survive at the ZP (see also [12]):

$$G^{(3)abc}_{\mu\nu\rho}(p,-p,0) = (g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}) p_\rho G^{(3)}(p^2) \tag{45}$$

Inserting eqs. (6,3,7) shows that indeed:

$$G^{(3)abc}_{\mu_1\mu_2\mu_3}(p,-p,0) \;=\; (g_{\mu_1\mu_2} - \frac{p_{\mu_1} p_{\mu_2}}{p^2}) p_{\mu_3} \frac{\left(T_1(p^2) - \frac{1}{2} T_2(p^2)\right)}{(1+\Pi^{(2)}(p^2))^2 (1+\Pi^{(2)}(0))}. \tag{46}$$

From this follows the relation of the coupling in the $\widetilde{\text{MOM}}$gg scheme to the $\overline{\text{MS}}$ coupling:

$$h^{\widetilde{\text{MOM}}\text{gg}}(\mu) = h^{\overline{\text{MS}}}(\mu) \frac{\left(T_1^{\overline{\text{MS}}}(-\mu^2) - \frac{1}{2} T_2^{\overline{\text{MS}}}(\mu^2)\right)^2}{\left(1+\Pi^{\overline{\text{MS}}}(-\mu^2)\right)^3}. \tag{47}$$

Inserting T_1, T_2 and Π gives

$$\begin{aligned}
h^{\widetilde{\text{MOM}}\text{gg}} \;=\;& h \;+\; h^2 \left[+ \frac{70}{3} - \frac{22}{9} n_f \right] \;+\; h^3 \;\Big[+ \frac{516217}{576} - \frac{4}{3} n_f \zeta_3 - \frac{7}{4} n_f \xi_L + \frac{1}{2} n_f \xi_L^2 \\
& - \frac{8125}{54} n_f + \frac{376}{81} n_f^2 + \frac{63}{4} \zeta_3 \xi_L - \frac{153}{4} \zeta_3 + \frac{225}{8} \xi_L + \frac{249}{32} \xi_L^2 + \frac{45}{16} \xi_L^3 + \frac{9}{64} \xi_L^4 \Big] \\
& + h^4 \;\Big[+ \frac{304676635}{6912} - \frac{1197}{8} n_f \zeta_3 \xi_L - \frac{21}{8} n_f \zeta_3 \xi_L^2 + \frac{13339}{27} n_f \zeta_3 - 15 n_f \zeta_5 \xi_L \\
& + \frac{1885}{9} n_f \zeta_5 - \frac{16663}{96} n_f \xi_L - \frac{23}{48} n_f \xi_L^2 - \frac{3}{32} n_f \xi_L^3 - \frac{7}{32} n_f \xi_L^4 - \frac{13203725}{1296} n_f \\
& + \frac{8}{3} n_f^2 \zeta_3 \xi_L + \frac{40}{9} n_f^2 \zeta_3 + \frac{44}{9} n_f^2 \xi_L - \frac{8}{9} n_f^2 \xi_L^2 + \frac{580495}{972} n_f^2 - \frac{5680}{729} n_f^3 \\
& + \frac{138171}{64} \zeta_3 \xi_L - \frac{63}{64} \zeta_3 \xi_L^2 + \frac{423}{64} \zeta_3 \xi_L^3 - \frac{27}{32} \zeta_3 \xi_L^4 - \frac{299961}{64} \zeta_3 - \frac{47295}{64} \zeta_5 \xi_L \\
& + \frac{3825}{64} \zeta_5 \xi_L^2 - \frac{1665}{64} \zeta_5 \xi_L^3 - \frac{81825}{64} \zeta_5 + \frac{188523}{128} \xi_L + \frac{118591}{256} \xi_L^2 + \frac{5175}{32} \xi_L^3 \\
& + \frac{5829}{256} \xi_L^4 - \frac{27}{64} \xi_L^5 - \frac{27}{256} \xi_L^6 \Big].
\end{aligned} \tag{48}$$

Following these examples and using the results in the appendix, it should be possible to construct the relations to relate the coupling constant in any $\widetilde{\text{MOM}}$ renormalization scheme to the $\overline{\text{MS}}$ one.

7 The $\widetilde{\text{MOM}}$ β-functions

The first two coefficients of the β-function in any massless renormalization scheme are scheme independent. When considering renormalization schemes in which the coupling constant depends on the generic covariant gauge parameter, this statement is not true, since the additional gauge dependence spoils these universality just as a mass parameter [25]. This means that we need not only to take into account the gauge parameter when writing down renormalization group equations, but also that there is a difference between the gauge parameter defined in the $\overline{\text{MS}}$ scheme and in any of the $\widetilde{\text{MOM}}$ schemes we are considering:

$$\xi_L^{\overline{\text{MS}}} = \frac{Z_3^{\widetilde{\text{MOM}}}}{Z_3^{\overline{\text{MS}}}} \xi_L^{\widetilde{\text{MOM}}}. \tag{49}$$

In Landau gauge we do not need to consider the additional μ dependence introduced by the gauge parameter nor the difference between $\xi_L^{\overline{\text{MS}}}$ and $\xi_L^{\widetilde{\text{MOM}}}$. So in Landau gauge the β-function in any $\widetilde{\text{MOM}}$ like scheme can be obtained in the following simple way once the $\widetilde{\text{MOM}}$ coupling is expressed as a series in the $\overline{\text{MS}}$ coupling

$$\beta^{\widetilde{\text{MOM}}}(h^{\widetilde{\text{MOM}}}) = \mu^2 \frac{\mathrm{d}h^{\widetilde{\text{MOM}}}}{\mathrm{d}\mu^2} = \frac{\partial h^{\widetilde{\text{MOM}}}}{\partial h^{\overline{\text{MS}}}} \mu^2 \frac{\mathrm{d}h^{\overline{\text{MS}}}}{\mathrm{d}\mu^2} = \frac{\partial h^{\widetilde{\text{MOM}}}}{\partial h^{\overline{\text{MS}}}} \beta^{\overline{\text{MS}}}(h^{\overline{\text{MS}}}). \tag{50}$$

where one has to invert the series $h^{\widetilde{\text{MOM}}}(h^{\overline{\text{MS}}})$ and insert this series on the right hand side to express the $\widetilde{\text{MOM}}$ β-function as a series in $h^{\widetilde{\text{MOM}}}$. The four-loop QCD β-function in the $\overline{\text{MS}}$ scheme was obtained in [20] and reads:

$$\begin{aligned}
\beta^{\overline{\text{MS}}} \;=\;& h^2 \left[-11 + \frac{2}{3} n_f\right] + h^3 \left[-102 + \frac{38}{3} n_f\right] + h^4 \left[-\frac{2857}{2} + \frac{5033}{18} n_f - \frac{325}{54} n_f^2\right] \\
&+ h^5 \left[-\frac{149753}{6} + \frac{6508}{27} n_f \zeta_3 + \frac{1078361}{162} n_f - \frac{6472}{81} n_f^2 \zeta_3 - \frac{50065}{162} n_f^2 \right. \\
&\left. -\frac{1093}{729} n_f^3 - 3564 \zeta_3\right],
\end{aligned} \tag{51}$$

Combining this result with the relations between the coupling constants, we can finally obtain the first four coefficients of the β-functions in the above introduced four $\widetilde{\text{MOM}}$ schemes, where we use the definition:

$$\beta(h) = \sum_{i=0}^{3} -h^{(i+2)} \beta_i \tag{52}$$

and in Landau gauge β_0 and β_1 are independent of the renormalization scheme. The three- and four-loop contribution in Landau gauge for the four schemes under consideration are:

$$\beta_3^{\widetilde{\text{MOMg}}} = \left[+\frac{58491}{16} - \frac{2277}{4}\zeta_3\right] + n_f \left[-\frac{15283}{24} + \frac{119}{6}\zeta_3\right] + n_f^2 \left[+\frac{481}{27} + \frac{8}{9}\zeta_3\right], \tag{53}$$

$$\begin{aligned}
\beta_4^{\widetilde{\text{MOMg}}} \;=\;& \left[+\frac{10982273}{64} - \frac{1425171}{32}\zeta_3 - \frac{36135}{2}\zeta_5\right] + n_f \left[-\frac{3830167}{96} + \frac{1075423}{144}\zeta_3 + \frac{60895}{9}\zeta_5\right] \\
&+ n_f^2 \left[+\frac{724445}{324} - \frac{12959}{54}\zeta_3 - \frac{9280}{27}\zeta_5\right] + n_f^3 \left[-\frac{788}{27} + \frac{16}{9}\zeta_3\right],
\end{aligned} \tag{54}$$

