Freiburg-THEP 00/3
DTP/00/16
TTP00-07
hep-ph/0003261

The tensor reduction and master integrals of the two-loop massless crossed box with light-like legs

C. Anastasiou[(a)1], T. Gehrmann[(b)2], C. Oleari[(a)3],
E. Remiddi[(c)4] and J. B. Tausk[(d)5]

[(a)] *Department of Physics, University of Durham, Durham DH1 3LE, England*
[(b)] *Institut für Theoretische Teilchenphysik, Univ. Karlsruhe, D-76128 Karlsruhe, Germany*
[(c)] *Dip. di Fisica, Università di Bologna and INFN, Sez. di Bologna, I-40126 Bologna, Italy*
[(d)] *Fakultät für Physik, Albert-Ludwigs-Universität Freiburg, D-79104 Freiburg, Germany*

Abstract

The class of the two-loop massless crossed boxes, with light-like external legs, is the final unresolved issue in the program of computing the scattering amplitudes of $2 \to 2$ massless particles at next-to-next-to-leading order.

In this paper, we describe an algorithm for the tensor reduction of such diagrams. After connecting tensor integrals to scalar ones with arbitrary powers of propagators in higher dimensions, we derive recurrence relations from integration-by-parts and Lorentz-invariance identities, that allow us to write the scalar integrals as a combination of two master crossed boxes plus simpler-topology diagrams.

We derive the system of differential equations that the two master integrals satisfy using two different methods, and we use one of these equations to express the second master integral as a function of the first one, already known in the literature. We then give the analytic expansion of the second master integral as a function of $\epsilon = (4-D)/2$, where D is the space-time dimension, up to order $\mathcal{O}(\epsilon^0)$.

[1]e-mail: Ch.Anastasiou@durham.ac.uk
[2]e-mail: Thomas.Gehrmann@phys.uni-karlsruhe.de
[3]e-mail: Carlo.Oleari@durham.ac.uk
[4]e-mail: Ettore.Remiddi@bo.infn.it
[5]e-mail: tausk@physik.uni-freiburg.de

1 Introduction

The increasing precision of high-energy scattering experiments has reached the level that next-to-next-to-leading theoretical results for $2 \to 2$ scattering processes of massless particles are demanded [1]. Several steps have already been taken towards this goal: in Refs. [2, 3] the tensor reduction of two-loop planar boxes and the computation of the relevant master integrals have been given. Reference [4] dealt with the tensor reduction of the pentabox, while other simpler two-loop diagrams obtained by one-loop insertion into one-loop four-point functions have been treated in Ref. [5].

The final unresolved issue is the tensor reduction of two-loop crossed boxes of Fig. 1 and the evaluation of the corresponding master integrals. One of the master integrals, corresponding to the scalar integral with all powers of propagators equal to unity, was computed in Ref. [6] as an analytic expansion in $\epsilon = (4 - D)/2$, where D is the space-time dimension.

A first result of this paper is to present a method to reduce tensor integrals of crossed two-loop boxes to two crossed-box master integrals, plus master integrals for simpler-topology diagrams. We choose as first master integral the already evaluated [6] crossed box with all powers of propagators equal to one, and as second master integral the scalar integral where the power of the second propagator is equal to two, all the others being one.

As it is well known [4, 7], tensor integrals can be related to scalar integrals with higher powers of the propagators in higher dimensions. We make use of recurrence relations obtained by integration-by-parts [8] and Lorentz-invariance identities [9] to connect integrals with different powers of the propagators and to derive a reduction algorithm that expresses the generic scalar integral as a function of the two master crossed boxes, plus simpler sub-topologies. We explicitly give the equations that connect the two master integrals in different dimensions (dimensional shift).

Quite in general, it has been shown that the master integrals for any Feynman graph topology satisfy a system of first-order differential equations on any of the Mandelstam variables on which they depend [10]. We derive a system of two coupled differential equations for the crossed-box master integrals by two independent methods: first using the raising and lowering operators and second by taking the on-shell limit of the differential equations for the crossed box with one off-shell leg [9].

Inserting the analytic expression of the first master crossed box into the differential equation for the first master integral, we obtain an algebraic equation for the second master crossed box, that can be solved to give the analytic expansion in ϵ of the second master integral. The differential equation for the second master integral provides an overall check of the calculation.

