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The three-loop on-shell renormalization of QCD and QED

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We describe a calculation of the *on-shell* renormalization factors in QCD and QED at the three loop level. Explicit results for the fermion mass renormalization factor Z_m and the on-shell fermion wave function renormalization constant Z_2 are given. We find that at $\mathcal{O}(\alpha_s^3)$ the wave function renormalization constant Z_2 in QCD becomes gauge dependent also in the on-shell scheme, thereby disproving the "gauge-independence" conjecture based on an earlier two-loop result. As a byproduct, we derive an $\mathcal{O}(\alpha_s^3)$ contribution to the anomalous dimension of the heavy quark field in HQET.

I. INTRODUCTION

Perturbative calculations in field theory are often instrumental in establishing accurate relations between theory and experiment. For example, in heavy quark physics, that now occupies the central place in the study of CP violation, perturbative effects play an important role alongside with non-perturbative effects. For heavy quarks, one separates soft and hard contributions through an expansion in the inverse quark mass; familiar examples being HQET (for a review see e.g. [1]) and NRQCD [2]. It often happens that the leading term in such an expansion is determined by a perturbative calculation in full QCD and for this reason perturbative calculations with heavy *on-shell* quarks in the initial and final state received significant attention in recent years. For many processes of interest we have witnessed a rapid computational progress and currently the "standard" level of accuracy is the next-to-next-to-leading or $\mathcal{O}(\alpha_s^2)$ order. Further progress in this direction will require three-loop computations.

In QED, an interesting recent development was an application of dimensionally regularized NRQED to the calculation of fundamental properties of simple atoms, like positronium and hydrogen. Here too, higher precision will require on-shell three loop calculations in many cases.

The renormalizability of QCD and QED implies that any multi-loop calculation can only be made meaningful if the corresponding renormalization constants are known[1]; hence the three-loop computations require the knowledge of the three-loop renormalization constants in QCD. As is well known, the renormalization constants are renormalization scheme dependent. In some schemes, as e.g. in the $\overline{\mathrm{MS}}$ scheme, they have been calculated up to the fourth order in perturbation theory. This is however insufficient if one is interested in the processes with heavy quarks being on-shell in the initial and final state – precisely as in QED, the renormalization constants in the on-shell scheme are required.

If one adopts the on-shell renormalization scheme for the quarks in QCD, two specific renormalization constants need to be calculated. The mass renormalization constant Z_m is defined by $m_0 = Z_m m$, where m_0 is the bare and m is the pole mass. The wave function renormalization factor Z_2 is defined as $\psi_0 = \sqrt{Z_2}\psi$, where ψ_0 and ψ are the bare and renormalized quark fields respectively. Both renormalization constants can be obtained from the expression for the heavy quark propagator close to the mass shell. The mass renormalization constant Z_m is determined from the position of the pole of the heavy quark propagator, while Z_2 is determined from the residue. Because of the infrared catastrophe, the residue does not exist and the introduction of the infrared regulator becomes necessary. In what follows dimensional regularization is used to regulate both ultraviolet *and* infrared divergences which appear in the calculation.

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[1] An immediate consequence of a more precise knowledge of just the on-shell renormalization factors is that it leads to an exact 3-loop result for the relation between the pole-quark-mass and the $\overline{\mathrm{MS}}$-quark-mass, which was recently presented in [3].

Let us remark that the choice of the infrared regulator is not important for the pole mass, since it is known to be infrared finite and gauge invariant to all orders in perturbation theory. On the contrary, Z_2 is not infrared finite already in the first non-trivial order in perturbative expansion. Hence, choosing the infrared regulator merely amounts to defining Z_2 (in addition to usual scheme definition) and one may expect that the question of gauge invariance may then depend on that choice[2]. Indeed, that is exactly what happens. An interesting phenomenon occurs if dimensional regularization is applied to infrared divergences. In this situation, in QED, Z_2 becomes gauge invariant to all orders in the coupling constant. This feature follows from dimensionally regulated Johnson-Zumino identity [4–6]:

$$\frac{\mathrm{d}\log Z_2}{\mathrm{d}\xi} = -ie_0^2 \int \frac{\mathrm{d}^D k}{(2\pi)^D k^4} = 0, \tag{1}$$

where ξ is the gauge fixing parameter in a general covariant gauge. Note that the right hand side of the above equation is zero by virtue of the fact that scale-less integrals in dimensional regularization are *defined* to be zero. We will derive Eq.(1) in Section VI using the path integral formalism.

Though no relation similar to Eq.(1) has been derived in QCD, the use of dimensional regularization for infra-red divergences resulted in some interesting observations. The one-loop Z_2 in QCD is gauge invariant since it is a trivial generalization of the QED result. The two-loop contribution to Z_2 is already sensitive to the non-abelian structure of QCD and there is no a priori reason to expect it to be gauge invariant. However, an explicit computation in Ref. [7] produced a gauge invariant result. The authors of that reference then conjectured that this fact may be true to all orders in perturbation theory, though no supporting arguments have been given.

The purpose of this paper is to present the three loop calculation of Z_m and Z_2 in dimensional regularization and hence to give a complete set of three loop renormalization constants that are specific to the on-shell scheme. The knowledge of Z_m and Z_2 with such an accuracy is an important step towards three loop calculations with heavy quarks close to the mass shell. The bulk of the paper is technical and attempts to provide some details about how the actual computation is done. Though the scale of this calculation makes it impossible to provide a detailed account of the algorithms used to obtain the result, we at least try to highlight some of its aspects which might be of importance for future related work. A peculiar feature of our result is that, in contrast to two first orders in perturbative expansion, the $\mathcal{O}(\alpha_s^3)$ contribution to Z_2 turns out to be *gauge dependent* thus disproving the all-orders gauge independence conjecture of Ref. [7].

The paper is organized as follows. In the next Section we introduce the necessary notations and describe the calculation. In Section III we discuss in detail how new master integrals that appear in non-abelian theory are computed. In Section IV the results for QCD renormalization constants are summarized. In Section V the anomalous dimension of the heavy quark field in HQET is derived. In Section VI we give the three loop renormalization constants for QED. In the appendices a complete list of master integrals for the three loop on-shell two-point functions, as well as some other useful formulas, are presented.

II. PRELIMINARIES

We are going to calculate the heavy quark mass and the heavy quark wave function renormalization constants in the on-shell renormalization scheme. The bare quark mass m_0 and the bare quark field ψ_0 are renormalized multiplicatively:

$$m_0 = Z_m m, \qquad \psi_0 = \sqrt{Z_2}\psi, \tag{2}$$

where m and ψ stand for the pole mass and the renormalized quark field, respectively. Both renormalization constants can be derived by considering one-particle irreducible quark self-energy operator $\hat{\Sigma}(p,m)$. We discuss this derivation in some detail since we believe that discussion given in [7] is slightly over-complicated in that place.

Because of the Lorentz invariance, the one-particle irreducible quark self energy operator $\hat{\Sigma}(p,m)$ can be parameterized by two independent functions:

$$\hat{\Sigma}(p,m) = m\Sigma_1(p^2,m) + (\hat{p}-m)\Sigma_2(p^2,m), \tag{3}$$

[2] In what follows, we loosely use the terms "gauge invariant", "gauge independent" and "gauge parameter independent" as equivalent.

so that complete fermion propagator reads:

$$\hat{S}_F(p) = \frac{i}{\hat{p} - m_0 + \hat{\Sigma}(p,m)}. \tag{4}$$

Let us expand the self energy operator in formal Taylor series around $p^2 = m^2$:

$$\begin{aligned}\hat{\Sigma}(p,m) &\approx m\ \Sigma_1(p^2,m)\big|_{p^2=m^2} + m\frac{\partial}{\partial p^2}\ \Sigma_1(p^2,m)\big|_{p^2=m^2}(p^2-m^2) + (\hat{p}-m)\ \Sigma_2(p^2,m)\big|_{p^2=m^2} \\ &\approx m\Sigma_1(p^2,m)|_{p^2=m^2} + (\hat{p}-m)\left(2m^2\frac{\partial}{\partial p^2}\ \Sigma_1(p^2,m)\big|_{p^2=m^2} + \Sigma_2(p^2,m)\big|_{p^2,m^2}\right). \end{aligned} \tag{5}$$

Substituting this expression into the fermion propagator Eq.(4), we identify the position of the pole with the pole mass and the residue with the wave function renormalization constant. We obtain:

$$\begin{aligned} Z_m &= 1 + \Sigma_1(p^2,m)\big|_{p^2=m^2}, \\ \frac{1}{Z_2} &= 1 + 2m^2\frac{\partial}{\partial p^2}\ \Sigma_1(p^2,m)\big|_{p^2=m^2} + \Sigma_2(p^2,m)\big|_{p^2=m^2}. \end{aligned} \tag{6}$$

Both of these constants can be obtained easily once an expansion of $\hat{\Sigma}(p,m)$ close to the mass shell is available. Let us introduce the Minkowski vector Q with $Q^2 = m^2$ and consider a one-parameter family of external momenta $p = Q(1+t)$. The quark self-energy operator reads:

$$\hat{\Sigma}(p,m) = m\Sigma_1(p^2,m) + (\hat{Q}-m)\Sigma_2(p^2,m) + t\hat{Q}\Sigma_2(p^2,m). \tag{7}$$

Taking the trace

$$T_1 = \mathrm{Tr}\left[\frac{\left(\hat{Q}+m\right)}{4m^2}\hat{\Sigma}(p,m)\right] = \Sigma_1(p^2,m) + t\Sigma_2(p^2,m), \tag{8}$$

and expanding in t up to $\mathcal{O}(t)$ we obtain:

$$T_1 = \Sigma_1(p^2,m)\big|_{p^2=m^2} + t\left[2m^2\frac{\partial}{\partial p^2}\ \Sigma_1(p^2,m)\big|_{p^2=m^2} + \Sigma_2(p^2,m)\big|_{p^2=m^2}\right] + \mathcal{O}(t^2). \tag{9}$$

Comparing Eqs.(6) with Eq.(9), we see that expanding the self energy operator in Qt along with projecting on the relevant structure by taking the trace as shown in Eq.(8) delivers an expression one needs to reconstruct the on-shell mass and wave function renormalization constants. The pole mass in the above equation is calculated iteratively by using mass counterterms where appropriate, i.e. by calculating lower order diagrams with the appropriate mass counterterm insertion.

Hence, both Z_m and Z_2 can be extracted from the heavy quark self-energy operator evaluated on-shell. This task is rather straightforward in low orders of perturbation theory but it becomes increasingly difficult when one goes to higher orders. Both the number of diagrams one has to calculate and the complexity of integrals increases dramatically. In the present case we have about sixty diagrams to be calculated and the question about an efficient way to do the calculation becomes of tremendous importance.

The most efficient way to evaluate those multiloop integrals is to utilize integration-by-parts identities within dimensional regularization [8,9] . The first thing to realize is that *any* integral that one can face in computing the on-shell fermion propagator can be expressed through eleven basic three loop integrals (topologies) shown in Fig.1. Any integral that belongs to a certain topology is considered to be a function of the powers of denominators (both positive and negative integer powers are allowed) and one irreducible numerator in each case. For each of the topologies one writes down a system of recurrence relations based on integration-by-parts identities that are derived using the fact that *in dimensional regularization* any integral of the total derivative vanishes:

$$0 = \int \frac{\partial}{\partial k_i^\mu}\left(l_j^\mu \times \text{propagators}\right). \tag{10}$$

Here k_i is one of the three loop momenta and l_j is either one of the loop momenta or the external momenta. Hence the starting set of equations consists of twelve recurrence relations for each of the topologies. The fact that not all of these relations are independent is not an obstacle since, when trying to solve them, some relations will vanish identically and for this reason will automatically be of no use.

The next step is to solve the system of these equations. Though there are several ways of thinking about what such a solution might be, we prefer to look for the most general one. We then want to construct an algorithm that, for a given topology and for any given initial set of powers of propagators, expresses an initial integral through a minimal set of "simpler" integrals. The simpler integrals are usually those that either have denominators raised to small powers or those that belong to simpler topologies. We then consider these "simpler", but still non-trivial topologies, write down a new set of recurrence relations for them, construct an algorithm that reduces any integral to even simpler topologies and continue along these lines until we have an algorithm that completely solves the initial problem in terms of a few master integrals. The final set of master integrals is found experimentally. There is no proof that the set we find is indeed minimal with respect to integration-by-parts relations in the strict mathematical sense, but for practical calculations this set of integrals is sufficient.

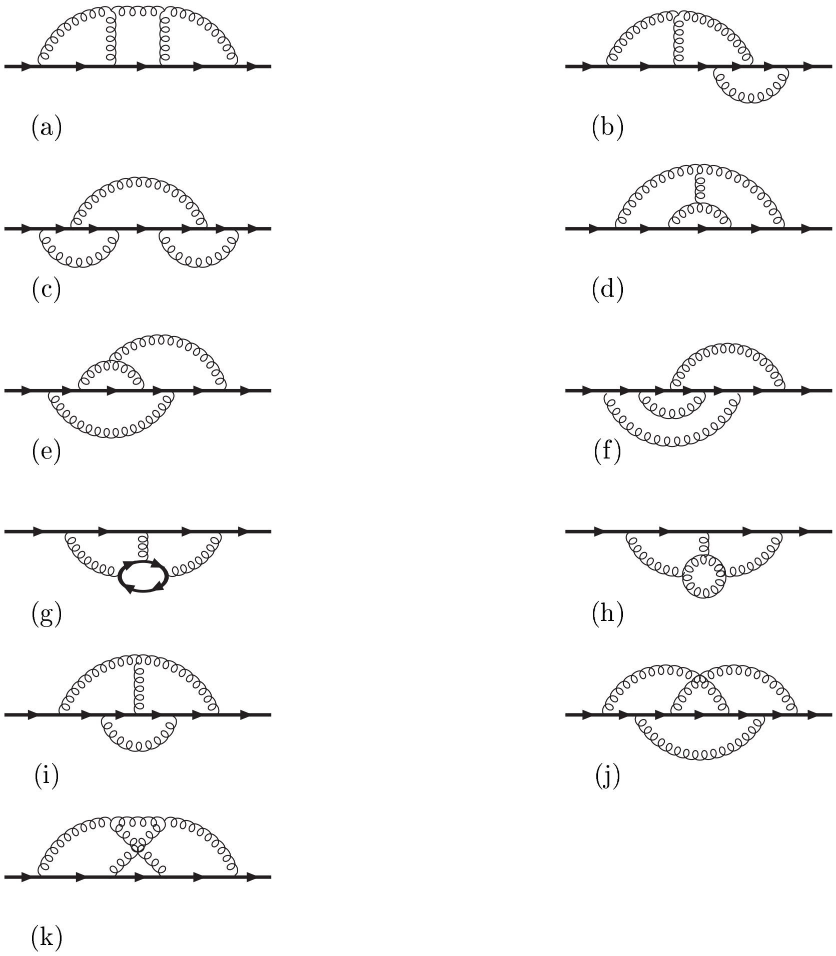

FIG. 1. Examples of three-loop quark propagator diagrams corresponding to eleven integration topologies.