$$\beta_3^{\widetilde{\text{MOM}}} = \left[+\frac{28965}{8} - \frac{3861}{8}\zeta_3\right] + n_f \left[-\frac{7715}{12} + \frac{175}{12}\zeta_3\right] + n_f^2 \left[+\frac{989}{54} + \frac{8}{9}\zeta_3\right], \tag{55}$$

$$\begin{aligned}
\beta_4^{\widetilde{\text{MOM}}} \;=\;& \left[+\frac{1380469}{8} - \frac{625317}{16}\zeta_3 - \frac{772695}{32}\zeta_5\right] + n_f \left[-\frac{970819}{24} + \frac{516881}{72}\zeta_3 + \frac{1027375}{144}\zeta_5\right] \\
&+ n_f^2 \left[+\frac{736541}{324} - \frac{6547}{27}\zeta_3 - \frac{9280}{27}\zeta_5\right] + n_f^3 \left[-\frac{800}{27} + \frac{16}{9}\zeta_3\right],
\end{aligned} \tag{56}$$

$$\beta_3^{\widetilde{\mathrm{MOM}}\mathrm{q}} = \left[+\frac{29559}{8} - 594\zeta_3\right] + n_f \left[-\frac{7769}{12} + \frac{64}{3}\zeta_3\right] + n_f^2 \left[+\frac{989}{54} + \frac{8}{9}\zeta_3\right], \tag{57}$$

$$\beta_4^{\widetilde{\mathrm{MOM}}\mathrm{q}} = \left[+\frac{2795027}{16} - \frac{174207}{4}\zeta_3 - \frac{734415}{32}\zeta_5\right] + n_f \left[-\frac{487751}{12} + \frac{67939}{9}\zeta_3 + \frac{1016935}{144}\zeta_5\right]$$
$$+ n_f^2 \left[+\frac{737837}{324} - \frac{6709}{27}\zeta_3 - \frac{9280}{27}\zeta_5\right] + n_f^3 \left[-\frac{800}{27} + \frac{16}{9}\zeta_3\right], \tag{58}$$

$$\beta_3^{\widetilde{\mathrm{MOM}}\mathrm{gg}} = \left[+\frac{186747}{64} - \frac{1683}{4}\zeta_3\right] + n_f \left[-\frac{35473}{96} + \frac{65}{6}\zeta_3\right] + n_f^2 \left[-\frac{829}{54} + \frac{8}{9}\zeta_3\right] + n_f^3 \left[+\frac{8}{9}\right]$$
$$, \tag{59}$$

$$\beta_4^{\widetilde{\mathrm{MOM}}\mathrm{gg}} = \left[+\frac{20783939}{128} - \frac{1300563}{32}\zeta_3 - \frac{900075}{32}\zeta_5\right]$$
$$+ n_f \left[-\frac{2410799}{64} + \frac{1323259}{144}\zeta_3 + \frac{908995}{144}\zeta_5\right]$$
$$+ n_f^2 \left[+\frac{1464379}{648} - \frac{12058}{27}\zeta_3 - \frac{7540}{27}\zeta_5\right]$$
$$+ n_f^3 \left[-\frac{3164}{27} + \frac{64}{9}\zeta_3\right] + n_f^4 \left[+\frac{320}{81}\right]. \tag{60}$$

8 Discussion

In numerical form the coupling constant relations and the corresponding β-functions for these four schemes in Landau gauge read:

$$h^{\widetilde{\mathrm{MOM}}} = h + h^2 \left[+14.0833 - 1.11111\, n_f\right] + h^3 \left[+475.475 - 72.4546\, n_f + 1.23457\, n_f^2\right]$$
$$+ h^4 \left[+18652.4 - 4109.72\, n_f + 209.401\, n_f^2 - 1.37174\, n_f^3\right], \tag{61}$$

$$h^{\widetilde{\mathrm{MOM}}\mathrm{q}} = h + h^2 \left[+14.0833 - 1.11111\, n_f\right] + h^3 \left[+470.054 - 72.4546\, n_f + 1.23457\, n_f^2\right]$$
$$+ h^4 \left[+18332.4 - 4089.25\, n_f + 209.401\, n_f^2 - 1.37174\, n_f^3\right], \tag{62}$$

$$h^{\widetilde{\mathrm{MOM}}\mathrm{g}} = h + h^2 \left[+14.0833 - 1.11111\, n_f\right] + h^3 \left[+469.196 - 71.7046\, n_f + 1.23457\, n_f^2\right]$$
$$+ h^4 \left[+18332 - 4036.86\, n_f + 206.568\, n_f^2 - 1.37174\, n_f^3\right], \tag{63}$$

$$h^{\widetilde{\mathrm{MOM}}\mathrm{gg}} = h + h^2 \left[+23.3333 - 2.44444\, n_f\right]$$
$$+ h^3 \left[+850.231 - 152.066\, n_f + 4.64198\, n_f^2\right]$$
$$+ h^4 \left[+37119.7 - 9377.02\, n_f + 602.56\, n_f^2 - 7.7915\, n_f^3\right], \tag{64}$$

$$\beta^{\widetilde{\mathrm{MOM}}} = \left(h^{\widetilde{\mathrm{MOM}}}\right)^2 \left[-11 + 0.666667\, n_f\right] + \left(h^{\widetilde{\mathrm{MOM}}}\right)^3 \left[-102 + 12.6667\, n_f\right]$$
$$+ \left(h^{\widetilde{\mathrm{MOM}}}\right)^4 \left[-3040.48 + 625.387\, n_f - 19.3833\, n_f^2\right]$$
$$+ \left(h^{\widetilde{\mathrm{MOM}}}\right)^5 \left[-100541 + 24423.3\, n_f - 1625.4\, n_f^2 + 27.4926\, n_f^3\right], \tag{65}$$

$$\begin{aligned}
\beta^{\widetilde{\mathrm{MOM}}\mathrm{g}} &= \left(h^{\widetilde{\mathrm{MOM}}\mathrm{g}}\right)^2 \left[-11 + 0.666667\, n_f\right] + \left(h^{\widetilde{\mathrm{MOM}}\mathrm{g}}\right)^3 \left[-102 + 12.6667\, n_f\right] \\
&+ \left(h^{\widetilde{\mathrm{MOM}}\mathrm{g}}\right)^4 \left[-2971.42 + 612.951\, n_f - 18.8833\, n_f^2\right] \\
&+ \left(h^{\widetilde{\mathrm{MOM}}\mathrm{g}}\right)^5 \left[-99327.8 + 23904.4\, n_f - 1591.07\, n_f^2 + 27.0482\, n_f^3\right], \qquad (66)
\end{aligned}$$

$$\begin{aligned}
\beta^{\widetilde{\mathrm{MOM}}\mathrm{q}} &= \left(h^{\widetilde{\mathrm{MOM}}\mathrm{q}}\right)^2 \left[-11 + 0.666667\, n_f\right] + \left(h^{\widetilde{\mathrm{MOM}}\mathrm{q}}\right)^3 \left[-102 + 12.6667\, n_f\right] \\
&+ \left(h^{\widetilde{\mathrm{MOM}}\mathrm{q}}\right)^4 \left[-2980.85 + 621.773\, n_f - 19.3833\, n_f^2\right] \\
&+ \left(h^{\widetilde{\mathrm{MOM}}\mathrm{q}}\right)^5 \left[-98539.5 + 24249\, n_f - 1622.19\, n_f^2 + 27.4926\, n_f^3\right], \qquad (67)
\end{aligned}$$

$$\begin{aligned}
\beta^{\widetilde{\mathrm{MOM}}\mathrm{gg}} &= \left(h^{\widetilde{\mathrm{MOM}}\mathrm{gg}}\right)^2 \left[-11 + 0.666667\, n_f\right] + \left(h^{\widetilde{\mathrm{MOM}}\mathrm{gg}}\right)^3 \left[-102 + 12.6667\, n_f\right] \\
&+ \left(h^{\widetilde{\mathrm{MOM}}\mathrm{gg}}\right)^4 \left[-2412.16 + 356.488\, n_f + 14.2834\, n_f^2 - 0.888889\, n_f^3\right] \\
&+ \left(h^{\widetilde{\mathrm{MOM}}\mathrm{gg}}\right)^5 \left[-84353.8 + 20077.1\, n_f - 1433.44\, n_f^2 + 108.637\, n_f^3 - 3.95062\, n_f^4\right]. \qquad (68)
\end{aligned}$$

The corresponding numbers for the $\overline{\mathrm{MS}}$ β-function are (as an expansion in $h = h^{\overline{\mathrm{MS}}}$):

$$\begin{aligned}
\beta^{\overline{\mathrm{MS}}} &= h^2 \left[-11 + 0.666667\, n_f\right] + h^3 \left[-102 + 12.6667\, n_f\right] \\
&+ h^4 \left[-1428.5 + 279.611\, n_f - 6.01852\, n_f^2\right] \\
&+ h^5 \left[-29243 + 6946.29\, n_f - 405.089\, n_f^2 - 1.49931\, n_f^3\right]. \qquad (69)
\end{aligned}$$

It has been argued [2] that momentum subtraction schemes will lead to couplings that are only slightly different. Comparing the $\widetilde{\mathrm{MOM}}$gg scheme with the other three schemes it is easy to see that this of course depends a lot on the linear combination used in the renormalization condition. On the other hand it is clearly seen that the couplings as well as the β-functions of the three schemes that are identical at one-loop order will also give very close-by values for the coupling constant at higher orders. Also the $\widetilde{\mathrm{MOM}}$gg scheme is still much closer to the other $\widetilde{\mathrm{MOM}}$ schemes than to the $\overline{\mathrm{MS}}$ one.

