Using differential equations to compute two-loop box integrals*

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The calculation of exclusive observables beyond the one-loop level requires elaborate techniques for the computation of multi-leg two-loop integrals. We discuss how the large number of different integrals appearing in actual two-loop calculations can be reduced to a small number of master integrals. An efficient method to compute these master integrals is to derive and solve differential equations in the external invariants for them. As an application of the differential equation method, we compute the $\mathcal{O}(\epsilon)$-term of a particular combination of on-shell massless planar double box integrals, which appears in the tensor reduction of $2 \to 2$ scattering amplitudes at two loops.

1. Introduction

Perturbative corrections to many inclusive quantities have been computed to the two- and three-loop level in past years. From the technical point of view, these inclusive calculations correspond to the computation of multi-loop two-point functions, for which many elaborate calculational tools have been developed. In contrast, corrections to exclusive observables, such as jet production rates, could up to now only be computed at the one-loop level. These calculations demand the computation of multi-leg amplitudes to the required number of loops, which beyond the one-loop level turn out to be a calculational challenge obstructing further progress. Despite considerable progress made in recent times, many of the two-loop integrals relevant for the calculation of jet observables beyond next-to-leading order are still unknown. One particular class of yet unknown integrals appearing in the two-loop corrections to three jet production in electron-positron collisions, to two-plus-one jet production in electron-proton collisions and to vector boson plus jet production in proton-proton collisions are two-loop four-point functions with massless internal propagators and one external leg off-shell.

We elaborate on in this talk several techniques to compute multi-leg amplitudes beyond one loop. We demonstrate how integration-by-parts identities (already known to be a very valuable tool in inclusive calculations) and identities following from Lorentz-invariance (which are non-trivial only for integrals depending on at least two independent external momenta) can be used to reduce the large number of different integrals appearing in an actual calculation to a small number of master integrals. This reduction can be carried out mechanically (by means of a small chain of computer programs), without explicit reference to the actual structure of the integrals under consideration and can also be used for the reduction of tensor integrals beyond one loop.

The master integrals themselves, however, can not be computed from these identities. We derive differential equations in the external momenta for them. Solving these differential equations, it is possible to compute the master integrals without explicitly carrying out any loop integration, so that this technique appears to be a valuable alternative to conventional approaches for the computation of multi-loop integrals.

We demonstrate the application of these tools on several examples.

2. Reduction to master integrals

Any scalar massless two-loop integral can be brought into the form

$I(p_1, \ldots, p_n) =$

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$$\int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{1}{D_1^{m_1} \dots D_t^{m_t}} S_1^{n_1} \dots S_q^{n_q} \ , \quad (1)$$

where the D_i are massless scalar propagators, depending on k, l and the external momenta $p_1, \dots, p_n$ while S_i are scalar products of a loop momentum with an external momentum or of the two loop momenta. The topology (interconnection of propagators and external momenta) of the integral is uniquely determined by specifying the set $(D_1, \dots, D_t)$ of t different propagators in the graph. The integral itself is then specified by the powers m_i of all propagators and by the selection $(S_1, \dots, S_q)$ of scalar products and their powers $(n_1, \dots, n_q)$ (all the m_i are positive integers greater or equal to 1, while the n_i are greater or equal to 0). Integrals of the same topology with the same dimension $r = \sum_i m_i$ of the denominator and same total number $s = \sum_i n_i$ of scalar products are denoted as a class of integrals $I_{t,r,s}$. The integration measure and scalar products appearing the above expression are in Minkowskian space, with the usual causal prescription for all propagators. The loop integrations are carried out for arbitrary space-time dimension d, which acts as a regulator for divergencies appearing due to the ultraviolet or infrared behaviour of the integrand (dimensional regularisation, [1,2]).

The number $N(I_{t,r,s})$ of the integrals grows quickly as r, s increase, but the integrals are related among themselves by various identities. One class of identities follows from the fact that the integral over the total derivative with respect to any loop momentum vanishes in dimensional regularisation

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\partial}{\partial k^\mu} J(k, \dots) = 0, \qquad (2)$$

where J is any combination of propagators, scalar products and loop momentum vectors. J can be a vector or tensor of any rank. The resulting identities [2,3] are called integration-by-parts (IBP) identities.