Our solution of the system of recurrence relations shows that it is possible to express any integral which belongs to the above topologies through 18 master integrals. Most of these integrals have been calculated in the course of the analytical calculation of the electron anomalous magnetic moment [10] and can be taken from there. It is remarkable that a transition from the abelian theory to the non-abelian theory does not result in a significant increase in the number of master integrals to be computed, although the number of basic topologies does. As compared to Ref. [10], we need one additional master integral that corresponds to topology A and we also need one of the master integrals of Ref. [10] to a higher order in the regularization parameter ε. For the QCD wave function renormalization constant we also need the constant C_1 (see [10]) which was not computed in [10], because it mysteriously canceled in the calculation of the electron anomalous magnetic moment[3]. The calculation of these new master integrals is described in the next Section.

There are two principal checks on our solution of the recurrence relations. First, we have computed the three-loop anomalous magnetic moment of the electron and confirmed the result of Ref. [10]. Second, the actual calculation of the on-shell quark mass renormalization constant Z_m has been performed in an arbitrary covariant gauge for the gluon field. The explicit cancellation of the gauge parameter in our result for Z_m is an important check of the correctness of the calculation. There are also some checks related to anomalous dimensions of the heavy quark field in HQET, that can be computed combining our result for Z_2 with the known result for $Z_2^{\overline{\rm MS}}$. We will elaborate on this issue in Section V.

III. NEW MASTER INTEGRALS

In this Section the calculation of three new master integrals is described. Note that throughout this paper we work with the integrals defined in Euclidean space. A pictorial representation of all master integrals is given in Fig.2 in the appendix. The space-time dimension in what follows is parameterized by $D = 4 - 2\varepsilon$. We will also use $C(\varepsilon) = [\pi^{2-\varepsilon}\Gamma(1+\varepsilon)]^3$, $\zeta_k = \sum\limits_{n=1}^{\infty} 1/n^k$ for the Riemann ζ-function and $a_4 = \sum\limits_{n=1}^{\infty} 1/(2^n n^4) \equiv \mathrm{Li}_4(1/2)$ in what follows.

A. Master integral I_{10}

Let us consider the master integral I_{10}. Our basic approach to computing on-shell master integrals is to perform a large mass expansion which yields an explicit power series representation for the on-shell integrals. The advantage of this approach is that in this way most multi-loop integrations can be performed rather easily, and the resulting representation of the on-shell master integral (expanded around the space-time dimension $D = 4$) takes the form of a nested harmonic sum which can be reduced to known mathematical constants.

This approach has been applied in earlier works [12] to obtain exact results for several on-shell integrals. It relies heavily on an ability to reduce specific classes of nested sums. A similar problem of reducing harmonic sums appears in calculations for deep-inelastic lepton-nucleon scattering, where one also deals with an infinite expansion (the light-cone expansion instead of the large mass expansion that we employ here). Collections of summation identities of the type that are needed for the present work can be found in the literature [13–17].

For I_{10} we start by taking the squared masses of the two particles inside the loop M^2 to be different from the square of the external momenta $p^2 = -m^2$. Eventually, we are interested in the result for $M^2 = m^2$.

The expression for Euclidean integral I_{10} reads:

$$I_{10} = \int\int\int \frac{\mathrm{d}^D k_1\ \mathrm{d}^D k_2\ \mathrm{d}^D k_3}{(k_1+p)^2\ (k_2^2+M^2)\ (k_3^2+M^2)\ (k_1+k_2)^2\ (k_3+k_2)^2} \tag{11}$$

and we want to perform a systematic expansion in (p^2/M^2). In the present case this can be achieved by making an ordinary Taylor expansion in the external momentum p in the integrand

$$\frac{1}{[(k_1+p)^2]^{\alpha_1}} \to \frac{1}{(k_1^2)^{\alpha_1}} \sum_{i=0}^{\infty} (-1)^i \left(\frac{2k_1\cdot p + p^2}{k_1^2}\right)^i \frac{\Gamma(\alpha_1+i)}{\Gamma(\alpha_1)\ i!}. \tag{12}$$

As a side remark, let us note that for integral I_{10} the same large mass expansion can be obtained via a Mellin-Barnes representation for the two massive lines [18]. The Mellin-Barnes method can be used successfully here because the underlying massless integral with arbitrary complex powers of the massless lines can be evaluated in a closed form in terms of a product of Euler Γ-functions. However the diagrammatic large mass expansion as we use it here (i.e. the expansion via a sequence of Taylor expansions in the integrand) can be used also for integrals where the underlying massless integral is not known for arbitrary complex powers of the propagators, and the diagrammatic approach to

[3]Moreover, we observed a similar cancellation in our calculation of the three loop slope of the Dirac form factor [11] as well as in the calculation of the relation between the $\overline{\rm MS}$ and the pole quark masses [3].

a large mass expansion is therefore more suitable for certain complicated integrals, even though it seems technically more involved for simple topologies.

After performing the Taylor expansion in the integrand in Eq. (11) one is left with a 3-loop vacuum (bubble) integral with a tensor numerator involving powers of $(k_1 \cdot p)$. This numerator is simplified after a bubble tensor reduction:

$$(k_1 \cdot p)^n \to \begin{cases} (k_1^2 p^2)^{\frac{n}{2}} \dfrac{n!}{2^n\,(n/2)!} \dfrac{\Gamma(D/2)}{\Gamma(D/2+n/2)}, & \text{for } n \text{ even}, \\ 0, & \text{for } n \text{ odd}, \end{cases} \tag{13}$$

so that Eq.(12) effectively becomes:

$$\frac{1}{[(k_1+p)^2]^{\alpha_1}} \to \frac{1}{(k_1^2)^{\alpha_1}} \sum_{i=0}^{\infty} \left(\frac{p^2}{k_1^2}\right)^i \frac{\Gamma(\alpha_1+i)\,\Gamma(\alpha_1+i+1-D/2)\Gamma(D/2)}{\Gamma(\alpha_1)\,i!\,\Gamma(\alpha_1+1-D/2)\,\Gamma(i+D/2)}. \tag{14}$$

The 3-loop vacuum bubble integral can now be evaluated in terms of Γ-functions

$$\begin{aligned} &\int\!\!\int\!\!\int \frac{\mathrm{d}^D k_1\ \mathrm{d}^D k_2\ \mathrm{d}^D k_3}{(k_1^2)^{\alpha_1}(k_2^2+M^2)\,(k_3^2+M^2)\,[(k_1-k_2)^2]\,[(k_3-k_2)^2]} = \\ &(-1)^{\alpha_1}\,\pi^{(3D/2)} M^{3D-8-2\alpha_1}\ \frac{\Gamma(-3D/2+4+\alpha_1)\Gamma^3(D/2-1)\Gamma(2-D/2)}{\Gamma(D/2-1+\alpha_1)\Gamma(D/2)} \\ &\times\left[\frac{\Gamma^2(3-D)}{\Gamma(6-2D)} - 2\sum_{j=1}^{\alpha_1} \frac{\Gamma(D/2-2+j)\Gamma(2+j-D)\Gamma(2-D/2)}{(j-1)!\,\Gamma(-3D/2+4+j)\Gamma(-1+D/2)}\right], \end{aligned} \tag{15}$$

where α_1 is any non-negative integer. In this way one obtains an explicit representation for I_{10} as an infinite power series in (p^2/M^2). To obtain the result on the mass shell, where $m^2/M^2 = 1$, we have to perform the infinite sum over the coefficients of the power series. This task becomes much easier once we expand the obtained representation for I_{10} around $D = 4$. For this purpose the Γ-functions are expanded as,

$$\Gamma(n+1+\varepsilon) = n!\,\Gamma(1+\varepsilon)\left[1+\varepsilon S_1(n) + \frac{\varepsilon^2}{2}\left(S_1^2(n) - S_2(n)\right) + \mathcal{O}(\varepsilon^3)\right], \tag{16}$$

where the harmonic sums S_k are defined as

$$S_k(n) = \sum_{i=1}^{n} \frac{1}{i^k}. \tag{17}$$

Expanding in ε we obtain the following off-shell expression for I_{10}, written in a form that allows a straightforward reduction of the remaining infinite sum as soon as we go on the mass shell

$$\begin{aligned} I_{10} = &-\frac{C(\varepsilon)}{M^{-2+6\varepsilon}}\left\{\frac{1}{3\varepsilon^3} + \frac{2}{\varepsilon^2} + \frac{1}{\varepsilon}\left(\frac{25}{3}+2\zeta_2\right) + \left(30+8\zeta_2-\frac{22}{3}\zeta_3\right) + \varepsilon\left(\frac{301}{3}+22\zeta_2-24\zeta_3+33\zeta_4\right)\right. \\ &+\varepsilon^2\left(322-4\zeta_2\zeta_3+52\zeta_2-\frac{130}{3}\zeta_3+108\zeta_4-102\zeta_5\right) + \frac{m^2}{M^2}\left[-\frac{1}{3\varepsilon^2}-\frac{5}{6\varepsilon}+\left(-\frac{19}{12}-\zeta_2\right)\right. \\ &\left.\left.+\varepsilon\left(-\frac{23}{8}-2\zeta_2+\frac{7}{3}\zeta_3\right)+\varepsilon^2\left(-\frac{85}{16}-\frac{7}{4}\zeta_2+\frac{10}{3}\zeta_3-\frac{21}{2}\zeta_4\right)\right]\right\} \\ &-\frac{C(\varepsilon)}{M^{-2+6\varepsilon}}\sum_{k=1}^{\infty}\left(\frac{m^2}{M^2}\right)^{k+1}\left\{\frac{1}{\varepsilon}\left[-\frac{1}{k}+\frac{1}{k+2}+\frac{2}{(k+1)^2}\right]+\varepsilon^0\left[-\frac{10}{k}-\frac{1}{1+k}+\frac{11}{2+k}-5\frac{S_1(k)}{k}\right.\right. \\ &-4\frac{S_1(1+k)}{1+k}+9\frac{S_1(2+k)}{2+k}+\frac{3}{k^2}+\frac{12}{(1+k)^2}-\frac{10}{(1+k)^3}+12\frac{S_1(1+k)}{(1+k)^2}-\frac{8}{(2+k)^2} \\ &\left.-\frac{2}{(2+k)^3}+2\frac{S_1(2+k)}{(2+k)^2}\right]+\varepsilon\left[\frac{-64-3\zeta_2}{k}+\frac{-13-4\zeta_2}{k+1}+\frac{77+7\zeta_2}{k+2}-13\frac{S_1^2(k)}{k}\right. \\ &+33\frac{S_1^2(k+2)}{k+2}-20\frac{S_1^2(k+1)}{k+1}-50\frac{S_1(k)}{k}-38\frac{S_1(k+1)}{k+1}+88\frac{S_1(k+2)}{k+2}+5\frac{S_2(k)}{k} \end{aligned}$$