Inserting the values $1, \ldots, 6$ for n_f shows that the coefficients for all the values of n_f are smaller for the $\overline{\mathrm{MS}}$ scheme than in any of the $\widetilde{\mathrm{MOM}}$ schemes. The difference between the $\overline{\mathrm{MS}}$ scheme and the $\widetilde{\mathrm{MOM}}$ schemes is only small for $n_f = 6$ and gets quite large for the small values of n_f.

Recently the momentum dependence (running) of the three-gluon asymmetrical vertex corresponding to the combination $T_{gg} = T_1(p^2) - \frac{1}{2}T_2$ was computed within the lattice approach in [1]. The resulting (nonperturbative!) behaviour of

$$\alpha_s(p^2) = 4\pi h^{\widetilde{\mathrm{MOM}}\mathrm{gg}}(p^2)$$

in the flavorless QCD has been found to be best described by an ansatz[5]

$$\alpha_s(p^2) = \alpha_s^{Pert}(p^2) + \frac{c}{p^2}, \qquad (70)$$

with

$$\Lambda_{\overline{\mathrm{MS}}} = 237 \pm 3\mathrm{MeV}, \quad c = 0.63 \pm 0.03\mathrm{GeV}^2. \qquad (71)$$

[5] To be consistent to [1] we use the Euclidean metrics below.

b_3/b_2	4 loops				4 loops + power					
	whole		>3		whole			>3		
	χ^2_{dof}	$\Lambda_{\overline{\rm MS}}$	χ^2_{dof}	$\Lambda_{\overline{\rm MS}}$	χ^2_{dof}	$\Lambda_{\overline{\rm MS}}$	c	χ^2_{dof}	$\Lambda_{\overline{\rm MS}}$	c
0	7.8	299	8.3	294	1.6	237	0.63	1.6	235	0.67
1	7.1	284	6.0	287	1.8	238	0.56	1.6	235	0.61
2	28	259	4.1	279	2.7	229	0.57	1.6	236	0.53
2.78	87.5	234.	3.2	272.	4.3	218.	0.63	1.6	238.	0.47
3	104	227	3.0	270	4.7	215	0.65	1.6	238	0.45
4	145	215	2.9	261	7.3	202	0.75	1.8	237	0.37
5	157	209	4.4	252	10.2	190	0.84	2.4	231	0.37

Table 1: *A collection of fitted parameters obtained by imposing different values for the ratio b_3/b_2 as defined in the text. "Whole" refers to the whole energy window (2 GeV $<\mu<$10 GeV) and ">3" to a momentum range above 3 GeV.*

Here $\Lambda_{\overline{\rm MS}} = exp(-70/66)\Lambda_{\widetilde{\rm MOM}{\rm gg}}$. The above results have been obtained by working at three-loop level and employing the $\widetilde{\rm MOM}$gg scheme. The authors of [1] have also investigated the dependence of the result on the (then unknown) four-loop contribution to $\beta^{\widetilde{\rm MOM}{\rm gg}}$. Their findings are summarized in Table 1, which we have copied from [1] with adding one more row obtained with the help of the linear extrapolation and corresponding to the true value of the parameter

$$\frac{b_3}{b_2} = \frac{\beta_3^{\widetilde{\rm MOM}{\rm gg}}/(4\pi)}{\beta_2^{\widetilde{\rm MOM}{\rm gg}}} = 2.78284\ldots \qquad (\text{for } n_f = 0).$$

Table 1 clearly demonstrates that taking into account the four-loop term in the $\beta^{\widetilde{\rm MOM}{\rm gg}}$-function leads to a significant decrease (around 30%) of the value of the non-perturbative $1/p^2$ correction.

9 Conclusions

In this paper we have analytically computed the full set of the three-loops propagators and fundamental three-linear vertexes with one of three external momenta set to zero in the massless QCD. The results have been used to find the NNNLO conversion factors transforming the $\overline{\rm MS}$ coupling constant to the ones defined corresponding to a set of regularization independent renormalization schemes based on momentum subtractions ($\widetilde{\rm MOM}$-schemes). Then we have used the conversion factors to evaluate the four-loop β functions in these schemes.

The newly computed corrections to the coupling constant running in $\widetilde{\rm MOM}$-schemes prove to be numerically significant. They should be taken into account when confronting the running obtained with the help of lattice simulations with the pQCD predictions.

Acknowledgments

K. G. Ch. is grateful A. I. Davydychev and J.H. Kühn for useful discussions. A. R. wants to thank Thorsten Seidensticker, Timo van Ritbergen and Dominik Stöckinger for valuable discussions. Thorsten Seidensticker also provided some preliminary results of work in progress for cross-checks. This work was supported by the DFG under Contract Ku 502/8-1 (*DFG-Forschergruppe "Quantenfeldtheorie, Computeralgebra und Monte-Carlo-Simulationen"*) and the *Graduiertenkolleg "Elementarteilchenphysik an Beschleunigern" at the Universität Karlsruhe.*

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A Technical Remarks

To evaluate the self-energies and vertex function at three loop order we made heavy use of computer programs. The program `QGRAF` [26] was used to generate the diagrams. For the self-energies and the vertex functions at the ZP we arrive at scalar integrals of the massless propagator type after applying some projectors. An algorithm which allows one to analytically evaluate divergent as well as finite parts of such integrals was elaborated in [27]. It has been implemented in an efficient way in the `MINCER` [28] package written for the symbolic manipulation program `FORM` [29]. The huge number of diagrams in the three-loop calculations requires a complete automation of the whole procedure which should include also the calculation of the color factors. This has been implemented and is described in more detail along with other similar installations in [30]. This setup also gives the user tools to perform series expansions in small parameters, a feature that was used to calculate the naive first order expansion in a small external momentum for the ghost vertex which is necessary to check the WST identity as described in Section 4. We had to evaluate the following diagrams:

Number of diagrams for	1 loops	2 loops	3 loops	approximate runtime per function
$\tilde{\Pi}$,Σ_V	1	7	106	6 hours
Π	4	27	494	12 hours
$2 \times \tilde{\Gamma}_i + 4 \times a_i^{(\prime)}$	2	40	1022	12 hours
$2 \times (\Lambda_i, \Lambda_i^T)$	2	40	1022	2 days
T_i	10	189	5526	4 weeks

All calculations were done on workstations using the Alpha 21164 processors running at 600 MHz. The approximate runtimes are only rough estimates and show that the calculation of the triple gluon vertex was by far the most demanding part. This is not only because of the number of diagrams but also because the complicated vertex and propagator structures generate a huge number of intermediate terms.

Wherever possible, we have cross-checked the following expressions with the known two- and three-loop results, the gluon, ghost and quark field anomalous dimensions can also be found in [31] and [32]. In most cases we found complete agreement. We only find different values than [3] for the two-loop contributions to Λ_g and Λ_g^T (their Γ_3 and Γ_4). For the two-loop triple gluon vertex we find complete agreement with [19], which contains a list of miss-prints in earlier publications, among them the gauge dependence of the one-loop T_2 in [3].

The latex code for all results in this paper has been created automatically. The code has then been retransformed to FORM input files and cross-checked against the original expressions. We hope that by doing these checks and avoiding any hand editing of the formulas we have circumvent the common problem of miss-prints in otherwise correct results. The price is of course that the layout of some of the formulas is not as fancy as it could be. Also note that these expressions will be made available in the World-Wide-Web.