In addition to the IBP identities, one can also exploit the fact that all integrals under consideration are Lorentz scalars (or, perhaps more precisely, "d-rotational" scalars) , which are invariant under a Lorentz (or d-rotational) transformation of the external momenta [4]. These Lorentz invariance (LI) identities are obtained from:

$$\left(p_1^\nu \frac{\partial}{\partial p_{1\mu}} - p_1^\mu \frac{\partial}{\partial p_{1\nu}} + \dots \right. \\ \left. \dots + p_n^\nu \frac{\partial}{\partial p_{n\mu}} - p_n^\mu \frac{\partial}{\partial p_{n\nu}} \right) I(p_1, \dots, p_n) = 0 \ . \qquad (3)$$

In the case of two-loop four-point functions, one has a total of 13 equations (10 IBP + 3 LI) for each integrand corresponding to an integral of class $I_{t,r,s}$, relating integrals of the same topology with up to $s+1$ scalar products and $r+1$ denominators, plus integrals of simpler topologies (*i.e.* with a smaller number of different denominators). The 13 identities obtained starting from an integral $I_{t,r,s}$ do contain integrals of the following types:

- $I_{t,r,s}$: the integral itself.
- $I_{t-1,r,s}$: simpler topology.
- $I_{t,r+1,s}, I_{t,r+1,s+1}$: same topology, more complicated than $I_{t,r,s}$.
- $I_{t,r-1,s}, I_{t,r-1,s-1}$: same topology, simpler than $I_{t,r,s}$.

Quite in general, single identities of the above kind can be used to obtain the reduction of $I_{t,r+1,s+1}$ or $I_{t,r+1,s}$ integrals in terms of $I_{t,r,s}$ and simpler integrals - rather than to get information on the $I_{t,r,s}$ themselves.

If one considers the set of all the identities obtained starting from the integrand of all the $N(I_{t,r,s})$ integrals of class $I_{t,r,s}$, one obtains $(N_{\mathrm{IBP}} + N_{\mathrm{LI}})N(I_{t,r,s})$ identities which contain $N(I_{t,r+1,s+1}) + N(I_{t,r+1,s})$ integrals of more complicated structure. It was first noticed by S. Laporta [5] that with increasing r and s the number of identities grows faster than the number of new unknown integrals. As a consequence, if for a given t-topology one considers the set of all the possible equations obtained by considering all the integrands up to certain values r^*, s^* of r, s, for large enough r^*, s^* the resulting system of equations, apparently overconstrained, can be used for expressing the more complicated integrals, with greater values of r, s in terms of simpler ones, with smaller values of r, s. An automatic procedure to preform this reduction by means of computer algebra is discussed in more detail in [6].

For any given four-point two-loop topology, this procedure can result either in a reduction towards a small number (typically one or two) of integrals of the topology under consideration and integrals of simpler topology (less different denominators), or even in a complete reduction of all integrals of the topology under consideration towards integrals with simpler topology. Leftover integrals of the topology under consideration are called irreducible master integrals or just master integrals.

3. Differential equations for master integrals

The IBP and LI identities allow to express integrals of the form (1) as a linear combination of a few master integrals, i.e. integrals which are not further reducible, but have to be computed by some different method.

At present, the complete set of master integrals for massless on-shell two-loop box integrals is known analytically [7–10] up to finite terms in $\epsilon = (4-d)/2$. For massless two-loop box integrals with one off-shell leg, several topologies are yet to be computed analytically. A purely numerical approach for computing these integrals order by order in ϵ has recently been proposed by Binoth and Heinrich in [11].

A method for the analytic computation of master integrals avoiding the explicit integration over the loop momenta is to derive differential equations in internal propagator masses or in external momenta for the master integral, and to solve these with appropriate boundary conditions. This method has first been suggested by Kotikov [12] to relate loop integrals with internal masses to massless loop integrals. It has been elaborated in detail and generalised to differential equations in external momenta in [13]; first applications were presented in [14]. In the case of four-point functions with one external off-shell leg and no internal masses, one has three independent invariants, resulting in three differential equations.