$$
\begin{aligned}
&+12\frac{S_2(k+1)}{k+1}-17\frac{S_2(k+2)}{k+2}+\frac{50+8\zeta_2}{(k+1)^2}+\frac{-77+2\zeta_2}{(k+2)^2}+76\frac{S_1(k+1)}{(k+1)^2}+\frac{38}{(k+1)^4}\\
&+15\frac{S_1(k)}{k^2}-\frac{9}{k^3}+\frac{30}{(k+2)^3}+\frac{12}{(k+2)^4}+10\frac{S_1^2(k+2)}{(k+2)^2}+36\frac{S_1^2(k+1)}{(k+1)^2}-\frac{60}{(k+1)^3}\\
&+\frac{30}{k^2}-56\frac{S_1(k+1)}{(k+1)^3}-16\frac{S_2(k+1)}{(k+1)^2}-45\frac{S_1(k+2)}{(k+2)^2}-16\frac{S_1(k+2)}{(k+2)^3}-6\frac{S_2(k+2)}{(k+2)^2}\Bigg]\\
&+\varepsilon^2\Bigg[\frac{7\zeta_3-30\zeta_2-336}{k}+\frac{20\zeta_3-36\zeta_2-103}{k+1}+\frac{-27\zeta_3+66\zeta_2+439}{k+2}+\frac{(-320-16\zeta_2)S_1(k)}{k}\\
&+\frac{(-236-32\zeta_2)S_1(k+1)}{k+1}+\frac{(556+48\zeta_2)S_1(k+2)}{k+2}-130\frac{S_1^2(k)}{k}-178\frac{S_1^2(k+1)}{k+1}\\
&+308\frac{S_1^2(k+2)}{k+2}-\frac{70}{3}\frac{S_1^3(k)}{k}-\frac{152}{3}\frac{S_1^3(k+1)}{k+1}+74\frac{S_1^3(k+2)}{k+2}+50\frac{S_2(k)}{k}+104\frac{S_2(k+1)}{k+1}\\
&-154\frac{S_2(k+2)}{k+2}-\frac{26}{3}\frac{S_3(k)}{k}-\frac{64}{3}\frac{S_3(k+1)}{k+1}+30\frac{S_3(k+2)}{k+2}-4\frac{S_{2,1}(k)}{k}-16\frac{S_{2,1}(k+1)}{k+1}\\
&+20\frac{S_{2,1}(k+2)}{k+2}+28\frac{S_1(k)S_2(k)}{k}+80\frac{S_1(k+1)S_2(k+1)}{k+1}-108\frac{S_1(k+2)S_2(k+2)}{k+2}\\
&+\frac{9\zeta_2+192}{k^2}+\frac{-24\zeta_3+52\zeta_2+186}{(k+1)^2}+\frac{-36\zeta_2-244}{(k+1)^3}+\frac{-10\zeta_3-29\zeta_2-479}{(k+2)^2}+\frac{-12\zeta_2+337}{(k+2)^3}\\
&+39\frac{S_1^2(k)}{k^2}+236\frac{S_1^2(k+1)}{(k+1)^2}-129\frac{S_1^2(k+2)}{(k+2)^2}-56\frac{S_1^2(k+2)}{(k+2)^3}-160\frac{S_1^2(k+1)}{(k+1)^3}+150\frac{S_1(k)}{k^2}\\
&-45\frac{S_1(k)}{k^3}-15\frac{S_2(k)}{k^2}-340\frac{S_1(k+1)}{(k+1)^3}+204\frac{S_1(k+1)}{(k+1)^4}-108\frac{S_2(k+1)}{(k+1)^2}+68\frac{S_2(k+1)}{(k+1)^3}\\
&+28\frac{S_3(k+1)}{(k+1)^2}+16\frac{S_{2,1}(k+1)}{(k+1)^2}+157\frac{S_1(k+2)}{(k+2)^3}+74\frac{S_1(k+2)}{(k+2)^4}+\frac{27}{k^4}-\frac{90}{k^3}+\frac{228}{(k+1)^4}\\
&-\frac{130}{(k+1)^5}+\frac{(330+48\zeta_2)S_1(k+1)}{(k+1)^2}+\frac{(-480+16\zeta_2)S_1(k+2)}{(k+2)^2}-40\frac{S_1(k+2)S_2(k+2)}{(k+2)^2}\\
&-96\frac{S_1(k+1)S_2(k+1)}{(k+1)^2}-\frac{98}{(k+2)^4}-\frac{50}{(k+2)^5}+55\frac{S_2(k+2)}{(k+2)^2}+28\frac{S_2(k+2)}{(k+2)^3}\\
&+\frac{32}{3}\frac{S_3(k+2)}{(k+2)^2}+8\frac{S_{2,1}(k+2)}{(k+2)^2}+72\frac{S_1^3(k+1)}{(k+1)^2}+\frac{76}{3}\frac{S_1^3(k+2)}{(k+2)^2}\Bigg]\Bigg\}+O(\varepsilon^3).
\end{aligned}
\tag{18}
$$

The nested harmonic sums $S_{n,\ldots,m}(i)$ appearing in this expression are defined recursively via

$$
S_{k,n,\ldots,m}(i)=\sum_{j=1}^{i}\frac{S_{n,\ldots,m}(j)}{j^k}. \tag{19}
$$

Finally, in the on-shell limit $m^2=M^2=1$, the infinite sum over k in Eq.(18) is performed to yield the following result

$$
\begin{aligned}
I_{10}=&\,C(\varepsilon)\Bigg(-\frac{1}{3\varepsilon^3}-\frac{5}{3\varepsilon^2}-\frac{1}{\varepsilon}\left(4+\frac{2}{3}\pi^2\right)+\frac{10}{3}-\frac{26}{3}\zeta_3-\frac{7}{3}\pi^2+\varepsilon\left(\frac{302}{3}-\frac{94}{3}\zeta_3-\pi^2-\frac{35}{18}\pi^4\right)\\
&-\varepsilon^2\left(-734+\frac{76}{3}\pi^2\zeta_3-\frac{101}{3}\pi^2+20\zeta_3+\frac{551}{90}\pi^4+462\zeta_5\right)+\mathcal{O}(\varepsilon^3)\Bigg).
\end{aligned}
\tag{20}
$$

The infinite sums that one needs to go from Eq.(18) to Eq.(20) are given in Appendix B.

B. Master integral I_5

Let us now consider a calculation of the constant C_1 (see Ref. [10]). The easiest way to extract it is to consider the integral I_5 of Ref. [10] (see Fig.2). We introduce $\Pi_1=k_1^2, \Pi_3=(k_1-k_2)^2, \Pi_4=k_1^2+2pk_1, \Pi_5=k_2^2+2pk_2, \Pi_6=k_3^2+M^2, \Pi_7=(k_2+k_3)^2+M^2$ with $p^2=-m^2$ and consider

$$
I_5(m,M) = \int \frac{\mathrm{d}^D k_1 \mathrm{d}^D k_2 \mathrm{d}^D k_3}{\Pi_1\Pi_3\Pi_4\Pi_5\Pi_6\Pi_7} \tag{21}
$$

for $m = M$. The result of Ref. [10] reads:

$$
\begin{aligned}
I_5(m,m) =& \; C(\varepsilon)\left\{\frac{1}{6\varepsilon^3}+\frac{3}{2\varepsilon^2}+\frac{1}{\varepsilon}\left(-\frac{\pi^2}{3}+\frac{55}{6}\right)-\frac{4}{45}\pi^4-\frac{14}{3}\zeta(3)-\frac{7}{3}\pi^2+\frac{95}{2}\right.\\
&\left.+\varepsilon\left(-\frac{2}{9}\pi^4-44\zeta_3-\frac{29}{3}\pi^2+\frac{1351}{6}+2C_1\right)\right\},
\end{aligned} \tag{22}
$$

consequently, in order to extract C_1 we have to compute I_5 to order $\mathcal{O}(\varepsilon)$.

For this purpose, it turns out to be useful to derive a suitable integral representation for the vacuum polarization subdiagram in I_5. This representation can be obtained by subsequent application of dispersion relation to "vacuum polarization" subgraph and the Mellin-Barnes transformation. Let us first write a dispersion representation for the vacuum polarization subgraph:

$$
\int \frac{\mathrm{d}^D k_3}{\Pi_6\Pi_7} = \frac{1}{\pi}\int\limits_{4M^2}^{\infty}\frac{\mathrm{d}\lambda^2}{\lambda^2+k_2^2}\frac{\pi^{7/2-\varepsilon}}{2^{1-2\varepsilon}\Gamma(3/2-\varepsilon)}\lambda^{-2\varepsilon}\left(1-\frac{4M^2}{\lambda^2}\right)^{1/2-\varepsilon}. \tag{23}
$$

Note that the above expression is written for an Euclidean momenta k_2^2. We then use the Mellin-Barnes representation for the propagator $1/(k_2^2+\lambda^2)$:

$$
\frac{1}{k_2^2+\lambda^2} = \frac{1}{2\pi i}\int\limits_C \mathrm{d}\sigma\Gamma(-\sigma)\Gamma(1+\sigma)(k_2^2)^\sigma(\lambda^2)^{-1-\sigma}, \tag{24}
$$

where the integration is performed over the contour C in the complex σ plane which goes from complex infinity to complex infinity and passes the real axes between the poles of $\Gamma(-\sigma)$ and $\Gamma(1+\sigma)$. It turns out to be more convenient to shift the integration contour to the right, crossing the singularity at $\sigma = 0$, so that the new integration contour $\tilde{C}$ crosses the real axes at e.g. $\sigma = 1/2$. We obtain:

$$
\frac{1}{k_2^2+\lambda^2} = \frac{1}{\lambda^2}+\frac{1}{2\pi i}\int\limits_{\tilde{C}} \mathrm{d}\sigma\Gamma(-\sigma)\Gamma(1+\sigma)(k_2^2)^\sigma(\lambda^2)^{-1-\sigma}. \tag{25}
$$

Let us study two terms in that equation separately. The first term produces a factorized expression where the integral over λ can be carried out independently of the integration over k_1 and k_2. The remaining two loop integral is an on-shell integral and for this reason can be easily done. The result of this calculation reads (we take the limit $m = M$):

$$
\begin{aligned}
I_{5a}(m,m) =& \; C(\varepsilon)\left\{\frac{1}{6\varepsilon^3}+\frac{3}{2\varepsilon^2}+\frac{1}{\varepsilon}\left(-\frac{\pi^2}{3}+\frac{55}{6}\right)-\frac{8}{3}\zeta_3-\pi^2+\frac{43}{2}\right.\\
&\left.+\varepsilon\left(-\frac{19}{45}\pi^4-16\zeta_3-\frac{5}{3}\pi^2+\frac{775}{6}\right)\right\}.
\end{aligned} \tag{26}
$$

In order to compute the contribution of the second term in Eq.(25) we insert it into dispersion integral Eq.(23) and integrate over λ. We obtain:

$$
\begin{aligned}
I_{5b}(m,M) =& \; \frac{1}{2\pi i}\frac{\pi^{5/2-\varepsilon}}{2^{1-2\varepsilon}\Gamma(3/2-\varepsilon)}\int\limits_{\tilde{C}}\mathrm{d}\sigma\Gamma(-\sigma)\Gamma(1+\sigma)\mathrm{B}\left(\sigma+\varepsilon,\frac{3}{2}-\varepsilon\right)(4M^2)^{-\varepsilon-\sigma}\\
&\times\int\frac{\mathrm{d}^D k_1\mathrm{d}^D k_2}{\Pi_1\Pi_3\Pi_4\Pi_5}(\Pi_2)^\sigma,
\end{aligned} \tag{27}
$$

where $\Pi_2 = k_2^2$.

To proceed further, we need to perform the integrals over k_1 and k_2. The easiest way to do that is to utilize the integration-by-parts identities. It turns out that there is an identity that allows to remove either Π_3 or Π_4 or Π_5 in which case the integrals become computable in a closed form for arbitrary σ. To write down this identity, we introduce

$$I(a_1,a_2,a_3,a_4,a_5)=\int\frac{\mathrm{d}^Dk_1\mathrm{d}^Dk_2}{\Pi_1^{a_1}\Pi_2^{a_2}\Pi_3^{a_3}\Pi_4^{a_4}\Pi_5^{a_5}}. \tag{28}$$

The general form of the identity is [19]:

$$\begin{aligned}I(\{a_i\})=&\frac{-1}{2(2D-2\sum_{i=1}^{5}a_i+a_4+a_5)}\Big\{\frac{2(D-1-a_3-a_1)(D-1-a_3-a_2)}{(D-1-2a_3)}\mathbf{3}^-\\&-\frac{(D-1-a_3-a_2)}{(D-1-2a_3)}\mathbf{4}^+\mathbf{5}^-\mathbf{1}^- -\frac{(D-1-a_3-a_1)}{(D-1-2a_3)}\mathbf{5}^+\mathbf{4}^-\mathbf{2}^-\\&+a_3\mathbf{3}^+(\mathbf{4}^- -\mathbf{5}^-)(\mathbf{2}^- -\mathbf{1}^-)-(a_2-1)\mathbf{5}^- -(a_1-1)\mathbf{4}^-\Big\}I(\{a_i\}),\end{aligned} \tag{29}$$

where as usual $\mathbf{1}^{\pm}\ I(a_1,...)=I(a_1\pm 1,...)$ and similar for other operators.

It is now easy to see that if we apply that recurrence relation to the integral $I(1,-\sigma,1,1,1)$, we immediately obtain a set of integrals that can be easily computed since the recurrence relation removes either one of the massive lines (in which case there appears a massless two point subgraph) or the line Π_3 (in which case the integral becomes a product of two one loop integrals). Therefore, we end up with the expression for $I(1,-\sigma,1,1,1)$ written in terms of Γ-functions:

$$\begin{aligned}I(1,-\sigma,1,1,1)=&\frac{1}{2+2\sigma-4\varepsilon}\Big[-\frac{1+\sigma}{2}S_2\left(1,-\sigma,1,1\right)-\frac{1}{2}S_2(1,2,-1-\sigma,1)\\&+\frac{1}{2}S_2(2,-1-\sigma,1,1)-\frac{1}{2}S_2(2,-\sigma,0,1)+\frac{1}{1-2\varepsilon}\Big(\left(1-\varepsilon+\frac{1}{2}\sigma\right)S_2(1,-\sigma,0,2)\\&+\left(\frac{1}{2}-\varepsilon\right)S_2(1,1,-1-\sigma,2)+\left(-2+2\varepsilon\sigma+6\varepsilon-4\varepsilon^2-\sigma\right)S_1(1,1)\ S_1(-\sigma,1)\Big)\Big],\end{aligned} \tag{30}$$

where the functions S_2 and S_1 are defined as follows:

$$\begin{aligned}S_2(a_1,a_2,a_3,a_4)&=O_1(a_1,a_2)\ S_1(a_1+a_2+a_3-2+\varepsilon,a_4),\\O_1(a_1,a_2)&=\frac{\Gamma(2-\varepsilon-a_1)\Gamma(2-\varepsilon-a_2)\Gamma(a_1+a_2-2+\varepsilon)}{\Gamma(a_1)\Gamma(a_2)\Gamma(4-2\varepsilon-a_1-a_2)},\\S_1(a_1,a_2)&=\frac{\Gamma(a_1+a_2-2+\varepsilon)\Gamma(4-2\varepsilon-2a_1-a_2)}{\Gamma(a_2)\Gamma(4-2\varepsilon-a_1-a_2)}.\end{aligned} \tag{31}$$

We use these expressions in Eq.(27) and integrate over σ by taking the residues located to the right of the integration contour, i.e. at the points $\sigma=n,\ n+\varepsilon,\ n+2\varepsilon$, where $n\geq 1$ is an integer[4]. Finally we arrive at the representation for I_5 written as an infinite series:

$$I_{5b}(m,M)=\sum_{n}^{\infty}d_n\left(\frac{m^2}{M^2}\right)^n. \tag{32}$$

The expression for the coefficient d_n is too long to be presented here; the essential point, however, is that d_n can be expressed through harmonic sums and the summation for $m=M$ can be carried out in a way similar to the one described in the previous subsection. Our final result for $I_{5b}(m,m)$ reads:

$$\begin{aligned}I_{5b}(m,m)=&-\frac{4}{45}\pi^4-2\zeta(3)-\frac{4}{3}\pi^2+26\\&+\varepsilon\left(96+\frac{25}{6}\pi^2\zeta_3+4\pi^2\log 2-\frac{26}{3}\pi^2-26\zeta_3-\frac{31}{90}\pi^4-\frac{49}{2}\zeta_5\right).\end{aligned} \tag{33}$$

Combining the results for I_{5a} and I_{5b} and comparing with Eq.(22), we obtain the constant C_1:

$$C_1=\zeta_3-\frac{1}{3}\pi^2-\frac{49}{180}\pi^4+\frac{25}{12}\pi^2\zeta_3+2\pi^2\log 2-\frac{49}{4}\zeta_5. \tag{34}$$

[4]It is interesting to note that different poles in σ correspond to contributions of different expansion regimes if I_5 is expanded in m/M according to the rules of the large mass expansion (for a review see e.g. [20]).