B Conventions

B.1 Color Factors

The following results are given for generic color factors and are valid for any semi-simple compact Lie group, where T^a_{ij} are the generators of the fundamental representation and f^{abc} are the structure constants of the Lie algebra.

	Definition	SU(N)	QCD	QED
C_A	$f^{acd}f^{bcd} = C_A\delta^{ab}$	N	3	0
C_F	$[T^aT^a]_{ij} = C_F\delta_{ij}$	$\frac{N^2-1}{2N}$	$\frac{4}{3}$	1
T	$\mathrm{Tr}(T^aT^b) = T\delta^{ab}$	$\frac{1}{2}$	$\frac{1}{2}$	1

B.2 Momentum Dependence

We give the $\overline{\text{MS}}$ renormalized result for all self-energies and vertex functions at the point $p^2 = -\mu^2$. The correct p^2 dependence for any of the self-energies and vertex functions Γ can be restored from the renormalization group equations, which are with our conventions:

$$\mu^2 \frac{\mathrm{d}}{\mathrm{d}\mu^2}\Gamma(h,q,\mu,\xi_L) = \left(\mu^2\frac{\partial}{\partial\mu^2} + \beta\frac{\partial}{\partial h} + \gamma_3\xi_L\frac{\partial}{\partial\xi_L}\right)\Gamma(h,q,\mu,\xi_L) = \gamma_\Gamma\Gamma(h,q,\mu,\xi_L) \tag{B.1}$$

For massless QCD one cam write

$$\Gamma = \Gamma_0 + \sum_{i>0}\Gamma_i \log^i(\frac{\mu^2}{-q^2}), \tag{B.2}$$

and as a direct consequence of (B.1) one gets (for $n > 0$)

$$\Gamma_n = \left(-\beta\frac{\partial}{\partial h} - \gamma_3\xi_L\frac{\partial}{\partial\xi_L} + \gamma_\Gamma\right)\frac{1}{n}\Gamma_{n-1} \tag{B.3}$$

The equation can be directly used to reconstruct the q-dependence of the scalar functions $1+\Pi$, $1+\tilde{\Pi}$ and $1+\Sigma_V$. For the vertex functions one needs to multiply the functions T_i, $\tilde{\Gamma}_i$ and Λ_i by $g = 4\pi\sqrt{h}$. The anomalous dimensions of any Γ is:

$$\gamma_\Gamma = -\left(\frac{n_g}{2}\gamma_3 + \frac{n_h}{2}\tilde{\gamma}_3 + \frac{n_q}{2}\gamma_2\right) \tag{B.4}$$

where n_g,n_h and n_q are the number of external gluon, ghost and quark fields, respectively, of the diagrams contributing to Γ. The QCD β-function and the field anomalous dimensions are given in part D of this appendix. Note that in our definitions for the anomalous dimensions a relative sign and a factor 2 is different from many other publications.

C Propagators and Vertices in massless QCD

C.1 The Gluon Self-Energy

$$
\begin{aligned}
\Pi^{\overline{\rm MS}} \;=\;& hC_A \left[-\frac{97}{36} - \frac{1}{2}\xi_L - \frac{1}{4}\xi_L^2\right] + hTn_f \left[+\frac{20}{9}\right] \\
&+ h^2 C_A T n_f \left[+\frac{59}{4} + 8\zeta_3 - \frac{10}{9}\xi_L - \frac{10}{9}\xi_L^2\right] + h^2 C_F T n_f \left[+\frac{55}{3} - 16\zeta_3\right] \\
&+ h^2 C_A^2 \left[-\frac{2381}{96} - 2\zeta_3\xi_L + 3\zeta_3 + \frac{463}{288}\xi_L + \frac{95}{144}\xi_L^2 - \frac{1}{16}\xi_L^3 + \frac{1}{16}\xi_L^4\right] \\
&+ h^3 C_A^3 \left[-\frac{10221367}{31104} - \frac{12071}{288}\zeta_3\xi_L - \frac{161}{96}\zeta_3\xi_L^2 + \frac{149}{96}\zeta_3\xi_L^3 + \frac{13}{96}\zeta_3\xi_L^4 + \frac{1549}{24}\zeta_3 + \frac{3}{8}\zeta_4\xi_L\right. \\
&\qquad + \frac{3}{32}\zeta_4\xi_L^2 + \frac{9}{32}\zeta_4 + \frac{115}{8}\zeta_5\xi_L + \frac{385}{96}\zeta_5\xi_L^2 - \frac{5}{24}\zeta_5\xi_L^3 - \frac{35}{192}\zeta_5\xi_L^4 + \frac{7025}{192}\zeta_5 \\
&\qquad \left. + \frac{13141}{1152}\xi_L + \frac{30835}{10368}\xi_L^2 - \frac{2813}{1152}\xi_L^3 - \frac{29}{48}\xi_L^4 + \frac{1}{16}\xi_L^5 - \frac{1}{64}\xi_L^6\right] \\
&+ h^3 C_A^2 T n_f \left[+\frac{1154561}{3888} + \frac{202}{9}\zeta_3\xi_L - \frac{25}{6}\zeta_3\xi_L^2 + \frac{871}{18}\zeta_3 - 9\zeta_4 - \frac{160}{3}\zeta_5 - \frac{241}{24}\xi_L\right. \\
&\qquad \left. - \frac{6137}{1296}\xi_L^2 - \frac{5}{18}\xi_L^3 + \frac{5}{12}\xi_L^4\right] \\
&+ h^3 C_A T^2 n_f^2 \left[-\frac{10499}{243} - \frac{64}{9}\zeta_3\xi_L - \frac{256}{9}\zeta_3 + \frac{16}{9}\xi_L - \frac{100}{81}\xi_L^2\right] \\
&+ h^3 C_A C_F T n_f \left[+\frac{96809}{324} + 8\zeta_3\xi_L + 8\zeta_3\xi_L^2 - \frac{1492}{9}\zeta_3 + 12\zeta_4 - 80\zeta_5 - \frac{55}{6}\xi_L - \frac{55}{6}\xi_L^2\right] \\
&+ h^3 C_F T^2 n_f^2 \left[-\frac{7402}{81} + \frac{608}{9}\zeta_3\right] + h^3 C_F^2 T n_f \left[-\frac{286}{9} - \frac{296}{3}\zeta_3 + 160\zeta_5\right]. \qquad \text{(C.1)}
\end{aligned}
$$

C.2 The Ghost Self-Energy

$$
\begin{aligned}
\tilde{\Pi}^{\overline{\rm MS}} \;=\;& hC_A\ [-1] + h^2 C_A T n_f \left[+\frac{95}{24}\right] \\
&+ h^2 C_A^2 \left[-\frac{1751}{192} - \frac{3}{8}\zeta_3\xi_L + \frac{3}{16}\zeta_3\xi_L^2 + \frac{15}{16}\zeta_3 + \frac{7}{64}\xi_L - \frac{3}{8}\xi_L^2\right] \\
&+ h^3 C_A^3 \left[-\frac{466373}{3888} - \frac{1429}{192}\zeta_3\xi_L + \frac{93}{64}\zeta_3\xi_L^2 + \frac{23}{64}\zeta_3\xi_L^3 + \frac{12403}{576}\zeta_3 - \frac{3}{16}\zeta_4\xi_L - \frac{3}{64}\zeta_4\xi_L^2\right. \\
&\qquad \left. - \frac{9}{64}\zeta_4 + \frac{65}{32}\zeta_5\xi_L - \frac{35}{32}\zeta_5\xi_L^2 + \frac{5}{32}\zeta_5\xi_L^3 + \frac{65}{32}\zeta_5 - \frac{61}{576}\xi_L - \frac{327}{128}\xi_L^2 - \frac{361}{384}\xi_L^3\right] \\
&+ h^3 C_A^2 T n_f \left[+\frac{150247}{1944} + \frac{13}{6}\zeta_3\xi_L + \frac{29}{9}\zeta_3 + \frac{9}{2}\zeta_4 - \frac{199}{144}\xi_L\right] \\
&+ h^3 C_A T^2 n_f^2 \left[-\frac{5161}{486} - \frac{8}{9}\zeta_3\right] + h^3 C_A C_F T n_f \left[+\frac{899}{24} - 22\zeta_3 - 6\zeta_4\right]. \qquad \text{(C.2)}
\end{aligned}
$$