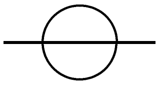
The derivatives in the invariants $s_{ij} = (p_i+p_j)^2$ can be expressed by derivatives in the external momenta:

$$\begin{aligned} s_{12}\frac{\partial}{\partial s_{12}} &= \frac{1}{2}\left(+p_1^\mu\frac{\partial}{\partial p_1^\mu}+p_2^\mu\frac{\partial}{\partial p_2^\mu}-p_3^\mu\frac{\partial}{\partial p_3^\mu}\right) , \\ s_{13}\frac{\partial}{\partial s_{13}} &= \frac{1}{2}\left(+p_1^\mu\frac{\partial}{\partial p_1^\mu}-p_2^\mu\frac{\partial}{\partial p_2^\mu}+p_3^\mu\frac{\partial}{\partial p_3^\mu}\right) , \\ s_{23}\frac{\partial}{\partial s_{23}} &= \frac{1}{2}\left(-p_1^\mu\frac{\partial}{\partial p_1^\mu}+p_2^\mu\frac{\partial}{\partial p_2^\mu}+p_3^\mu\frac{\partial}{\partial p_3^\mu}\right) . \end{aligned} \tag{4}$$

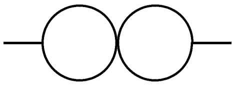
It is evident that acting with the right hand sides of (4) on a master integral $I_{t,t,0}$ will, after interchange of derivative and integration, yield a combination of integrals of the same type as appearing in the IBP and LI identities for $I_{t,t,0}$, including integrals of type $I_{t,t+1,1}$ and $I_{t,t+1,0}$. Consequently, the scalar derivatives (on left hand side of (4)) of $I_{t,t,0}$ can be expressed by a linear combination of integrals up to $I_{t,t+1,1}$ and $I_{t,t+1,0}$. These can all be reduced (for topologies containing only one master integral) to $I_{t,t,0}$ and to integrals of simpler topology by applying the IBP and LI identities. As a result, we obtain for the master integral $I_{t,t,0}$ an inhomogeneous linear first order differential equation in each invariant. For topologies with more than one master integral, one finds a coupled system of first order differential equations. The inhomogeneous term in these differential equations contains only topologies simpler than $I_{t,t,0}$, which are considered to be known if working in a bottom-up approach.

The master integral $I_{t,t,0}$ is obtained by matching the general solution of its differential equation to an appropriate boundary condition. Quite in general, finding a boundary condition is a simpler problem than evaluating the whole integral, since it depends on a smaller number of kinematical variables. In some cases, the boundary condition can even be determined from the differential equation itself.

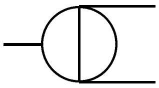
In writing down the differential equations for all master integrals appearing in the reduction of two-loop four-point functions with up to one off-shell leg, one finds that all boundary conditions can be related to the following four one-scale two-loop integrals:



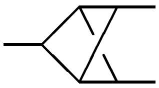
$$= A_3\,(-s)^{d-3}$$



$$= A_{2,LO}^2\,(-s)^{d-4}$$



$$= A_4\,(-s)^{d-4}$$



$$= A_6\,(-s)^{d-6} \qquad (5)$$

These integrals fulfill homogeneous differential equations in s; their normalisation can therefore not be calculated in the differential equation method. They can however be evaluated explicitly using Feynman parameters [15].

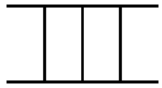
4. Applications

Using the differential equation method, we computed in [4] all master integrals with up to $t = 5$ different denominators that can appear in the reduction of two-loop four-point functions with one off-shell leg.

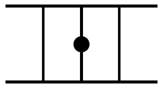
Differential equations were also used in the computation of two-loop on-shell master integrals with planar [9] and crossed [10] topologies. These topologies contain two master integrals each. The differential equations were applied only to obtain a relation among both master integrals, while one of the master integrals was calculated by Smirnov for the planar case [7] and Tausk for the non-planar case [8] using a different method.

At this workshop, Nigel Glover reported first progress towards the computation of massless on-shell $2 \to 2$ scattering amplitudes at two loops [16]. In the reduction of planar amplitudes, it turned out that it is not sufficient to know the planar double box integrals up to their finite terms in ϵ. In the tensor reduction procedure, the following combination of master integrals arises:

$$\frac{1}{\epsilon}\Big[$$



$$-\,t$$



$$\Big]$$

$$\sim \quad \frac{1}{\epsilon}\left[K_1(x,\epsilon) + K_2(x,\epsilon)\right] \;, \qquad (6)$$

where $x = t/s$. The functions $K_1(x,\epsilon)$ and $K_2(x,\epsilon)$ were computed only up to finite terms in ϵ by Smirnov and Veretin in [9]. The structure of the up to now unknown $\mathcal{O}(\epsilon)$-term was subject of debate at this workshop. We did therefore decide after the workshop to attempt its computation using the differential equation method.