C. Master integral I_{18}

For integral I_{18} we proceed as for I_{10} and perform an expansion in p^2/M^2 of a more general Euclidean integral

$$I_{18} = \int\int\int \frac{\mathrm{d}^D k_1\, \mathrm{d}^D k_2\, \mathrm{d}^D k_3}{(k_1+p)^2\,(k_3+p)^2\,(k_1-k_2)^2\,(k_3-k_2)^2\,(k_1^2+M^2)\,(k_2^2+M^2)\,(k_3^2+M^2)}. \quad (35)$$

As is the case for I_{10}, the expansion of I_{18} can be performed[5] by making an ordinary Taylor expansion in the external momentum p in the integrand of Eq. (35), i.e. by expanding both $1/(k_1+p)^2$ and $1/(k_3+p)^2$ in small p as in Eq.(12).

The resulting integral is formally a 3-loop vacuum bubble integral that, after the necessary tensor reductions and partial fractioning of massive and massless lines, can be evaluated in terms of Γ-functions using Eqs.(67-71) in the Appendix A. In this way one obtains an explicit representation for I_{18} as an infinite series in p^2/M^2 with coefficients being a nested sum over a product of Γ-functions. After expanding the expressions in ε as in Eq.(16) and writing the nested sums in a form that allows a straightforward reduction on the mass shell we obtain a fairly simple expression

$$\begin{aligned} I_{18} = & \frac{C(\varepsilon)}{M^{2+6\varepsilon}} \sum_{k=1}^{\infty} \left(\frac{m^2}{M^2}\right)^k \left[2\frac{\zeta_3}{k^2} - 2\frac{\zeta_2}{k^3} + 4\zeta_2 \frac{S_1(k)}{k^2} + 2\frac{S_1(k)}{k^4} - 2\frac{S_1^2(k)}{k^3} \right. \\ & \left. + 4\frac{S_1(k)S_2(k)}{k^2} - 4\frac{S_{2,1}(k)}{k^2} + O(\varepsilon) \right]. \end{aligned} \quad (36)$$

The on-shell result is now easily recovered by putting $m^2 = M^2 = 1$ and using the expressions for the infinite sums Eqs. (72-83). We finally obtain

$$I_{18} = C(\varepsilon)\left[2\pi^2\zeta_3 - 5\zeta_5 + O(\varepsilon)\right]. \quad (37)$$

Having derived the last unknown contribution for the three loop on-shell two-point master integrals, we are able to write down a complete set of master integrals for the three loop on-shell two point functions without unknown constants. The list updates the result of Ref. [10] and can be found in Appendix C.

IV. THREE LOOP QCD RENORMALIZATION CONSTANTS IN THE ON-SHELL SCHEME

Performing an explicit computation along the lines outlined in the previous Sections, we are able to derive an expression for the QCD on-shell renormalization constants up to the three loop order. We work here in a general covariant gauge with a gauge fixing parameter ξ; the gluon propagator (omitting the color indices) therefore reads:

$$D_{\mu\nu}(p) = -\frac{i}{p^2}\left(g_{\mu\nu} - \xi\frac{p_\mu p_\nu}{p^2}\right). \quad (38)$$

In our previous article [3] we have given a relation between the pole and the $\overline{\mathrm{MS}}$ mass, from where the value of Z_m can be determined. Here we give the result for that renormalization constant explicitly. We write:

$$Z_m = 1 + a_0 C_F\left(-\frac{3}{4\varepsilon} - \frac{1}{1-2\varepsilon}\right) + a_0^2 C_F Z_m^{(2)} + a_0^3 C_F Z_m^{(3)}, \quad (39)$$

where

$$a_0 = \frac{\alpha_s^{(0)}}{\pi}\frac{\Gamma(1+\varepsilon)m^{-2\varepsilon}}{(4\pi)^{-\varepsilon}},$$

---

[5]We should note that a large mass expansion of integrals I_{18} and I_{10} is rather special since no contributions arise other than the Taylor expansion of the whole integrand in p. However, for other types of integrals without a through-going massive line the combinatorics of the large mass expansion gives rise to several more subgraphs that must be Taylor expanded.

and $\alpha_s^{(0)}$ is the bare coupling constant and $Z_m^{(2)}$ and $Z_m^{(3)}$ are the two and the three loop coefficients respectively:

$$
\begin{aligned}
Z_m^{(2)} =& \; C_F\, d_1^{(2)} + C_A\, d_2^{(2)} + T_R N_L\, d_3^{(2)} + T_R N_H\, d_4^{(2)} , \\
Z_m^{(3)} =& \; C_F^2\, d_1^{(3)} + C_F C_A\, d_2^{(3)} + C_A^2\, d_3^{(3)} + C_F T_R N_L\, d_4^{(3)} + C_F T_R N_H\, d_5^{(3)} + C_A T_R N_L\, d_6^{(3)} \\
& + C_A T_R N_H\, d_7^{(3)} + T_R^2 N_L N_H\, d_8^{(3)} + T_R^2 N_H^2\, d_9^{(3)} + T_R^2 N_L^2\, d_{10}^{(3)} .
\end{aligned} \tag{40}
$$

Here C_F and C_A are the Casimir operators of the fundamental and the adjoint representation of the color gauge group (the group is SU(3) for QCD) and T_R is the trace normalization of the fundamental representation. N_L is the number of massless quark flavors and N_H is the number of quark flavors with a pole mass equal to m.

Our results for the coefficients $d_j^{(i)}$ are:

$$
\begin{aligned}
d_1^{(2)} =& \frac{9}{32\varepsilon^2} + \frac{45}{64\varepsilon} + \frac{199}{128} - \frac{3}{4}\,\zeta_3 + \frac{1}{2}\,\pi^2\log 2 - \frac{5}{16}\pi^2 \\
& + \varepsilon\left(\frac{677}{256} - \frac{33}{4}\zeta_3 + 3\pi^2\log 2 - \pi^2\log^2 2 - \frac{55}{32}\pi^2 + \frac{7}{40}\pi^4 - \frac{1}{2}\log^4 2 - 12a_4\right), \\
d_2^{(2)} =& -\frac{11}{32\varepsilon^2} - \frac{91}{64\varepsilon} - \frac{605}{128} + \frac{3}{8}\,\zeta_3 - \frac{1}{4}\,\pi^2\log 2 + \frac{1}{12}\,\pi^2 \\
& + \varepsilon\left(-\frac{3799}{256} + \frac{13}{4}\zeta_3 - \frac{3}{2}\pi^2\log 2 + \frac{1}{2}\pi^2\log^2 2 + \frac{19}{48}\pi^2 - \frac{7}{80}\pi^4 + \frac{1}{4}\log^4 2 + 6a_4\right), \\
d_3^{(2)} =& \frac{1}{8\varepsilon^2} + \frac{7}{16\varepsilon} + \frac{45}{32} + \frac{1}{12}\,\pi^2 + \varepsilon\left(\frac{279}{64} + \zeta_3 + \frac{7}{24}\pi^2\right), \\
d_4^{(2)} =& \frac{1}{8\varepsilon^2} + \frac{7}{16\varepsilon} + \frac{69}{32} - \frac{1}{6}\,\pi^2 + \varepsilon\left(\frac{463}{64} - \frac{7}{2}\zeta_3 + \pi^2\log 2 - \frac{5}{6}\pi^2\right), \\
d_1^{(3)} =& -\frac{9}{128\varepsilon^3} - \frac{63}{256\varepsilon^2} + \frac{1}{\varepsilon}\left(-\frac{457}{512} + \frac{9}{16}\zeta_3 - \frac{3}{8}\,\pi^2\log 2 + \frac{15}{64}\,\pi^2\right) - \frac{14225}{3072} \\
& - \frac{1}{16}\zeta_3\pi^2 + \frac{9}{8}\zeta_3 + \frac{5}{8}\zeta_5 + 5\pi^2\log 2 + \frac{5}{4}\pi^2\log^2 2 - \frac{731}{384}\pi^2 - \frac{73}{480}\pi^4 - \frac{1}{8}\log^4 2 - 3a_4, \\
d_2^{(3)} =& \frac{33}{128\varepsilon^3} + \frac{1039}{768\varepsilon^2} + \frac{1}{\varepsilon}\left(\frac{2563}{512} - \frac{53}{32}\zeta_3 + \frac{53}{48}\pi^2\log 2 - \frac{61}{96}\pi^2\right) \\
& + \frac{144929}{9216} - \frac{19}{16}\zeta_3\pi^2 - \frac{2459}{96}\zeta_3 + \frac{45}{16}\zeta_5 + \frac{223}{36}\pi^2\log 2 - \frac{221}{72}\pi^2\log^2 2 \\
& - \frac{1477}{576}\pi^2 + \frac{4639}{8640}\pi^4 - \frac{167}{144}\log^4 2 - \frac{167}{6}a_4, \\
d_3^{(3)} =& -\frac{121}{576\varepsilon^3} - \frac{167}{108\varepsilon^2} + \frac{1}{\varepsilon}\left(-\frac{81797}{10368} + \frac{11}{16}\zeta_3 - \frac{11}{24}\,\pi^2\log 2 + \frac{11}{72}\,\pi^2\right) \\
& - \frac{2188703}{62208} + \frac{51}{64}\zeta_3\pi^2 + \frac{3059}{288}\zeta_3 - \frac{65}{32}\zeta_5 - \frac{313}{72}\pi^2\log 2 \\
& + \frac{11}{9}\pi^2\log^2 2 + \frac{553}{3456}\pi^2 - \frac{3667}{17280}\pi^4 + \frac{11}{18}\log^4 2 + \frac{44}{3}a_4, \\
d_4^{(3)} =& -\frac{3}{32\varepsilon^3} - \frac{77}{192\varepsilon^2} + \frac{1}{\varepsilon}\left(-\frac{415}{384} + \frac{1}{4}\zeta_3 - \frac{1}{3}\pi^2\log 2 + \frac{7}{48}\,\pi^2\right) - \frac{2911}{2304} + \frac{169}{24}\zeta_3 \\
& - \frac{29}{9}\pi^2\log 2 + \frac{8}{9}\pi^2\log^2 2 + \frac{475}{288}\pi^2 - \frac{371}{2160}\pi^4 + \frac{4}{9}\log^4 2 + \frac{32}{3}a_4, \\
d_5^{(3)} =& -\frac{3}{32\varepsilon^3} - \frac{77}{192\varepsilon^2} + \frac{1}{\varepsilon}\left(-\frac{631}{384} + \frac{1}{4}\zeta_3 - \frac{1}{3}\,\pi^2\log 2 + \frac{1}{3}\,\pi^2\right) - \frac{8743}{2304} + \frac{71}{12}\zeta_3 \\
& - \frac{67}{36}\pi^2\log 2 + \frac{5}{9}\pi^2\log^2 2 + \frac{425}{432}\pi^2 - \frac{161}{2160}\pi^4 + \frac{4}{9}\log^4 2 + \frac{32}{3}a_4, \\
d_6^{(3)} =& \frac{11}{72\varepsilon^3} + \frac{877}{864\varepsilon^2} + \frac{1}{\varepsilon}\left(\frac{3125}{648} + \frac{1}{6}\pi^2\log 2 + \frac{7}{72}\pi^2\right) + \frac{317381}{15552} + \frac{41}{144}\zeta_3 \\
& + \frac{29}{18}\pi^2\log 2 - \frac{4}{9}\pi^2\log^2 2 + \frac{73}{108}\pi^2 + \frac{29}{432}\pi^4 - \frac{2}{9}\log^4 2 - \frac{16}{3}a_4,
\end{aligned}
$$

$$
\begin{aligned}
d_7^{(3)} &= \frac{11}{72\varepsilon^3} + \frac{877}{864\varepsilon^2} + \frac{1}{\varepsilon}\left(\frac{502}{81} + \frac{1}{6}\pi^2\log 2 - \frac{13}{36}\pi^2\right) + \frac{473549}{15552} + \frac{1}{8}\zeta_3\pi^2 - \frac{1345}{144}\zeta_3 \\
&\quad -\frac{5}{8}\zeta_5 + \frac{115}{18}\pi^2\log 2 - \frac{5}{18}\pi^2\log^2 2 - \frac{707}{144}\pi^2 + \frac{1}{54}\pi^4 - \frac{2}{9}\log^4 2 - \frac{16}{3}a_4, \\
d_8^{(3)} &= -\frac{1}{18\varepsilon^3} - \frac{17}{54\varepsilon^2} + \frac{1}{\varepsilon}\left(-\frac{152}{81} + \frac{1}{18}\pi^2\right) - \frac{2032}{243} + \frac{17}{9}\zeta_3 - \frac{2}{3}\pi^2\log 2 + \frac{13}{27}\pi^2, \\
d_9^{(3)} &= -\frac{1}{36\varepsilon^3} - \frac{17}{108\varepsilon^2} + \frac{1}{\varepsilon}\left(-\frac{385}{324} + \frac{1}{9}\pi^2\right) - \frac{5441}{972} + \frac{53}{18}\zeta_3 - \frac{2}{3}\pi^2\log 2 + \frac{79}{135}\pi^2, \\
d_{10}^{(3)} &= -\frac{1}{36\varepsilon^3} - \frac{17}{108\varepsilon^2} + \frac{1}{\varepsilon}\left(-\frac{223}{324} - \frac{1}{18}\pi^2\right) - \frac{2687}{972} - \frac{19}{18}\zeta_3 - \frac{17}{54}\pi^2.
\end{aligned} \tag{41}
$$

We have used here $\zeta_k = \sum_{n=1}^{\infty} 1/n^k$ for the Riemann ζ-function and also $a_4 = \sum_{n=1}^{\infty} 1/(2^n n^4)$.