C.3 The Quark Self-Energy

$$
\begin{aligned}
\Sigma_V^{\overline{\rm MS}} \;=\;& hC_F\;[+\xi_L] + \; h^2C_AC_F\;\left[+\frac{41}{4} - 3\zeta_3\xi_L - 3\zeta_3 + \frac{13}{2}\xi_L + \frac{9}{8}\xi_L^2\right] \\
&+ h^2C_FTn_f\;\left[-\frac{7}{2}\right] + \; h^2C_F^2\;\left[-\frac{5}{8}\right] \\
&+ h^3C_A^2C_F\;\left[+\frac{159257}{648} - 35\zeta_3\xi_L - \frac{39}{8}\zeta_3\xi_L^2 - \frac{1}{3}\zeta_3\xi_L^3 - \frac{3139}{24}\zeta_3 + \frac{3}{8}\zeta_4\xi_L + \frac{3}{16}\zeta_4\xi_L^2 \right.\\
&\qquad \left. -\frac{69}{16}\zeta_4 + \frac{5}{2}\zeta_5\xi_L + \frac{5}{4}\zeta_5\xi_L^2 + \frac{165}{4}\zeta_5 + \frac{39799}{576}\xi_L + \frac{787}{64}\xi_L^2 + \frac{55}{24}\xi_L^3\right] \\
&+ h^3C_AC_F^2\;\left[-\frac{997}{24} - 17\zeta_3\xi_L + \zeta_3\xi_L^3 + 44\zeta_3 + 6\zeta_4 + 20\zeta_5\xi_L - 20\zeta_5 + 4\xi_L + \frac{3}{2}\xi_L^2 - \frac{1}{8}\xi_L^3\right] \\
&+ h^3C_AC_FTn_f\;\left[-\frac{11887}{81} + 8\zeta_3\xi_L + \frac{52}{3}\zeta_3 - \frac{1723}{72}\xi_L\right] + \; h^3C_FT^2n_f^2\;\left[+\frac{1570}{81}\right] \\
&+ h^3C_F^2Tn_f\;\left[-\frac{79}{6} + 16\zeta_3 - \frac{3}{2}\xi_L\right] + \; h^3C_F^3\;\left[-\frac{73}{12} - \frac{2}{3}\zeta_3\xi_L^3 + \frac{7}{8}\xi_L\right]. \qquad \text{(C.3)}
\end{aligned}
$$

C.4 The Triple Gluon Vertex

$$
\begin{aligned}
T_1^{\overline{\rm MS}} \;=\;& 1 \;+\; hC_A\;\left[-\frac{61}{36} - \frac{1}{4}\xi_L^2\right] + \; hTn_f\;\left[+\frac{20}{9}\right] \\
&+ h^2C_A^2\;\left[-\frac{9907}{576} - \frac{15}{8}\zeta_3\xi_L + \frac{13}{8}\zeta_3 + \frac{153}{64}\xi_L + \frac{35}{36}\xi_L^2 - \frac{3}{16}\xi_L^3 + \frac{1}{16}\xi_L^4\right] \\
&+ h^2C_ATn_f\;\left[+\frac{955}{72} + 8\zeta_3 - \frac{10}{9}\xi_L^2\right] + \; h^2C_FTn_f\;\left[+\frac{55}{3} - 16\zeta_3\right] \\
&+ h^3C_A^3\;\left[-\frac{155555}{648} - \frac{48047}{1152}\zeta_3\xi_L - \frac{1481}{384}\zeta_3\xi_L^2 + \frac{145}{128}\zeta_3\xi_L^3 + \frac{13}{96}\zeta_3\xi_L^4 + \frac{68537}{1152}\zeta_3 \right.\\
&\qquad +\frac{9}{16}\zeta_4\xi_L + \frac{9}{64}\zeta_4\xi_L^2 + \frac{27}{64}\zeta_4 + \frac{2345}{128}\zeta_5\xi_L + \frac{2995}{384}\zeta_5\xi_L^2 + \frac{55}{128}\zeta_5\xi_L^3 - \frac{35}{192}\zeta_5\xi_L^4 \\
&\qquad \left. +\frac{2065}{128}\zeta_5 + \frac{44555}{2304}\xi_L + \frac{127487}{20736}\xi_L^2 - \frac{271}{192}\xi_L^3 - \frac{149}{192}\xi_L^4 + \frac{3}{32}\xi_L^5 - \frac{1}{64}\xi_L^6\right] \\
&+ h^3C_A^2C_F\;\left[-2 + \frac{43}{32}\zeta_3\xi_L + \frac{101}{32}\zeta_3\xi_L^2 + \frac{17}{32}\zeta_3\xi_L^3 - \frac{1497}{32}\zeta_3 - \frac{325}{32}\zeta_5\xi_L - \frac{215}{32}\zeta_5\xi_L^2 \right.\\
&\qquad \left. -\frac{35}{32}\zeta_5\xi_L^3 + \frac{1695}{32}\zeta_5 - \frac{1}{2}\xi_L + \frac{1}{4}\xi_L^2\right] + \; h^3C_FT^2n_f^2\;\left[-\frac{7402}{81} + \frac{608}{9}\zeta_3\right] \\
&+ h^3C_A^2Tn_f\;\left[+\frac{682607}{2592} + \frac{1889}{72}\zeta_3\xi_L - \frac{25}{6}\zeta_3\xi_L^2 + \frac{3733}{72}\zeta_3 - \frac{27}{2}\zeta_4 - \frac{160}{3}\zeta_5 - \frac{521}{96}\xi_L \right.\\
&\qquad \left. -\frac{9511}{2592}\xi_L^2 - \frac{5}{6}\xi_L^3 + \frac{5}{12}\xi_L^4\right] + \; h^3C_F^2Tn_f\;\left[-\frac{286}{9} - \frac{296}{3}\zeta_3 + 160\zeta_5\right] \\
&+ h^3C_AC_FTn_f\;\left[+\frac{181711}{648} + 8\zeta_3\xi_L^2 - \frac{1438}{9}\zeta_3 + 18\zeta_4 - 80\zeta_5 - \frac{55}{6}\xi_L^2\right] \\
&+ h^3C_AT^2n_f^2\;\left[-\frac{3415}{81} - \frac{64}{9}\zeta_3\xi_L - \frac{248}{9}\zeta_3 + \frac{16}{9}\xi_L - \frac{100}{81}\xi_L^2\right]. \qquad \text{(C.4)}
\end{aligned}
$$

$$
\begin{aligned}
T_2^{\overline{\rm MS}} \;=\;& hC_A \left[-\frac{37}{12} + \frac{3}{2}\xi_L + \frac{1}{4}\xi_L^2 \right] + \; hTn_f \; \left[+\frac{8}{3} \right] \\
& + h^2 C_A^2 \left[-\frac{443}{24} + \frac{1}{2}\zeta_3\xi_L - \frac{3}{2}\zeta_3 + \frac{31}{48}\xi_L + \frac{59}{72}\xi_L^2 - \frac{11}{16}\xi_L^3 - \frac{1}{8}\xi_L^4 \right] \\
& + h^2 C_A T n_f \left[+\frac{91}{6} + \frac{5}{2}\xi_L - \frac{2}{9}\xi_L^2 \right] + \; h^2 C_F T n_f \; [+8] \\
& + h^3 C_A^3 \left[-\frac{2108863}{6912} - \frac{787}{36}\zeta_3\xi_L - \frac{1093}{96}\zeta_3\xi_L^2 - \frac{95}{48}\zeta_3\xi_L^3 + \frac{7}{48}\zeta_3\xi_L^4 - \frac{19255}{96}\zeta_3 \right. \\
& \qquad + \frac{3805}{96}\zeta_5\xi_L + \frac{845}{48}\zeta_5\xi_L^2 + \frac{355}{96}\zeta_5\xi_L^3 + \frac{35}{192}\zeta_5\xi_L^4 + \frac{46985}{192}\zeta_5 + \frac{9869}{2304}\xi_L \\
& \qquad \left. + \frac{178183}{20736}\xi_L^2 + \frac{7039}{2304}\xi_L^3 - \frac{57}{128}\xi_L^4 + \frac{5}{32}\xi_L^5 + \frac{3}{64}\xi_L^6 \right] \\
& + h^3 C_A^2 C_F \left[+\frac{53}{2} + \frac{245}{8}\zeta_3\xi_L + \frac{49}{2}\zeta_3\xi_L^2 - \frac{5}{8}\zeta_3\xi_L^3 - \frac{9}{16}\zeta_3\xi_L^4 + \frac{7249}{16}\zeta_3 - \frac{125}{2}\zeta_5\xi_L \right. \\
& \qquad \left. - 50\zeta_5\xi_L^2 - 5\zeta_5\xi_L^3 - \frac{1025}{2}\zeta_5 - \frac{61}{4}\xi_L + \frac{3}{4}\xi_L^3 \right] \\
& + h^3 C_A^2 T n_f \left[+\frac{279701}{864} + \frac{5}{18}\zeta_3\xi_L + \frac{19}{4}\zeta_3\xi_L^2 + \frac{749}{12}\zeta_3 + \frac{10}{3}\zeta_5\xi_L - \frac{110}{3}\zeta_5 + \frac{197}{12}\xi_L \right. \\
& \qquad \left. + \frac{3637}{2592}\xi_L^2 - \frac{131}{36}\xi_L^3 - \frac{1}{3}\xi_L^4 \right] \\
& + h^3 C_A C_F T n_f \left[+\frac{1105}{6} - 24\zeta_3\xi_L - 8\zeta_3\xi_L^2 - 208\zeta_3 + \frac{320}{3}\zeta_5 + 25\xi_L + \frac{31}{6}\xi_L^2 \right] \\
& + h^3 C_A T^2 n_f^2 \left[-\frac{1741}{27} + \frac{32}{9}\zeta_3\xi_L - 16\zeta_3 - \frac{14}{9}\xi_L - \frac{140}{81}\xi_L^2 \right] \\
& + h^3 C_F T^2 n_f^2 \left[-\frac{176}{3} + \frac{128}{3}\zeta_3 \right] + \; h^3 C_F^2 T n_f \; [-4] \, .
\end{aligned} \tag{C.5}
$$