Starting from the differential equations for the massless double box integrals with one off-shell leg, which were obtained using the algorithms described in Section 3, we can derive the differential equations for the above on-shell integrals by constraining $s_{12} + s_{13} + s_{23} = 0$ (see also [10]). Using s and x as independent variables, we obtain two homogeneous differential equations in s (corresponding to a rescaling relation, [14]) and two coupled inhomogeneous equations in x, which can be employed to compute the two master integrals. Using the fact that the master integrals and their derivatives are regular in t at $t = -s$, the boundary condition at $x = -1$ is inferred from the differential equations. This condition is however not yet sufficient to determine the boundary conditions for both master integrals, since it only fixes one combination of them. It does however turn out that it is sufficient to match the non-logarithmic terms in $K_1(x,\epsilon)$, obtained by Smirnov in [7], up to finite order in ϵ to find the boundary condition for $K_1(x,\epsilon) + K_2(x,\epsilon)$ up to $\mathcal{O}(\epsilon)$. The differential equations are then solved order by order in ϵ by expressing the unknown as sum of harmonic polylogarithms [17]. Requiring the differential equation to be fulfilled and the boundary conditions to be matched, the coefficients of the harmonic polylogarithms in the ansatz can all be determined.

Using the same normalisation factor as [9], the coefficient of the $\mathcal{O}(\epsilon)$-term of the above combination of double box integrals reads:

$$\left[K_1(x,\epsilon) + K_2(x,\epsilon)\right]\Big|_{\mathcal{O}(\epsilon)}$$

$$
\begin{aligned}
= \quad & -\frac{4}{3}\ln^4 x + \frac{4}{3}\ln^3 x - 4(4+3\pi^2)\ln^2 x \\
& -\Big(46 + 21\pi^2 - \frac{16}{3}\zeta(3)\Big)\ln x \\
& -50 - 8\pi^2 - \frac{139}{60}\pi^4 + \frac{698}{3}\zeta(3) \\
& -32\,S_{2,2}(-x) + 32\ln x\,S_{1,2}(-x) \\
& -128\,\mathrm{Li}_4(-x) + 32\Big(3\ln x - \ln(1+x) \\
& -2\Big)\,\mathrm{Li}_3(-x) + 32\Big(- \ln^2 x + 2\ln x \\
& + \ln x \ln(1+x) - \frac{5}{6}\pi^2\Big)\mathrm{Li}_2(-x) \\
& +8\left(\ln^2 x + \pi^2\right)\ln^2(1+x) \\
& +\frac{16}{3}\Big(- \ln^3 x + 6\ln^2 x + 6\pi^2 - 2\pi^2\ln x \\
& +6\zeta(3)\Big)\ln(1+x) \\
& +x\Big[+ \frac{16}{3}\ln^4 x - \frac{52}{3}\ln^3 x + \frac{46}{3}\pi^2\ln^2 x \\
& +\Big(2 - \frac{169}{3}\pi^2 - \frac{496}{3}\zeta(3)\Big)\ln x \\
& -84 - \frac{46}{3}\pi^2 + \frac{823}{360}\pi^4 + 536\,\zeta(3) \\
& -56\,S_{2,2}(-x) + 56\ln x\,S_{1,2}(-x) \\
& +40\,\mathrm{Li}_4(-x) + \Big(16\ln x - 56\ln(1+x) \\
& -104\Big)\,\mathrm{Li}_3(-x) - \Big(36\ln^2 x + \frac{8}{3}\pi^2 \\
& -56\ln x\ln(1+x) - 104\ln x\Big)\mathrm{Li}_2(-x) \\
& +14(\ln^2 x + \pi^2)\ln^2(1+x) \\
& +\frac{4}{3}\Big(- 16\ln^3 x + 39\ln^2 x - 23\pi^2\ln x \\
& +39\pi^2 + 42\zeta(3)\Big)\ln(1+x)\Big]\ . \qquad (7)
\end{aligned}
$$

5. Outlook

We have demonstrated how techniques developed for multi-loop calculation of two-point functions can be extended towards integrals with a larger number of external legs. In particular, we have shown that the use of differential equations in external invariants can be used as a powerful method to compute master integrals without carrying out explicit loop integrations. As a first example of the application of these tools in practice, we computed some up to now unknown two-loop four-point functions, relevant for jet calculus beyond the next-to-leading order. The most important potential application of these tools is the yet outstanding derivation of two-loop virtual corrections to exclusive quantities, such as jet observables.

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