It is more cumbersome to present the result for the wave function renormalization Z_2 since, as we mentioned already, it becomes gauge dependent at $\mathcal{O}(\alpha_s^3)$. We parameterize Z_2 as:

$$
Z_2 = 1 + a_0 C_F\left(-\frac{3}{4\varepsilon} - \frac{1}{1-2\varepsilon}\right) + a_0^2 C_F Z_2^{(2)} + a_0^3 C_F Z_2^{(3)}, \tag{42}
$$

where

$$
\begin{aligned}
Z_2^{(2)} &= C_F\, f_1^{(2)} + C_A\, f_2^{(2)} + T_R N_L\, f_3^{(2)} + T_R N_H\, f_4^{(2)}, \\
Z_2^{(3)} &= C_F^2\, f_1^{(3)} + C_F C_A\, f_2^{(3)} + C_A^2\, f_3^{(3)} + C_F T_R N_L\, f_4^{(3)} + C_F T_R N_H\, f_5^{(3)} + C_A T_R N_L\, f_6^{(3)} \\
&\quad + C_A T_R N_H\, f_7^{(3)} + T_R^2 N_L N_H\, f_8^{(3)} + T_R^2 N_H^2\, f_9^{(3)} + T_R^2 N_L^2\, f_{10}^{(3)}.
\end{aligned} \tag{43}
$$

Our result for the coefficients $f_j^{(i)}$ reads:

$$
\begin{aligned}
f_1^{(2)} &= \frac{9}{32\varepsilon^2} + \frac{51}{64\varepsilon} + \frac{433}{128} - \frac{3}{2}\zeta_3 + \pi^2\log 2 - \frac{13}{16}\pi^2 \\
&\quad +\varepsilon\left(\frac{211}{256} - \frac{147}{8}\zeta_3 + \frac{23}{4}\pi^2\log 2 - 2\pi^2\log^2 2 - \frac{89}{32}\pi^2 + \frac{7}{20}\pi^4 - \log^4 2 - 24a_4\right), \\
f_2^{(2)} &= -\frac{11}{32\varepsilon^2} - \frac{101}{64\varepsilon} - \frac{803}{128} + \frac{3}{4}\zeta_3 - \frac{1}{2}\pi^2\log 2 + \frac{5}{16}\pi^2 \\
&\quad +\varepsilon\left(-\frac{4241}{256} + \frac{129}{16}\zeta_3 - \frac{23}{8}\pi^2\log 2 + \pi^2\log^2 2 + \frac{41}{48}\pi^2 - \frac{7}{40}\pi^4 + \frac{1}{2}\log^4 2 + 12a_4\right), \\
f_3^{(2)} &= \frac{1}{8\varepsilon^2} + \frac{9}{16\varepsilon} + \frac{59}{32} + \frac{1}{12}\pi^2 + \varepsilon\left(\frac{369}{64} + \zeta_3 + \frac{3}{8}\pi^2\right), \\
f_4^{(2)} &= \frac{1}{4\varepsilon^2} + \frac{19}{48\varepsilon} + \frac{1139}{288} - \frac{1}{3}\pi^2 + \varepsilon\left(\frac{20275}{1728} - 7\zeta_3 + 2\pi^2\log 2 - \frac{19}{12}\pi^2\right), \\
f_1^{(3)} &= -\frac{9}{128\varepsilon^3} - \frac{81}{256\varepsilon^2} + \frac{1}{\varepsilon}\left(-\frac{1039}{512} + \frac{9}{8}\zeta_3 - \frac{3}{4}\pi^2\log 2 + \frac{39}{64}\pi^2\right) - \frac{10823}{3072} + \frac{1}{8}\zeta_3\pi^2 \\
&\quad -\frac{187}{32}\zeta_3 - \frac{5}{16}\zeta_5 + \frac{685}{48}\pi^2\log 2 + 3\pi^2\log^2 2 - \frac{7199}{1152}\pi^2 - \frac{41}{120}\pi^4 - \frac{5}{12}\log^4 2 - 10a_4, \\
f_2^{(3)} &= \frac{33}{128\varepsilon^3} + \frac{1217}{768\varepsilon^2} + \frac{1}{\varepsilon}\left(\frac{14887}{1536} - \frac{27}{8}\zeta_3 + \frac{53}{24}\pi^2\log 2 - \frac{331}{192}\pi^2\right) \\
&\quad +\frac{150871}{9216} - \frac{45}{16}\zeta_3\pi^2 - \frac{9941}{192}\zeta_3 + \frac{145}{16}\zeta_5 + \frac{2281}{288}\pi^2\log 2 \\
&\quad -\frac{499}{72}\pi^2\log^2 2 - \frac{1169}{576}\pi^2 + \frac{20053}{17280}\pi^4 - \frac{319}{144}\log^4 2 - \frac{319}{6}a_4, \\
f_3^{(3)} &= -\frac{121}{576\varepsilon^3} - \frac{1501}{864\varepsilon^2} + \frac{1}{\varepsilon}\left(-\frac{55945}{5184} + \frac{173}{128}\zeta_3 - \frac{11}{12}\pi^2\log 2 + \frac{55}{96}\pi^2 - \frac{\pi^4}{1080}\right. \\
&\quad \left. +\xi\left(-\frac{3}{256}\zeta_3 + \frac{\pi^4}{4320} - \frac{1}{768}\right)\right) - \frac{2551697}{62208} + \frac{127}{72}\zeta_3\pi^2 + \frac{14371}{576}\zeta_3 - \frac{37}{6}\zeta_5
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{271}{36}\pi^2\log 2+\frac{391}{144}\pi^2\log^2 2+\frac{1073}{3456}\pi^2-\frac{10811}{23040}\pi^4+\frac{349}{288}\log^4 2+\frac{349}{12}a_4\\
&+\xi\left(-\frac{13}{768}+\frac{1}{144}\zeta_3\pi^2-\frac{13}{256}\zeta_3+\frac{7}{384}\zeta_5-\frac{1}{256}\pi^2+\frac{17}{27648}\pi^4\right),\\
f_4^{(3)}=&-\frac{3}{32\varepsilon^3}-\frac{103}{192\varepsilon^2}+\frac{1}{\varepsilon}\left(-\frac{351}{128}+\frac{3}{4}\zeta_3-\frac{2}{3}\pi^2\log 2+\frac{23}{48}\pi^2\right)-\frac{3773}{2304}+\frac{413}{24}\zeta_3\\
&-\frac{58}{9}\pi^2\log 2+\frac{16}{9}\pi^2\log^2 2+\frac{905}{288}\pi^2-\frac{733}{2160}\pi^4+\frac{8}{9}\log^4 2+\frac{64}{3}a_4,\\
f_5^{(3)}=&-\frac{3}{16\varepsilon^3}-\frac{91}{192\varepsilon^2}+\frac{1}{\varepsilon}\left(-\frac{1525}{384}+\zeta_3-\frac{2}{3}\pi^2\log 2+\frac{19}{24}\pi^2\right)-\frac{78967}{6912}+\frac{5273}{288}\zeta_3\\
&-\frac{31}{9}\pi^2\log 2+\frac{5}{6}\pi^2\log^2 2+\frac{865}{648}\pi^2-\frac{137}{720}\pi^4+\frac{7}{6}\log^4 2+28a_4,\\
f_6^{(3)}=&\frac{11}{72\varepsilon^3}+\frac{1069}{864\varepsilon^2}+\frac{1}{\varepsilon}\left(\frac{550}{81}-\frac{1}{4}\zeta_3+\frac{1}{3}\pi^2\log 2-\frac{1}{18}\pi^2\right)+\frac{416405}{15552}-\frac{145}{36}\zeta_3+\frac{29}{9}\pi^2\log 2\\
&-\frac{8}{9}\pi^2\log^2 2+\frac{7}{18}\pi^2+\frac{29}{216}\pi^4-\frac{4}{9}\log^4 2-\frac{32}{3}a_4,\\
f_7^{(3)}=&\frac{1}{\varepsilon^3}\left(\frac{15}{64}-\xi\frac{1}{192}\right)+\frac{1}{\varepsilon^2}\left(\frac{353}{288}+\xi\frac{1}{64}\right)+\frac{1}{\varepsilon}\left(\frac{503}{48}+\frac{1}{3}\pi^2\log 2-\frac{59}{72}\pi^2-\frac{1}{2}\zeta_3-\xi\frac{35}{576}\right)\\
&+\frac{32257}{648}+\frac{11}{48}\zeta_3\pi^2-\frac{13301}{576}\zeta_3-\frac{15}{16}\zeta_5+\frac{521}{36}\pi^2\log 2-\frac{1}{3}\pi^2\log^2 2-\frac{13567}{1296}\pi^2\\
&+\frac{5}{72}\pi^4-\frac{2}{3}\log^4 2-16a_4+\xi\left(\frac{407}{1728}-\frac{1}{24}\zeta_3\right),\\
f_8^{(3)}=&-\frac{1}{12\varepsilon^3}-\frac{7}{18\varepsilon^2}+\frac{1}{\varepsilon}\left(-\frac{31}{9}+\frac{1}{6}\pi^2\right)-\frac{1168}{81}+4\zeta_3-\frac{4}{3}\pi^2\log 2+\frac{7}{6}\pi^2,\\
f_9^{(3)}=&-\frac{1}{12\varepsilon^3}-\frac{5}{36\varepsilon^2}+\frac{1}{\varepsilon}\left(-\frac{131}{54}+\frac{2}{9}\pi^2\right)-\frac{6887}{648}+7\zeta_3-\frac{4}{3}\pi^2\log 2+\frac{11}{10}\pi^2,\\
f_{10}^{(3)}=&-\frac{1}{36\varepsilon^3}-\frac{23}{108\varepsilon^2}+\frac{1}{\varepsilon}\left(-\frac{325}{324}-\frac{1}{18}\pi^2\right)-\frac{4025}{972}-\frac{19}{18}\zeta_3-\frac{23}{54}\pi^2.
\end{aligned}
\tag{44}
$$

Here the gauge parameter ξ is defined in Eq.(38) and we again stress that the three loop contribution to Z_2 is gauge dependent. This fact is not too surprising by itself, since the wave function renormalization constants are known to be gauge dependent in general (as one can see for instance in the $\overline{\rm MS}$ scheme). The on-shell renormalization constant computed in dimensional regularization, however, unexpectedly turned out to be gauge independent in two first orders in perturbation theory. This fact, observed for the first time in [7], resulted in "all orders gauge independence" conjecture by the authors of that reference. Our calculation shows that things "return back to normal" in the third order of perturbative expansion, where gauge parameter dependence explicitly appears. Let us note that the "strongest" gauge dependence appears in $C_A T_R N_H$ color structures. We note in this respect that the N_H-dependent divergences permit a strong check related to decoupling of heavy quark loops from HQET. We discuss this issue in detail in the next Section.

V. HQET WAVE FUNCTION RENORMALIZATION

There is an interesting application of the results derived above which also serves as an additional check on its correctness. Namely, the on-shell wave function renormalization constant and the known $\overline{\rm MS}$ wave function renormalization constant can be combined to derive the heavy quark wave function renormalization in HQET. To two loops that has been done in Ref. [7]. Moreover, the HQET wave function renormalization constant[6] can be shown to

[6]When we refer to HQET wave function renormalization constant we mean the $\overline{\rm MS}$ wave function renormalization constant in HQET.

exponentiate , i.e. it can be written as[7]

$$Z_2^{\rm HQET} = e^{\left(\frac{\alpha_s}{\pi}C_F x_1 + \left(\frac{\alpha_s}{\pi}\right)^2 C_F(C_A x_2 + T_R N_L x_3) + \left(\frac{\alpha_s}{\pi}\right)^3 C_F\left(C_A^2 x_4 + C_A T_R N_L x_5 + N_L^2 T_R^2 x_6 + C_F N_L T_R x_7\right) + \mathcal{O}(\alpha_s^4)\right)}. \tag{45}$$

For this reason, only few genuinely new color structures appear at the three loop level and the C_F^3 and $C_F^2 C_A$ coefficients in $Z_2^{\rm HQET}$ are completely determined by the two loop expression for $Z_2^{\rm HQET}$. This remarkable feature provides an additional strong cross check on our result.