C.5 The Ghost Gluon Vertex

$$
\begin{aligned}
\tilde{\Gamma}_{\rm h}^{\overline{\rm MS}} \;=\;& 1 \;+\; hC_A \left[+\frac{1}{2}\xi_L \right] + \; h^2 C_A^2 \left[-\frac{9}{16}\zeta_3\xi_L + \frac{3}{16}\zeta_3\xi_L^2 + \frac{43}{16}\xi_L + \frac{7}{16}\xi_L^2 \right] \\
& + h^3 C_A^3 \left[+\frac{725}{192}\zeta_3\xi_L + \frac{267}{64}\zeta_3\xi_L^2 - 2\zeta_3\xi_L^3 - \frac{105}{8}\zeta_5\xi_L - \frac{95}{16}\zeta_5\xi_L^2 + \frac{35}{16}\zeta_5\xi_L^3 \right. \\
& \qquad \left. + \frac{3631}{128}\xi_L + \frac{2089}{384}\xi_L^2 + \frac{17}{24}\xi_L^3 \right] + \; h^3 C_A^2 T n_f \left[+\frac{29}{12}\zeta_3\xi_L - \frac{493}{48}\xi_L \right] \\
& + h^3 C_A^2 C_F \left[-27\zeta_3\xi_L - 6\zeta_3\xi_L^2 + \frac{9}{2}\zeta_3\xi_L^3 + \frac{225}{8}\zeta_5\xi_L + \frac{75}{8}\zeta_5\xi_L^2 - \frac{15}{4}\zeta_5\xi_L^3 \right] .
\end{aligned} \tag{C.6}
$$

$$
\begin{aligned}
\tilde{\Gamma}_{\mathrm{g}}^{\overline{\mathrm{MS}}} \;=\; & 1 \;+\; hC_A \left[+\frac{3}{4}+\frac{1}{4}\xi_L\right] \\
& + h^2C_A^2 \left[+\frac{599}{96}-\frac{9}{16}\zeta_3\xi_L+\frac{3}{16}\zeta_3\xi_L^2+\frac{97}{32}\xi_L+\frac{1}{4}\xi_L^2\right] + \; h^2C_ATn_f \left[-\frac{29}{12}\right] \\
& + h^3C_A^3 \left[+\frac{43273}{432}-\frac{817}{192}\zeta_3\xi_L+\frac{293}{64}\zeta_3\xi_L^2+\frac{3}{64}\zeta_3\xi_L^3+\frac{783}{64}\zeta_3-\frac{85}{16}\zeta_5\xi_L-\frac{325}{64}\zeta_5\xi_L^2\right. \\
& \qquad \left. +\frac{5}{16}\zeta_5\xi_L^3-\frac{875}{64}\zeta_5+\frac{28039}{768}\xi_L+\frac{4091}{768}\xi_L^2+\frac{31}{48}\xi_L^3\right] \\
& + h^3C_A^2C_F \left[+\frac{27}{4}-\frac{249}{16}\zeta_3\xi_L-\frac{111}{16}\zeta_3\xi_L^2+\frac{3}{16}\zeta_3\xi_L^3-\frac{639}{16}\zeta_3+\frac{165}{8}\zeta_5\xi_L+\frac{15}{2}\zeta_5\xi_L^2\right. \\
& \qquad \left. +\frac{225}{8}\zeta_5-\frac{21}{4}\xi_L+\frac{3}{4}\xi_L^2\right] + \; h^3C_AC_FTn_f \left[-16+12\zeta_3\right] \\
& + h^3C_A^2Tn_f \left[-\frac{15143}{216}+\frac{55}{24}\zeta_3\xi_L-\frac{49}{8}\zeta_3-\frac{357}{32}\xi_L\right] + \; h^3C_AT^2n_f^2 \left[+\frac{280}{27}\right] .
\end{aligned} \tag{C.7}
$$

C.6 The Quark Gluon Vertex

$$
\begin{aligned}
\Lambda_{\mathrm{q}}^{\overline{\mathrm{MS}}} \;=\; & 1 \;+\; hC_A \left[+1+\frac{1}{2}\xi_L\right] + \; hC_F \left[+\xi_L\right] \\
& + h^2C_A^2 \left[+\frac{2015}{192}-\frac{3}{2}\zeta_3+\frac{181}{64}\xi_L+\frac{13}{16}\xi_L^2\right] + \; h^2C_ATn_f \left[-\frac{95}{24}\right] \\
& + h^2C_AC_F \left[+\frac{41}{4}-3\zeta_3\xi_L-3\zeta_3+\frac{15}{2}\xi_L+\frac{13}{8}\xi_L^2\right] + \; h^2C_FTn_f \left[-\frac{7}{2}\right] + \; h^2C_F^2 \left[-\frac{5}{8}\right] \\
& + h^3C_A^3 \left[+\frac{2255345}{15552}-\frac{119}{192}\zeta_3\xi_L-\frac{19}{32}\zeta_3\xi_L^2-\frac{37}{64}\zeta_3\xi_L^3-\frac{1817}{72}\zeta_3+\frac{3}{16}\zeta_4\xi_L+\frac{3}{64}\zeta_4\xi_L^2\right. \\
& \qquad +\frac{9}{64}\zeta_4-\frac{105}{32}\zeta_5\xi_L+\frac{5}{32}\zeta_5\xi_L^2+\frac{15}{32}\zeta_5\xi_L^3-\frac{335}{32}\zeta_5+\frac{6511}{192}\xi_L+\frac{3295}{384}\xi_L^2 \\
& \qquad \left. +\frac{235}{128}\xi_L^3\right] + \; h^3C_A^2Tn_f \left[-\frac{169525}{1944}-\frac{1}{6}\zeta_3\xi_L-\frac{2}{9}\zeta_3-\frac{9}{2}\zeta_4-\frac{491}{48}\xi_L\right] \\
& + h^3C_A^2C_F \left[+\frac{331393}{1296}-41\zeta_3\xi_L-\frac{45}{8}\zeta_3\xi_L^2+\frac{5}{12}\zeta_3\xi_L^3-\frac{3631}{24}\zeta_3+\frac{3}{8}\zeta_4\xi_L+\frac{3}{16}\zeta_4\xi_L^2\right. \\
& \qquad \left. -\frac{69}{16}\zeta_4+\frac{35}{8}\zeta_5\xi_L+\frac{5}{4}\zeta_5\xi_L^2-\frac{5}{8}\zeta_5\xi_L^3+\frac{125}{2}\zeta_5+\frac{13117}{144}\xi_L+\frac{311}{16}\xi_L^2+\frac{11}{3}\xi_L^3\right] \\
& + h^3C_AC_F^2 \left[-\frac{253}{6}-17\zeta_3\xi_L+\zeta_3\xi_L^3+44\zeta_3+6\zeta_4+20\zeta_5\xi_L-20\zeta_5+\frac{59}{16}\xi_L+\frac{3}{2}\xi_L^2\right. \\
& \qquad \left. -\frac{1}{8}\xi_L^3\right] + \; h^3C_AC_FTn_f \left[-\frac{121637}{648}+8\zeta_3\xi_L+\frac{118}{3}\zeta_3+6\zeta_4-\frac{1067}{36}\xi_L\right] \\
& + h^3C_AT^2n_f^2 \left[+\frac{5161}{486}+\frac{8}{9}\zeta_3\right] + \; h^3C_FT^2n_f^2 \left[+\frac{1570}{81}\right] \\
& + h^3C_F^2Tn_f \left[-\frac{79}{6}+16\zeta_3-\frac{3}{2}\xi_L\right] + \; h^3C_F^3 \left[-\frac{73}{12}-\frac{2}{3}\zeta_3\xi_L^3+\frac{7}{8}\xi_L\right] .
\end{aligned} \tag{C.8}
$$

$$
\begin{aligned}
\Lambda_{\mathrm{q}}^{T\,\overline{\mathrm{MS}}} \;=\;& hC_A\ \left[+\frac{9}{4} - \xi_L - \frac{1}{4}\xi_L^2\right] +\ hC_F\ [-2] \\
&+ h^2C_A^2\ \left[+\frac{523}{24} - \frac{11}{8}\zeta_3\xi_L - \frac{1}{16}\zeta_3\xi_L^2 + \frac{11}{16}\zeta_3 - \frac{145}{48}\xi_L - \frac{215}{144}\xi_L^2 - \frac{1}{16}\xi_L^3 + \frac{1}{16}\xi_L^4\right] \\
&+ h^2C_AC_F\ \left[-\frac{505}{18} + \frac{13}{4}\xi_L - \frac{1}{2}\xi_L^2 - \frac{1}{4}\xi_L^3\right] +\ h^2C_FTn_f\ \left[+\frac{52}{9}\right] \\
&+ h^2C_ATn_f\ \left[-\frac{16}{3} - 4\zeta_3 - \frac{2}{3}\xi_L - \frac{5}{9}\xi_L^2\right] +\ h^2C_F^2\ [+9 - 2\xi_L] \\
&+ h^3C_A^3\ \left[+\frac{2844547}{6912} - \frac{2221}{96}\zeta_3\xi_L - \frac{203}{192}\zeta_3\xi_L^2 + \frac{89}{48}\zeta_3\xi_L^3 + \frac{29}{192}\zeta_3\xi_L^4 + \frac{565}{9}\zeta_3\right. \\
&\qquad + \frac{655}{48}\zeta_5\xi_L + \frac{385}{96}\zeta_5\xi_L^2 - \frac{35}{48}\zeta_5\xi_L^3 - \frac{35}{192}\zeta_5\xi_L^4 - \frac{23575}{192}\zeta_5 - \frac{268897}{6912}\xi_L \\
&\qquad \left. - \frac{40489}{2304}\xi_L^2 - \frac{10445}{2304}\xi_L^3 - \frac{155}{576}\xi_L^4 + \frac{1}{16}\xi_L^5 - \frac{1}{64}\xi_L^6\right] \\
&+ h^3C_A^2C_F\ \left[-\frac{1450057}{2592} + \frac{389}{48}\zeta_3\xi_L + \frac{35}{12}\zeta_3\xi_L^2 - \frac{1}{16}\zeta_3\xi_L^3 - \frac{1939}{8}\zeta_3 - \frac{265}{24}\zeta_5\xi_L\right. \\
&\qquad \left. + \frac{5}{24}\zeta_5\xi_L^2 + \frac{5}{8}\zeta_5\xi_L^3 + \frac{8665}{24}\zeta_5 + \frac{10469}{288}\xi_L - \frac{299}{96}\xi_L^2 - \frac{149}{36}\xi_L^3 - \frac{15}{32}\xi_L^4 + \frac{1}{16}\xi_L^5\right] \\
&+ h^3C_A^2Tn_f\ \left[-\frac{171143}{864} + \frac{275}{18}\zeta_3\xi_L - \frac{3}{2}\zeta_3\xi_L^2 - \frac{880}{9}\zeta_3 + \frac{20}{3}\zeta_5 + \frac{415}{72}\xi_L - \frac{575}{288}\xi_L^2\right. \\
&\qquad \left. + \frac{1}{36}\xi_L^3 + \frac{5}{18}\xi_L^4\right] +\ h^3C_AT^2n_f^2\ \left[+\frac{700}{27} - \frac{32}{9}\zeta_3\xi_L + \frac{112}{9}\zeta_3 + \frac{40}{27}\xi_L\right] \\
&+ h^3C_AC_FTn_f\ \left[+\frac{78625}{648} - 4\zeta_3\xi_L + 4\zeta_3\xi_L^2 + \frac{196}{3}\zeta_3 + \frac{400}{3}\zeta_5 - \frac{73}{18}\xi_L - \frac{113}{24}\xi_L^2 - \frac{5}{9}\xi_L^3\right] \\
&+ h^3C_AC_F^2\ \left[+\frac{76339}{288} + 6\zeta_3\xi_L + 316\zeta_3 - \frac{1240}{3}\zeta_5 - \frac{3235}{72}\xi_L - \frac{107}{32}\xi_L^2 + \frac{1}{2}\xi_L^3\right] \\
&+ h^3C_FT^2n_f^2\ \left[-\frac{2000}{81}\right] +\ h^3C_F^2Tn_f\ \left[+\frac{821}{9} - \frac{160}{3}\zeta_3 - 160\zeta_5 + \frac{52}{9}\xi_L\right] \\
&+ h^3C_F^3\ \left[-\frac{973}{12} - \frac{496}{3}\zeta_3 + \frac{640}{3}\zeta_5 + 9\xi_L\right]. \qquad \text{(C.9)}
\end{aligned}
$$

$$
\begin{aligned}
\Lambda_{\mathrm{g}}^{\overline{\mathrm{MS}}} \;=\;& 1 + hC_A \left[+\frac{1}{4} + \frac{3}{4}\xi_L\right] + hC_F\ [-\xi_L] \\
&+ h^2C_A^2 \left[+\frac{1075}{192} - 3\zeta_3 + \frac{181}{64}\xi_L + \xi_L^2\right] + h^2C_AC_F \left[+\frac{3}{4} - 3\zeta_3\xi_L - 3\zeta_3 - \frac{3}{2}\xi_L - \frac{3}{8}\xi_L^2\right] \\
&+ h^2C_ATn_f \left[-\frac{55}{24}\right] + h^2C_FTn_f \left[+\frac{1}{2}\right] + h^2C_F^2 \left[+\frac{19}{8} - 2\xi_L^2\right] \\
&+ h^3C_A^3 \left[+\frac{59815}{1944} - \frac{1235}{48}\zeta_3\xi_L - \frac{73}{32}\zeta_3\xi_L^2 - \frac{1}{4}\zeta_3\xi_L^3 - \frac{10145}{288}\zeta_3 + \frac{3}{16}\zeta_4\xi_L + \frac{3}{64}\zeta_4\xi_L^2\right. \\
&\qquad \left. + \frac{9}{64}\zeta_4 + \frac{35}{4}\zeta_5\xi_L - \frac{5}{16}\zeta_5\xi_L^2 + \frac{365}{16}\zeta_5 + \frac{5695}{128}\xi_L + \frac{4033}{384}\xi_L^2 + \frac{141}{64}\xi_L^3\right] \\
&+ h^3C_A^2C_F \left[+\frac{21971}{648} + \frac{55}{2}\zeta_3\xi_L - \frac{5}{4}\zeta_3\xi_L^2 + \frac{5}{12}\zeta_3\xi_L^3 - \frac{527}{3}\zeta_3 + \frac{3}{8}\zeta_4\xi_L + \frac{3}{16}\zeta_4\xi_L^2 - \frac{69}{16}\zeta_4\right. \\
&\qquad \left. - \frac{35}{2}\zeta_5\xi_L + \frac{5}{2}\zeta_5\xi_L^2 + 140\zeta_5 - \frac{13117}{288}\xi_L - \frac{159}{16}\xi_L^2 - \frac{173}{96}\xi_L^3\right] \\
&+ h^3C_A^2Tn_f \left[-\frac{89777}{3888} + \frac{17}{6}\zeta_3\xi_L + \frac{359}{18}\zeta_3 - \frac{9}{2}\zeta_4 - 20\zeta_5 - 14\xi_L\right] \\
&+ h^3C_AC_F^2 \left[+\frac{1259}{32} - 17\zeta_3\xi_L + \frac{9}{2}\zeta_3\xi_L^2 + \frac{1}{2}\zeta_3\xi_L^3 + 82\zeta_3 + 6\zeta_4 + 20\zeta_5\xi_L - 100\zeta_5\right. \\
&\qquad \left. - \frac{747}{32}\xi_L - \frac{29}{2}\xi_L^2 - \frac{23}{8}\xi_L^3\right] + h^3C_AT^2n_f^2 \left[+\frac{1741}{486} + \frac{8}{9}\zeta_3\right] \\
&+ h^3C_AC_FTn_f \left[-\frac{6823}{324} - 2\zeta_3\xi_L + \frac{58}{3}\zeta_3 + 6\zeta_4 + \frac{761}{72}\xi_L\right] + h^3C_FT^2n_f^2 \left[-\frac{302}{81}\right] \\
&+ h^3C_F^2Tn_f \left[-\frac{45}{2} + 16\zeta_3 + \frac{19}{2}\xi_L\right] + h^3C_F^3 \left[-\frac{109}{12} - \frac{2}{3}\zeta_3\xi_L^3 + \frac{41}{8}\xi_L\right].
\end{aligned}
\tag{C.10}
$$