Let us first explain why is it at all possible to determine the HQET wave function renormalization from the $\overline{\rm MS}$ and the on-shell wave function renormalization constants in full QCD. The explanation is apparent if one asks what HQET implies for the computation of Feynman diagrams. In fact, all underlying assumptions of HQET are satisfied if we try to compute the quark self energy in the vicinity of the particle mass shell. If we denote the off-shellness by $-\Lambda^2 = p^2 - m^2$, the HQET Lagrangian gives a prescription of how non-analytic contributions in Λ to the quark self energy should be computed. Such a computation should clearly be supplemented by a calculation of the contribution coming from momenta region $k \gg \Lambda$, which is analytic in Λ. This contribution is provided by the Taylor expansion of the quark self energy in Λ in full QCD. At this point the computation is reduced to the computation of the on-shell integrals in full QCD and this is precisely the piece of work we do when we compute the on-shell wave function renormalization constant in full QCD (i.e. we compute the quark self-energy operator to the first non-trivial order in Λ). The on-shell wave function renormalization constant in full QCD is both ultraviolet and infra-red divergent. The ultraviolet divergences can be removed by means of the $\overline{\rm MS}$ renormalization; the infra-red divergences then cancel out once the HQET contribution to the quark self energy is computed since the off-shell self energy is infra-red finite. We therefore see, that once the infra-red divergences of the on-shell Z_2 are isolated, they should match the ultraviolet divergences in HQET and this is the way how the HQET wave function renormalization constant and the HQET anomalous dimension of the heavy quark field are determined.

We now turn to the computation of this HQET quantity. In order to isolate the infra-red divergences, we consider the ratio of the $\overline{\rm MS}$ and the on-shell wave function renormalization constants in full QCD. Before giving the complete result, let us first discuss what we expect to happen to the contribution of heavy fermion loops.

To begin with, it is well known that the contribution of heavy quark loops in HQET vanishes identically due to the fact that the anti-particle pole is removed from the heavy quark propagator. Therefore, a quark-antiquark pair can not be created and this is the reason why the heavy fermion loops are disregarded in HQET. Hence, it is impossible to obtain the color structures proportional to N_H out of any HQET computation. As it follows from the preceding discussion, this should imply that in the ratio $Z_2^{\overline{\rm MS}}/Z_2^{\rm OS}$, computed in full QCD, any N_H-dependent structure should become finite since otherwise it would be impossible to remove the remaining divergence by performing a HQET calculation. Note, that this requirement on N_H-dependent structures provides a check on the gauge dependence of Z_2 computed in this paper since the color structure $C_A T_R N_H$ is gauge dependent. By explicit computation of the ratio $Z_2^{\overline{\rm MS}}/Z_2^{\rm OS}$ we find, however, that a divergent pieces proportional to N_H remain in this ratio. One should recall at this point that the decoupling of heavy degrees of freedom in field theories within the minimal subtraction scheme is not automatic. In order to observe the decoupling of heavy particles, one should express *all* $\overline{\rm MS}$ renormalized parameters of the theory in terms of their effective parameters whose renormalization scale dependence is governed by only the light degrees of freedom. In QCD this is done using well established decoupling relations [21,22]. Since the ratio $Z_2^{\overline{\rm MS}}/Z_2^{\rm OS}$ depends on the coupling constant and the gauge parameter, one should take into account the decoupling relations for these two quantities in order to obtain a proper low energy result. Once $\alpha_s^{(N_H+N_L)}$ and $\xi^{(N_H+N_L)}$ are expressed in terms of $\alpha_s^{(N_L)}$ and $\xi^{(N_L)}$, one observes explicitly that all divergences in the N_H-depending terms disappear and one is left with the expression that can match the structure of divergences in HQET. Let us emphasize that this argument alone proves that the on-shell wave function renormalization constant Z_2 in full QCD, that we compute in this paper, should be gauge dependent !

After this discussion, we are ready to present our result for the HQET anomalous dimension of the heavy quark field in the $\overline{\rm MS}$ scheme. The expression for the wave function renormalization is then easily derived from the condition $(Z_2^{\overline{\rm MS}}/Z_2^{\rm OS})Z_2^{\rm HQET}$ = finite. We define:

$$\frac{{\rm d}\log Z_2^{\rm HQET}}{{\rm d}\log\mu^2} = \gamma_{\rm HQET}. \tag{46}$$

---

[7] We are not aware of any systematic discussion of this point within HQET. The clue that there is such an exponentiation, can be taken from earlier studies of the eikonal approximation in QCD [23]. We thank A.G. Grozin for discussions on this point.

Our result for the HQET anomalous dimension $\gamma_{\rm HQET}$ then reads:

$$\gamma_{\rm HQET} = \gamma_1 \frac{\alpha_s}{\pi} + \gamma_2 \left(\frac{\alpha_s}{\pi}\right)^2 + \gamma_3 \left(\frac{\alpha_s}{\pi}\right)^3 + \mathcal{O}(\alpha_s^4), \tag{47}$$

where

$$\begin{aligned}
\gamma_1 =& C_F \left[-\frac{1}{4}\xi - \frac{1}{2} \right], \\
\gamma_2 =& C_F \left[C_A \left(-\frac{19}{24} - \frac{5}{32}\xi + \frac{1}{64}\xi^2 \right) + \frac{1}{3} T_R N_L \right], \\
\gamma_3 =& C_F \Bigg[C_A^2 \left(-\frac{3}{16}\zeta_3 - \frac{19495}{27648} - \frac{1}{360}\pi^4 + \xi \left(-\frac{379}{2048} - \frac{15}{256}\zeta_3 + \frac{1}{1440}\pi^4 \right) \right. \\
& \left. + \xi^2 \left(\frac{69}{2048} + \frac{3}{512}\zeta_3 \right) - \frac{5}{1024}\xi^3 \right) + T_R^2 N_L^2 \left(\frac{5}{108} \right) + C_F T_R N_L \left(\frac{51}{64} - \frac{3}{4}\zeta_3 \right) \\
& + C_A T_R N_L \left(\frac{1105}{6912} + \frac{3}{4}\zeta_3 + \frac{17}{256}\xi \right) \Bigg].
\end{aligned} \tag{48}$$

Here α_s is the $\overline{\rm MS}$ coupling constant in the theory with N_L massless flavors and the gauge parameter ξ is defined in Eq.(38). The above set of equations gives the result for the $\overline{\rm MS}$ anomalous dimension of the heavy quark field in HQET up to three loops. The two-loop result was obtained directly within HQET in [24]. The N_L-dependent terms in Eq.(48) agree with preliminary results of the ongoing three loop calculation within HQET.[8]

VI. SUMMARY OF QED RENORMALIZATION CONSTANTS

In this Section we summarize the results necessary for the on-shell renormalization of QED up to the three loop order and also demonstrate that Z_2 in QED is gauge parameter independent in a general covariant ξ gauge, if dimensional regularization is used for both ultraviolet and infra-red divergences. This result for Z_2 was previously mentioned in [7]; in other regularization schemes the gauge dependence of Z_2 was studied in [4–6]. While we do not add anything new to this question in essence, we think that the derivation presented below is relatively simple and transparent and so we decided to include it into our discussion for completeness.

Let us start with deriving an equation which determines a dependence of the fermion propagator on the gauge parameter ξ in a general covariant gauge. Such an equation is easiest to derive in the path integral formulation of QED. In a general ξ gauge the expression for the two-point electron Green function reads:

$$\langle {\rm T}\bar{\psi}(x)\psi(0)\rangle_\xi = N^{-1} \int [{\rm d}B]\, e^{\frac{-i}{2\eta}\int dx B(x)^2} \int [d\bar{\psi}][d\psi][dA]\delta(\partial_\mu A_\mu - B)\bar{\psi}(x)\psi(0)\, e^{iS}, \tag{49}$$

where $\eta = 1-\xi$, S is the standard QED action

$$S = \int {\rm d}^4 x \left(-\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \bar{\psi}\left(i\hat{\partial} - m \right)\psi - e_0 \bar{\psi}\gamma_\mu \psi A_\mu \right), \tag{50}$$

and N is the normalization factor:

$$N = \int [{\rm d}B]\, e^{\frac{-i}{2\eta}\int dx B(x)^2} \int [d\bar{\psi}][d\psi][dA]\delta(\partial_\mu A_\mu - B)\, e^{iS}. \tag{51}$$

If the integration over B is performed first, one recovers the familiar expression for the path integral representation of the fermion two-point function in a covariant gauge. We now perform a gauge transformation in the integral over matter fields:

---

[8] We were informed by A.Grozin that he is calculating the three-loop HQET anomalous dimension using a recently constructed algorithm for 3-loop calculations in HQET [25]. He has presently obtained all 3-loop terms except for the $C_F C_A^2$ one. The result coincides with Eq. (48).

$$A_\mu \to A_\mu + \partial_\mu f, \qquad \psi \to e^{-ie_0 f}\psi, \tag{52}$$

and choose $f(x)$ to satisfy the equation $\partial^2 f = B(x)$. We obtain:

$$\begin{aligned}\langle \mathrm{T}\bar{\psi}(x)\psi(0)\rangle_\xi = & \langle \mathrm{T}\bar{\psi}(x)\psi(0)\rangle_{B=0} \left[\int [\mathrm{d}B]\; e^{\frac{-i}{2\eta}\int dx B(x)^2}\right]^{-1} \\ & \times \int [\mathrm{d}B(x)]\; \exp\left\{\frac{-i}{2\eta}\int (dx)B^2 + ie_0 \int (dy)\,(G(x-y) - G(-y))\,B(y)\right\}\end{aligned} \tag{53}$$

where $G(x)$ is the solution of the equation $\partial^2 G(x) = \delta^{(4)}(x)$ and $\langle \mathrm{T}\bar{\psi}(x)\psi(0)\rangle_{B=0}$ is the fermion Green function in the Lorentz gauge $\partial_\mu A_\mu = 0$.

We can now integrate over $B(x)$ and a convenient way to do this is to perform a Fourier transform and integrate over its Fourier components $B(k)$. We then obtain:

$$\langle \mathrm{T}\bar{\psi}(x)\psi(0)\rangle_\xi = e^{\frac{i\eta e_0^2}{2}\int (dk) J_k(x) J_{-k}(x)} \langle \mathrm{T}\bar{\psi}(x)\psi(0)\rangle_{B=0}, \tag{54}$$

where $(dk) = \mathrm{d}^D k/(2\pi)^D$ and J_k is defined as:

$$J_k(x) = \frac{1}{k^2}\left(e^{ikx} - 1\right). \tag{55}$$

By differentiating with respect to ξ in Eq.(54), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\langle \mathrm{T}\bar{\psi}(x)\psi(0)\rangle_\xi = -\frac{ie_0^2}{2}\int (dk) J_k(x) J_{-k}(x)\; \langle \mathrm{T}\bar{\psi}(x)\psi(0)\rangle_\xi. \tag{56}$$

In momentum space this equation reads:

$$\frac{\mathrm{d}}{\mathrm{d}\xi} S_F(p)_\xi = ie_0^2 \int \frac{(dq)}{q^4} S_F(p-q)_\xi - ie_0^2 \int \frac{(dk)}{k^4} S_F(p)_\xi. \tag{57}$$

We now analyze Eq.(57) close to the mass shell where we approximate the fermion propagator by:

$$S_F(p) \approx \frac{Z_2}{\hat{p} - m}. \tag{58}$$

The fact that the right hand side of Eq.(57) does not develop a double pole in the limit $p^2 \to m^2$ implies that $\mathrm{d}m/\mathrm{d}\xi$ is zero, in other words the pole mass of the fermion in QED is gauge independent. If we equate the single pole contributions on both sides of Eq.(57), we obtain:

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\log Z_2 = -ie_0^2 \int \frac{(dk)}{k^4}. \tag{59}$$

Note that this result is derived under the assumption that the integral

$$\int \frac{(dq)}{q^4} S_F(p-q)_\xi \tag{60}$$

does not develop a pole if $p^2 \to m^2$ and one can make sure by power counting that this is a non-trivial statement. The correct approach, which matches perturbative calculations, is to take the limit $p^2 \to m^2$ first assuming that the integrals remain finite because of the regularization. In this sense the integral in Eq.(60) does not develop a single pole and can be neglected. We now return to Eq.(59) and note that in dimensional regularization the integral over k is zero. Hence we obtain:

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\log Z_2 = 0. \tag{61}$$

We thus proved that the on-shell fermion wave function renormalization constant in QED is gauge parameter independent.

Let us now turn to the presentation of the three-loop QED renormalization constants. We consider QED of a single electron; that implies that $N_H = 1$ and $N_L = 0$ should be used in general formulas of Section IV.

We write the mass renormalization constant Z_m as

$$Z_m = 1 + \sum_i \left(\frac{\alpha_0}{\pi} \frac{\Gamma(1+\varepsilon) m^{-2\varepsilon}}{(4\pi)^{-\varepsilon}} \right)^i Z_m^{(i)}, \tag{62}$$

where α_0 is the bare QED coupling constant. For the expansion coefficients $Z_m^{(i)}$ we obtain:

$$\begin{aligned}
Z_m^{(1)} &= -\frac{3}{4\varepsilon} - \frac{1}{1-2\varepsilon}, \\
Z_m^{(2)} &= \frac{13}{32\varepsilon^2} + \frac{73}{64\varepsilon} + \frac{475}{128} - \frac{3}{4}\zeta_3 + \frac{1}{2}\pi^2 \log 2 - \frac{23}{48}\pi^2 \\
&\quad + \varepsilon \left(\frac{2529}{256} - \frac{47}{4}\zeta_3 + 4\pi^2 \log 2 - \pi^2 \log^2 2 - \frac{245}{96}\pi^2 + \frac{7}{40}\pi^4 - \frac{1}{2}\log^4 2 - 12 a_4 \right), \\
Z_m^{(3)} &= -\frac{221}{1152\varepsilon^3} - \frac{5561}{6912\varepsilon^2} + \frac{1}{\varepsilon} \left(-\frac{154445}{41472} + \frac{13}{16}\zeta_3 - \frac{17}{24}\pi^2 \log 2 + \frac{391}{576}\pi^2 \right) \\
&\quad - \frac{3489365}{248832} - \frac{1}{16}\zeta_3 \pi^2 + \frac{719}{72}\zeta_3 + \frac{5}{8}\zeta_5 + \frac{89}{36}\pi^2 \log 2 + \frac{65}{36}\pi^2 \log^2 2 \\
&\quad - \frac{5783}{17280}\pi^2 - \frac{979}{4320}\pi^4 + \frac{23}{72}\log^4 2 + \frac{23}{3} a_4.
\end{aligned} \tag{63}$$

The wave function renormalization constant is parameterized as

$$Z_2 = 1 + \sum_i \left(\frac{\alpha_0}{\pi (4\pi)^{-\varepsilon}} \Gamma(1+\varepsilon) \right)^i Z_2^{(i)}, \tag{64}$$

where

$$\begin{aligned}
Z_2^{(1)} &= -\frac{3}{4\varepsilon} - \frac{1}{1-2\varepsilon}, \\
Z_2^{(2)} &= \frac{17}{32\varepsilon^2} + \frac{229}{192\varepsilon} + \frac{8453}{1152} - \frac{55}{48}\pi^2 - \frac{3}{2}\zeta_3 + \pi^2 \log 2 \\
&\quad + \varepsilon \left(\frac{31}{4}\pi^2 \log 2 - \frac{419}{96}\pi^2 + \frac{86797}{6912} - \frac{203}{8}\zeta_3 - 2\pi^2 \log^2 2 - \log^4 2 - 24 a_4 + \frac{7}{20}\pi^4 \right), \\
Z_2^{(3)} &= -\frac{131}{384\varepsilon^3} - \frac{2141}{2304\varepsilon^2} + \frac{1}{\varepsilon} \left(\frac{17}{8}\zeta_3 - \frac{17}{12}\pi^2 \log 2 + \frac{935}{576}\pi^2 - \frac{116489}{13824} \right) \\
&\quad - \frac{2121361}{82944} + \frac{1}{8}\zeta_3 \pi^2 + \frac{2803}{144}\zeta_3 - \frac{5}{16}\zeta_5 + \frac{1367}{144}\pi^2 \log 2 \\
&\quad + \frac{23}{6}\pi^2 \log^2 2 - \frac{197731}{51840}\pi^2 - \frac{383}{720}\pi^4 + \frac{3}{4}\log^4 2 + 18 a_4.
\end{aligned} \tag{65}$$

We explicitly see that the QED result for both the mass and the wave function renormalization constants does not depend on the gauge parameter, in agreement with Johnson-Zumino identity Eq.(61) and in variance with the QCD calculation of the previous section.

Finally, for completeness, we give the relation between the bare α_0 and the physical QED coupling constant α defined in the on-shell scheme (i.e. through the photon propagator at zero momentum transfer). The result reads [7]:

$$\frac{\alpha}{\alpha_0} = 1 - \frac{4}{3} \left(\frac{\alpha_0 \Gamma(\varepsilon) m^{-2\varepsilon}}{\pi (4\pi)^{-\varepsilon}} \right) - \frac{4\varepsilon}{(2-\varepsilon)(1-2\varepsilon)(1+2\varepsilon)} \left(1 + \varepsilon(7 - 4\varepsilon)\right) \left(\frac{\alpha_0 \Gamma(\varepsilon) m^{-2\varepsilon}}{\pi (4\pi)^{-\varepsilon}} \right)^2, \tag{66}$$

and completes the set of the renormalization constants required for the renormalization of the QED up to three loops in the on-shell scheme.

VII. CONCLUSIONS

We have presented a calculation of the three loop on-shell renormalization constants both in QED and QCD. Explicit result for the mass and the fermion wave function renormalization constants are derived. Dimensional regularization is used to regulate both ultraviolet and infrared divergences.

Our calculation represents an important step towards three loop calculations with heavy on-shell quarks in QCD (semileptonic b decays, NRQCD, etc.). Furthermore, these results are important for applications in QED where the on-shell scheme is clearly superior as compared to any other scheme.

The use of dimensional regularization has some interesting consequences for the gauge dependence of the fermion wave function renormalization constant. First, in QED Z_2 turns out to be gauge invariant to all orders in α. In QCD, only the first two orders of the perturbative expansion of Z_2 in α_s are gauge independent; unfortunately, the dependence on the gauge parameter explicitly appears in the third order of the perturbative expansion.

As a byproduct of this analysis, we derived the anomalous dimension for the heavy quark field in HQET.

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APPENDIX A: ELEMENTARY MULTILOOP INTEGRALS

In this appendix expressions are given for a number of integrals in dimensional regularization for which exact analytic results are known. We present these below in Minkowski space.

For a 1-loop massive bubble integral one has [8]

$$\int \frac{\mathrm{d}^D p}{(p^2 - m^2 + i\epsilon)^{\alpha_1}(p^2 + i\epsilon)^{\alpha_2}} = \\ i\,\pi^{D/2}(-1)^{-\alpha_1-\alpha_2}(m^2)^{D/2-\alpha_1-\alpha_2}\frac{\Gamma(\alpha_1+\alpha_2-D/2)\Gamma(D/2-\alpha_2)}{\Gamma(\alpha_1)\Gamma(D/2)}. \tag{67}$$

A 1-loop massless propagator-type integral has the simple form

$$\int \frac{\mathrm{d}^D p}{(p^2 + i\epsilon)^{\alpha_1}[(p+Q)^2 + i\epsilon]^{\alpha_2}} = \\ i^{1-D}\,\pi^{D/2}\,(Q^2)^{D/2-\alpha_1-\alpha_2}\frac{\Gamma(D/2-\alpha_1)\Gamma(D/2-\alpha_2)\Gamma(\alpha_1+\alpha_2-D/2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(D-\alpha_1-\alpha_2)}. \tag{68}$$

A compact expression for this integral with a general tensor numerator can be found in Ref. [9]. For a 1-loop on-shell propagator-type integral one derives

$$\int \frac{\mathrm{d}^D p}{(p^2 + i\epsilon)^{\alpha_1}(p^2 + 2p\cdot Q + i\epsilon)^{\alpha_2}} = \\ i\,\pi^{D/2}(Q^2)^{D/2-\alpha_1-\alpha_2}(-1)^{-\alpha_1-\alpha_2}\frac{\Gamma(\alpha_1+\alpha_2-D/2)\Gamma(D-2\alpha_1-\alpha_2)}{\Gamma(\alpha_2)\Gamma(D-\alpha_1-\alpha_2)}, \tag{69}$$

and for a two-loop bubble integral with one massless and two massive lines one finds [26]

$$\int\int \frac{\mathrm{d}^D p\,\mathrm{d}^D k}{(p^2 - m^2 + i\epsilon)^{\alpha_1}\,[(p+k)^2 + i\epsilon]^{\alpha_2}\,(k^2 - m^2 + i\epsilon)^{\alpha_3}} = \\ \pi^D (m^2)^{D-\alpha_1-\alpha_2-\alpha_3}\,(-1)^{1-\alpha_1-\alpha_2-\alpha_3}\frac{\Gamma(-D+\alpha_1+\alpha_2+\alpha_3)}{\Gamma(\alpha_1)\Gamma(\alpha_3)} \\ \times\frac{\Gamma(-D/2+\alpha_1+\alpha_2)\Gamma(-D/2+\alpha_2+\alpha_3)\Gamma(D/2-\alpha_2)}{\Gamma(D/2)\Gamma(\alpha_1+2\alpha_2+\alpha_3-D)}. \tag{70}$$

Also for this integral with a general tensor numerator a compact expression is known [27].

Several more simple cases follow by using these expressions recursively as the powers α_1,α_2 and α_3, are allowed to be non-integer, possibly containing D. In this way one can obtain, for instance,

$$\int\int \frac{\mathrm{d}^D p\,\mathrm{d}^D k}{(p^2 + i\epsilon)^{\alpha_1}\,(k^2 - m^2 + i\epsilon)^{\alpha_2}\,(k^2 + i\epsilon)^{\alpha_4}\,[(p+k)^2 + i\epsilon]^{\alpha_3}} = \\ \pi^D (m^2)^{D-\alpha_1-\alpha_2-\alpha_3-\alpha_4}\,(-1)^{1-\alpha_1-\alpha_2-\alpha_3-\alpha_4}\frac{\Gamma(\alpha_1+\alpha_2+\alpha_3+\alpha_4-D)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)} \\ \times\frac{\Gamma(\alpha_1+\alpha_3-D/2)\Gamma(D/2-\alpha_1)\Gamma(D/2-\alpha_3)\Gamma(D-\alpha_1-\alpha_3-\alpha_4)}{\Gamma(D/2)\Gamma(D-\alpha_1-\alpha_3)}. \tag{71}$$

APPENDIX B: SOME HARMONIC SUMS

In this Appendix we present a partial list of harmonic sums which we used in this paper. These results can be found e.g. in [14–17].

$$\sum_{i=1}^{\infty} \frac{1}{i^k} = S_k(\infty) = \zeta_k \qquad (k > 1) \tag{72}$$

$$\sum_{i=1}^{\infty} \frac{S_1(i)}{i^2} \equiv S_{2,1}(\infty) = 2\zeta_3, \tag{73}$$

$$\sum_{i=1}^{\infty} \frac{S_1(i)}{i^3} \equiv S_{3,1}(\infty) = \frac{5}{4}\zeta_4, \tag{74}$$

$$\sum_{i=1}^{\infty} \frac{S_1(i)}{i^4} \equiv S_{4,1}(\infty) = 3\zeta_5 - \zeta_2\zeta_3, \tag{75}$$

$$\sum_{i=1}^{\infty} \frac{S_2(i)}{i^2} \equiv S_{2,2}(\infty) = \frac{7}{4}\zeta_4, \tag{76}$$

$$\sum_{i=1}^{\infty} \frac{S_2(i)}{i^3} \equiv S_{3,2}(\infty) = 3\zeta_2\zeta_3 - \frac{9}{2}\zeta_5, \tag{77}$$

$$\sum_{i=1}^{\infty} \frac{S_{2,1}(i)}{i^2} \equiv S_{2,2,1}(\infty) = 2\zeta_5, \tag{78}$$

$$\sum_{i=1}^{\infty} \frac{S_1^2(i)}{i^2} = \frac{17}{4}\zeta_4, \tag{79}$$

$$\sum_{i=1}^{\infty} \frac{S_1^3(i)}{i^2} = \zeta_2\zeta_3 + 10\zeta_5, \tag{80}$$

$$\sum_{i=1}^{\infty} \frac{S_1^2(i)}{i^3} = \frac{7}{2}\zeta_5 - \zeta_2\zeta_3, \tag{81}$$

$$\sum_{i=1}^{\infty} \frac{S_3(i)}{i^2} = \frac{11}{2}\zeta_5 - 2\zeta_2\zeta_3, \tag{82}$$

$$\sum_{i=1}^{\infty} \frac{S_1(i)S_2(i)}{i^2} = \zeta_5 + \zeta_2\zeta_3. \tag{83}$$

APPENDIX C: LIST OF MASTER INTEGRALS

In this Appendix we present the list of master integrals we have used in our calculation. The list updates the result that can be found in [10]. The Euclidean master integrals are defined as (for a pictorial representation see Fig.2):

$$I_1 = \left\langle \frac{(-1)\; p \cdot k_2}{D_1 D_2 D_3 D_4 D_5 D_6 D_7 D_8} \right\rangle,$$

$$I_2 = \left\langle \frac{1}{D_1 D_2 D_3 D_4 D_7 D_8} \right\rangle, \qquad I_3 = \left\langle \frac{1}{D_1 D_2 D_4 D_5 D_6 D_8} \right\rangle,$$

$$I_4 = \left\langle \frac{1}{D_2 D_3 D_4 D_6 D_7 D_8} \right\rangle, \qquad I_5 = \left\langle \frac{1}{D_1 D_3 D_4 D_5 D_7 D_8} \right\rangle,$$

$$I_6 = \left\langle \frac{1}{D_1 D_3 D_5 D_6 D_7 D_8} \right\rangle, \qquad I_7 = \left\langle \frac{1}{D_2 D_4 D_5 D_6 D_7 D_8} \right\rangle,$$

$$I_8 = \left\langle \frac{1}{D_1 D_2 D_3 D_4 D_5} \right\rangle, \qquad I_9 = \left\langle \frac{1}{D_2 D_3 D_5 D_6 D_7} \right\rangle,$$

$$I_{10} = \left\langle \frac{1}{D_2 D_4 D_6 D_7 D_8} \right\rangle, \qquad I_{11} = \left\langle \frac{1}{D_1 D_3 D_5 D_7} \right\rangle,$$

$$I_{12} = \left\langle \frac{1}{D_1 D_2 D_4 D_5} \right\rangle, \qquad I_{13} = \left\langle \frac{1}{D_3 D_5 D_6 D_7} \right\rangle,$$

$$
\begin{aligned}
I_{14} &= \left\langle \frac{1}{D_2 D_3 D_4 D_5} \right\rangle, & I_{15} &= \left\langle \frac{1}{D_3 D_4 D_7 D_8} \right\rangle, \\
I_{16} &= \left\langle \frac{1}{D_3 D_6 D_7 D_8} \right\rangle, & I_{17} &= \left\langle \frac{1}{D_1 D_4 D_5} \right\rangle, \\
I_{18} &= \left\langle \frac{1}{D_1 D_2 D_5 D_6 D_7 D_8 D_9} \right\rangle, \qquad (84)
\end{aligned}
$$

where

$$
\langle ... \rangle = \int \mathrm{d}^D k_1 \mathrm{d}^D k_2 \mathrm{d}^D k_3 ......, \qquad (85)
$$

the momentum p is always considered as *incoming* and

$$
\begin{aligned}
D_1 &= (p-k_1)^2+1, & D_2 &= (p-k_1-k_2)^2+1, \\
D_3 &= (p-k_1-k_2-k_3)^2+1, & D_4 &= (p-k_2-k_3)^2+1, \\
D_5 &= (p-k_3)^2+1, & D_6 &= k_1^2, \\
D_7 &= k_2^2, & D_8 &= k_3^2, \\
D_9 &= (k_3-k_1-k_2)^2. \qquad (86)
\end{aligned}
$$

where $p^2 = -1$.