$$
\begin{aligned}
\Lambda_{\mathrm{g}}^{T\,\overline{\mathrm{MS}}} \;=\;& hC_A \left[+\frac{1}{2} - \frac{1}{2}\xi_L\right] + hC_F\ [+2\xi_L] + h^2C_A^2 \left[+\frac{167}{36} - \frac{1}{2}\zeta_3\xi_L - \frac{5}{2}\zeta_3 - \frac{29}{24}\xi_L - \frac{5}{8}\xi_L^2\right] \\
&+ h^2C_AC_F \left[+\frac{43}{6} + 4\zeta_3 + 9\xi_L + \frac{3}{2}\xi_L^2\right] + h^2C_ATn_f \left[-\frac{7}{9}\right] + h^2C_FTn_f\ [-4] \\
&+ h^2C_F^2\ [-3 + 2\xi_L^2] + h^3C_AC_FTn_f \left[-\frac{15827}{108} + 8\zeta_3\xi_L - \frac{40}{9}\zeta_3 - \frac{112}{3}\xi_L\right] \\
&+ h^3C_A^3 \left[+\frac{1017553}{10368} + \frac{857}{72}\zeta_3\xi_L + \frac{7}{16}\zeta_3\xi_L^2 - \frac{7373}{144}\zeta_3 - \frac{115}{12}\zeta_5\xi_L - \frac{205}{12}\zeta_5 - \frac{36727}{1728}\xi_L\right. \\
&\qquad \left. - \frac{2309}{384}\xi_L^2 - \frac{43}{32}\xi_L^3\right] \\
&+ h^3C_A^2C_F \left[+\frac{39011}{216} - 63\zeta_3\xi_L - \frac{9}{4}\zeta_3\xi_L^2 + \frac{3917}{36}\zeta_3 + 25\zeta_5\xi_L - \frac{305}{3}\zeta_5 + \frac{4091}{32}\xi_L\right. \\
&\qquad \left. + \frac{2263}{96}\xi_L^2 + \frac{63}{16}\xi_L^3\right] \\
&+ h^3C_A^2Tn_f \left[-\frac{61085}{1296} - \frac{2}{9}\zeta_3\xi_L - \frac{14}{9}\zeta_3 + \frac{80}{3}\zeta_5 + \frac{3197}{432}\xi_L\right] + h^3C_F^3 \left[+3 - \frac{17}{4}\xi_L\right] \\
&+ h^3C_AC_F^2 \left[-\frac{3515}{48} + 6\zeta_3\xi_L - 6\zeta_3\xi_L^2 - \frac{184}{3}\zeta_3 + \frac{280}{3}\zeta_5 + \frac{361}{16}\xi_L + 17\xi_L^2 + \frac{11}{4}\xi_L^3\right] \\
&+ h^3C_AT^2n_f^2 \left[+\frac{260}{81}\right] + h^3C_FT^2n_f^2 \left[+\frac{208}{9}\right] + h^3C_F^2Tn_f \left[+\frac{28}{3} - 11\xi_L\right].
\end{aligned}
\tag{C.11}
$$

D Renormalization Group Coefficients

D.1 The β-function

$$\begin{aligned}\beta &= h^2 \left[-\frac{11}{3}C_A + \frac{4}{3}Tn_f\right] + h^3 \left[+\frac{20}{3}C_A T n_f - \frac{34}{3}C_A^2 + 4C_F T n_f\right] \\ &\quad + h^4 \left[+\frac{205}{9}C_A C_F T n_f - \frac{158}{27}C_A T^2 n_f^2 + \frac{1415}{27}C_A^2 T n_f - \frac{2857}{54}C_A^3 \right. \\ &\qquad \left. -\frac{44}{9}C_F T^2 n_f^2 - 2C_F^2 T n_f\right]. \end{aligned} \tag{D.1}$$

D.2 The Gluon Field Anomalous Dimension

$$\begin{aligned}\gamma_3 &= hC_A \left[+\frac{13}{6} - \frac{1}{2}\xi_L\right] + hn_f T \left[-\frac{4}{3}\right] \\ &\quad + h^2 C_A^2 \left[+\frac{59}{8} - \frac{11}{8}\xi_L - \frac{1}{4}\xi_L^2\right] + h^2 C_A n_f T\ [-5] + h^2 C_F n_f T\ [-4] \\ &\quad + h^3 C_A^3 \left[+\frac{9965}{288} - \frac{3}{4}\zeta_3\xi_L - \frac{3}{16}\zeta_3\xi_L^2 - \frac{9}{16}\zeta_3 - \frac{167}{32}\xi_L - \frac{33}{32}\xi_L^2 - \frac{7}{32}\xi_L^3\right] \\ &\quad + h^3 C_A^2 n_f T \left[-\frac{911}{18} + 18\zeta_3 + 2\xi_L\right] + h^3 C_A C_F n_f T \left[-\frac{5}{18} - 24\zeta_3\right] \\ &\quad + h^3 C_A n_f^2 T^2 \left[+\frac{76}{9}\right] + h^3 C_F n_f^2 T^2 \left[+\frac{44}{9}\right] + h^3 C_F^2 n_f T\ [+2]. \end{aligned} \tag{D.2}$$

D.3 The Ghost Field Anomalous Dimension

$$\begin{aligned}\tilde{\gamma}_3 &= hC_A \left[+\frac{3}{4} - \frac{1}{4}\xi_L\right] + h^2 C_A^2 \left[+\frac{95}{48} + \frac{1}{16}\xi_L\right] + h^2 C_A n_f T \left[-\frac{5}{6}\right] \\ &\quad + h^3 C_A C_F n_f T \left[-\frac{45}{4} + 12\zeta_3\right] + h^3 C_A n_f^2 T^2 \left[-\frac{35}{27}\right] + h^3 C_A^2 n_f T \left[-\frac{97}{108} - 9\zeta_3 + \frac{7}{8}\xi_L\right] \\ &\quad + h^3 C_A^3 \left[+\frac{15817}{1728} + \frac{3}{8}\zeta_3\xi_L + \frac{3}{32}\zeta_3\xi_L^2 + \frac{9}{32}\zeta_3 - \frac{17}{32}\xi_L - \frac{3}{32}\xi_L^2 - \frac{3}{64}\xi_L^3\right]. \end{aligned} \tag{D.3}$$

D.4 The Quark Field Anomalous Dimension

$$\begin{aligned}\gamma_2 &= hC_F\ [-\xi_L] + h^2 C_A C_F \left[-\frac{25}{4} - 2\xi_L - \frac{1}{4}\xi_L^2\right] + h^2 C_F n_f T\ [+2] + h^2 C_F^2 \left[+\frac{3}{2}\right] \\ &\quad + h^3 C_A^2 C_F \left[-\frac{9155}{144} - \frac{3}{4}\zeta_3\xi_L - \frac{3}{8}\zeta_3\xi_L^2 + \frac{69}{8}\zeta_3 - \frac{263}{32}\xi_L - \frac{39}{32}\xi_L^2 - \frac{5}{16}\xi_L^3\right] \\ &\quad + h^3 C_A C_F n_f T \left[+\frac{287}{9} + \frac{17}{4}\xi_L\right] + h^3 C_A C_F^2 \left[+\frac{143}{4} - 12\zeta_3\right] + h^3 C_F n_f^2 T^2 \left[-\frac{20}{9}\right] \\ &\quad + h^3 C_F^2 n_f T\ [-3] + h^3 C_F^3 \left[-\frac{3}{2}\right]. \end{aligned} \tag{D.4}$$