$$
\begin{aligned}
I_1 &= C(\varepsilon)\left(-5\zeta_5 + \frac{1}{2}\pi^2\zeta_3 + \mathcal{O}(\varepsilon)\right), \\
I_2 &= C(\varepsilon)\left(2\frac{\zeta_3}{\varepsilon} + 10\zeta_3 - \frac{1}{3}\pi^2 - \frac{13}{90}\pi^4 + \varepsilon\left(\frac{49}{6}\pi^2\zeta_3 + 24\zeta_3 - \frac{85}{2}\zeta_5 + 4\pi^2\log 2 - \frac{4}{3}\pi^2 - \frac{13}{18}\pi^4\right) + \mathcal{O}(\varepsilon^2)\right), \\
I_3 &= C(\varepsilon)\left(\frac{1}{3\varepsilon^3} + \frac{7}{3\varepsilon^2} + \frac{31}{3\varepsilon} + \frac{103}{3} - \frac{4}{3}\zeta_3 - \frac{2}{15}\pi^4 + \varepsilon\left(\frac{235}{3} + \frac{28}{3}\zeta_3\pi^2 + \frac{32}{3}\zeta_3 - 78\zeta_5 + \frac{8}{3}\pi^2 - \frac{3}{5}\pi^4\right) + \mathcal{O}(\varepsilon^2)\right), \\
I_4 &= C(\varepsilon)\left(2\frac{\zeta_3}{\varepsilon} + 2\zeta_3 + \frac{1}{3}\pi^2 - \frac{7}{90}\pi^4 + \varepsilon\left(-\frac{2}{3}\zeta_3\pi^2 - 12\zeta_3 + 44\zeta_5 + \frac{14}{3}\pi^2 - \frac{41}{90}\pi^4\right) + \mathcal{O}(\varepsilon^2)\right), \\
I_5 &= C(\varepsilon)\left(\frac{1}{6\varepsilon^3} + \frac{3}{2\varepsilon^2} + \frac{1}{\varepsilon}\left(\frac{55}{6} - \frac{1}{3}\pi^2\right) + \frac{95}{2} - \frac{14}{3}\zeta_3 - \frac{7}{3}\pi^2 - \frac{4}{45}\pi^4\right. \\
&\quad \left.+\varepsilon\left(\frac{1351}{6} + \frac{25}{6}\zeta_3\pi^2 - 42\zeta_3 - \frac{49}{2}\zeta_5 + 4\pi^2\log 2 - \frac{31}{3}\pi^2 - \frac{23}{30}\pi^4\right) + \mathcal{O}(\varepsilon^2)\right), \\
I_6 &= C(\varepsilon)\left(\frac{1}{3\varepsilon^3} + \frac{7}{3\varepsilon^2} + \frac{31}{3\varepsilon} + \frac{103}{3} + \frac{2}{3}\zeta_3 + \frac{1}{3}\pi^2 - \frac{4}{45}\pi^4 + \varepsilon\left(\frac{235}{3} + \frac{2}{3}\zeta_3\pi^2 + \frac{20}{3}\zeta_3 - 2\zeta_5 + 4\pi^2 - \frac{3}{10}\pi^4\right) + \mathcal{O}(\varepsilon^2)\right), \\
I_7 &= C(\varepsilon)\left(\frac{1}{6\varepsilon^3} + \frac{3}{2\varepsilon^2} + \frac{1}{\varepsilon}\left(\frac{55}{6} - \frac{1}{3}\pi^2\right) + \frac{95}{2} - \frac{8}{3}\zeta_3 - 2\pi^2 - \frac{1}{15}\pi^4\right. \\
&\quad \left.+\varepsilon\left(\frac{1351}{6} + 6\zeta_3\pi^2 - 14\zeta_3 - 64\zeta_5 - \frac{17}{3}\pi^2 - \frac{47}{45}\pi^4\right) + \mathcal{O}(\varepsilon^2)\right), \\
I_8 &= C(\varepsilon)\left(-\frac{1}{\varepsilon^3} - \frac{16}{3\varepsilon^2} - \frac{16}{\varepsilon} - 20 + 2\zeta_3 - \frac{8}{3}\pi^2 + \varepsilon\left(\frac{364}{3} - \frac{200}{3}\zeta_3 + 16\pi^2\log 2 - 28\pi^2 - \frac{3}{10}\pi^4\right)\right. \\
&\quad \left.+\varepsilon^2\left(1244 + 21\zeta_3\pi^2 - 776\zeta_3 - 126\zeta_5 + 168\pi^2\log 2 - \frac{80}{3}\pi^2\log^2 2 - 188\pi^2 + \frac{46}{15}\pi^4 - \frac{64}{3}\log^4 2 - 512a_4\right) + \mathcal{O}(\varepsilon^3)\right), \\
I_9 &= C(\varepsilon)\left(-\frac{2}{3\varepsilon^3} - \frac{10}{3\varepsilon^2} + \frac{1}{\varepsilon}\left(-\frac{26}{3} - \frac{1}{3}\pi^2\right) - 2 - \frac{16}{3}\zeta_3 - \frac{11}{3}\pi^2 + \varepsilon\left(\frac{398}{3} - \frac{248}{3}\zeta_3 + 16\pi^2\log 2 - \frac{73}{3}\pi^2 - \frac{13}{45}\pi^4\right)\right. \\
&\quad \left.+\varepsilon^2\left(1038 - \frac{8}{3}\zeta_3\pi^2 - \frac{1888}{3}\zeta_3 - 96\zeta_5 + 160\pi^2\log 2 - \frac{128}{3}\pi^2\log^2 2 - 129\pi^2 + \frac{3}{5}\pi^4 - \frac{64}{3}\log^4 2 - 512a_4\right) + \mathcal{O}(\varepsilon^3)\right), \\
I_{10} &= C(\varepsilon)\left(-\frac{1}{3\varepsilon^3} - \frac{5}{3\varepsilon^2} + \frac{1}{\varepsilon}\left(-4 - \frac{2}{3}\pi^2\right) + \frac{10}{3} - \frac{26}{3}\zeta_3 - \frac{7}{3}\pi^2 + \varepsilon\left(\frac{302}{3} - \frac{94}{3}\zeta_3 - \pi^2 - \frac{35}{18}\pi^4\right)\right. \\
&\quad \left.-\varepsilon^2\left(-734 + \frac{76}{3}\pi^2\zeta_3 - \frac{101}{3}\pi^2 + 20\zeta_3 + \frac{551}{90}\pi^4 + 462\zeta_5\right) + \mathcal{O}(\varepsilon^3)\right),
\end{aligned}
$$

$$
\begin{aligned}
I_{11}= &\, C(\varepsilon)\left(\frac{1}{\varepsilon^3}+\frac{7}{2\varepsilon^2}+\frac{253}{36\varepsilon}+\frac{2501}{216}+\varepsilon\left(\frac{59437}{1296}-\frac{64}{9}\pi^2\right)+\varepsilon^2\left(\frac{2831381}{7776}-\frac{1792}{9}\zeta_3+\frac{256}{3}\pi^2\log 2-\frac{2272}{27}\pi^2\right)\right.\\
&\left.+\varepsilon^3\left(\frac{117529021}{46656}-\frac{63616}{27}\zeta_3+\frac{9088}{9}\pi^2\log 2-\frac{3584}{9}\pi^2\log^2 2-\frac{49840}{81}\pi^2+\frac{2752}{135}\pi^4-\frac{1024}{9}\log^4 2-\frac{8192}{3}a_4\right)+\mathcal{O}(\varepsilon^4)\right),\\
I_{12}= &\, C(\varepsilon)\left(\frac{2}{\varepsilon^3}+\frac{23}{3\varepsilon^2}+\frac{35}{2\varepsilon}+\frac{275}{12}+\varepsilon\left(-\frac{189}{8}+\frac{112}{3}\zeta_3\right)\right.\\
&\left.+\varepsilon^2\left(-\frac{14917}{48}+280\zeta_3-\frac{32}{3}\pi^2\log^2 2-\frac{136}{45}\pi^4+\frac{32}{3}\log^4 2+256a_4\right)+\mathcal{O}(\varepsilon^3)\right),\\
I_{13}= &\, C(\varepsilon)\left(\frac{1}{3\varepsilon^3}+\frac{7}{6\varepsilon^2}+\frac{25}{12\varepsilon}-\frac{5}{24}+\frac{8}{3}\zeta_3+\varepsilon\left(-\frac{959}{48}+\frac{28}{3}\zeta_3-\frac{2}{15}\pi^4\right)\right.\\
&\left.+\varepsilon^2\left(-\frac{10493}{96}+\frac{50}{3}\zeta_3+48\zeta_5-\frac{7}{15}\pi^4\right)+\mathcal{O}(\varepsilon^3)\right),\\
I_{14}= &\, C(\varepsilon)\left(\frac{3}{2\varepsilon^3}+\frac{23}{4\varepsilon^2}+\frac{105}{8\varepsilon}+\frac{275}{16}+\frac{4}{3}\pi^2+\varepsilon\left(-\frac{567}{32}+28\zeta_3-8\pi^2\log 2+10\pi^2\right)\right.\\
&\left.+\varepsilon^2\left(-\frac{14917}{64}+210\zeta_3-60\pi^2\log 2+16\pi^2\log^2 2+\frac{145}{3}\pi^2-\frac{62}{45}\pi^4+8\log^4 2+192a_4\right)+\mathcal{O}(\varepsilon^3)\right),\\
I_{15}= &\, C(\varepsilon)\left(\frac{1}{2\varepsilon^3}+\frac{7}{4\varepsilon}+\frac{1}{\varepsilon}\left(\frac{25}{8}+\frac{1}{3}\pi^2\right)-\frac{5}{16}+4\zeta_3+\frac{7}{6}\pi^2+\varepsilon\left(-\frac{959}{32}+14\zeta_3+\frac{25}{12}\pi^2+\frac{16}{45}\pi^4\right)\right.\\
&\left.+\varepsilon^2\left(-\frac{10493}{64}+\frac{8}{3}\zeta_3\pi^2+25\zeta_3+72\zeta_5-\frac{5}{24}\pi^2+\frac{56}{45}\pi^4\right)+\mathcal{O}(\varepsilon^3)\right),\\
I_{16}= &\, C(\varepsilon)\left(-\frac{1}{6\varepsilon^3}-\frac{5}{6\varepsilon^2}+\frac{1}{\varepsilon}\left(-\frac{11}{6}-\frac{1}{3}\pi^2\right)+\frac{23}{6}-\frac{16}{3}\zeta_3-\frac{5}{3}\pi^2+\varepsilon\left(\frac{135}{2}-\frac{80}{3}\zeta_3-\frac{11}{3}\pi^2-\frac{37}{45}\pi^4\right)\right.\\
&\left.+\varepsilon^2\left(\frac{949}{2}-\frac{32}{3}\pi^2\zeta_3-\frac{176}{3}\zeta_3-208\zeta_5+\frac{23}{3}\pi^2-\frac{37}{9}\pi^4\right)+\mathcal{O}(\varepsilon^3)\right),\\
I_{17}= &\, C(\varepsilon)\left(-\frac{1}{\varepsilon^3}-\frac{3}{\varepsilon^2}-\frac{6}{\varepsilon}-10-15\varepsilon-21\varepsilon^2-28\varepsilon^3+\mathcal{O}(\varepsilon^4)\right),\\
I_{18}= &\, C(\varepsilon)\left(2\pi^2\zeta_3-5\zeta_5+\mathcal{O}(\varepsilon)\right).
\end{aligned}
$$

Finally, we should note that integrals I_{12} and I_{14} are known to a higher order in ε than we have written here; the corresponding higher order terms in ε can be found in Ref. [28]. Furthermore, integrals I_{13}, I_{15}, I_{16} and I_{17} can be performed exactly in terms of Euler gamma functions, as one can see for instance by combining the exact forms in Appendix A. This means that these four integrals are known in principle to an arbitrary high order in ε.

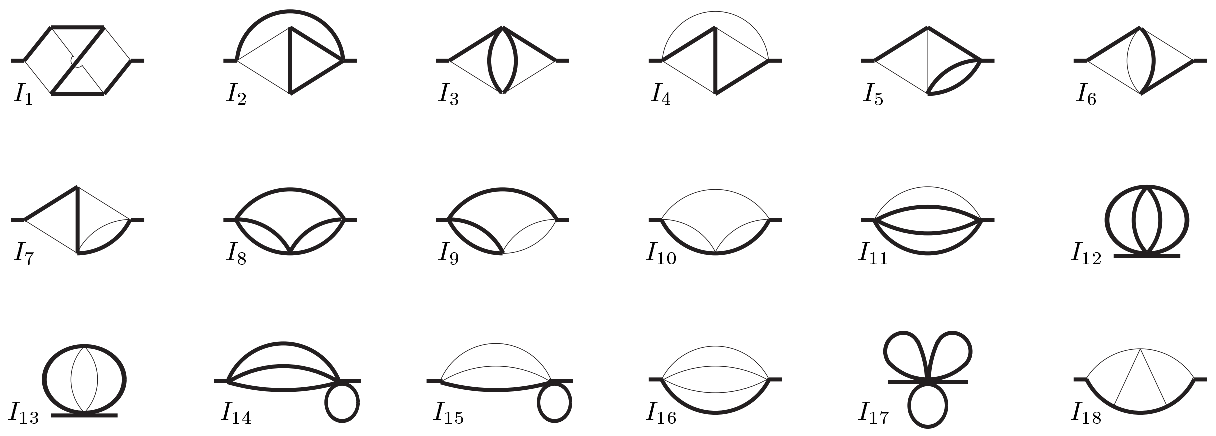

FIG. 2. Pictorial representation of the complete set of primitive 3-loop on-shell integrals I_1 – I_{18}. Thick lines are used to indicate massive scalar propagators and thin lines indicate massless propagators.

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