Our paper is organized as follows: in Section 2 we introduce the notation we are going to use. In Section 3 we briefly review how tensor integrals are related to scalar ones, and we give the reduction algorithm to express these integrals as a combination of the two master crossed boxes, plus simpler master integrals for topologies with fewer propagators. In Sections 4 and 5 we derive the differential equations that the two master integrals satisfy in arbitrary dimensions D, using the two different methods. The analytic expansion of the second master integral is given in Section 6. For completeness, in Appendix A we present the algorithm for the tensor reduction and for the dimensional shift of the crossed triangle, since this is one of the non-trivial sub-topologies produced by the reduction procedure. Finally we conclude with Section 7.

2 Notation

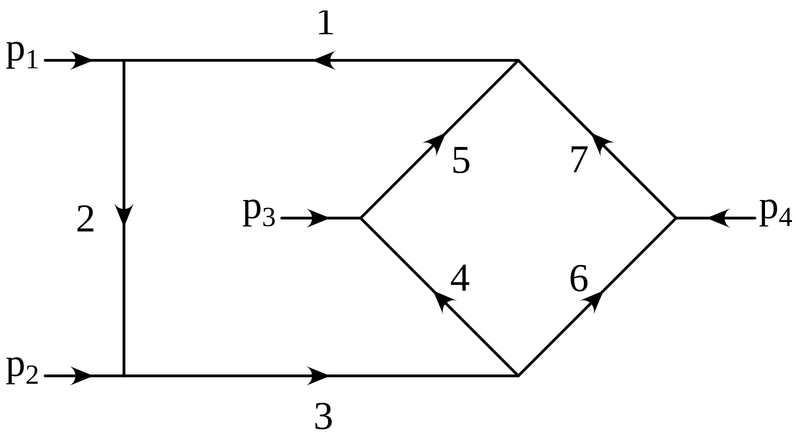

Figure 1: *The generic two-loop crossed box.*

We denote the generic two-loop tensor crossed (or non-planar) four-point function in D dimensions of Fig. 1 with seven propagators A_i raised to arbitrary powers ν_i as

$$\begin{aligned} &\mathrm{Xbox}^D\left(\nu_1,\nu_2,\nu_3,\nu_4,\nu_5,\nu_6,\nu_7;s,t\right)[1;\, k^\mu;\, l^\mu;\, k^\mu k^\nu;\, k^\mu l^\nu;\, \ldots] \\ &\qquad = \int \frac{d^D k}{i\pi^{D/2}} \int \frac{d^D l}{i\pi^{D/2}}\, \frac{[1;\, k^\mu;\, l^\mu;\, k^\mu k^\nu;\, k^\mu l^\nu;\, \ldots]}{A_1^{\nu_1}A_2^{\nu_2}A_3^{\nu_3}A_4^{\nu_4}A_5^{\nu_5}A_6^{\nu_6}A_7^{\nu_7}}, \end{aligned} \tag{2.1}$$

where the propagators are

$$\begin{aligned} A_1 &= (k+l+p_{34})^2 + i0, \\ A_2 &= (k+l+p_{134})^2 + i0, \\ A_3 &= (k+l)^2 + i0, \end{aligned}$$

$$
\begin{aligned}
A_4 &= l^2 + i0, \\
A_5 &= (l+p_3)^2 + i0, \\
A_6 &= k^2 + i0, \\
A_7 &= (k+p_4)^2 + i0.
\end{aligned}
\tag{2.2}
$$

The external momenta p_j are in-going and light-like, $p_j^2 = 0$, $j = 1 \dots 4$, so that the only momentum scales are the usual Mandelstam variables $s = (p_1 + p_2)^2$ and $t = (p_2 + p_3)^2$, together with $u = -s - t$. For ease of notation, we define $p_{ij} = p_i + p_j$ and $p_{ijk} = p_i + p_j + p_k$. In the square brackets we keep trace of the tensor structure that may be present in the numerator of Eq. (2.1). When the numerator is unity we have the scalar integral

$$
\mathrm{Xbox}^D\left(\nu_1,\nu_2,\nu_3,\nu_4,\nu_5,\nu_6,\nu_7;s,t\right)[1] \equiv \mathrm{Xbox}^D\left(\nu_1,\nu_2,\nu_3,\nu_4,\nu_5,\nu_6,\nu_7;s,t\right). \tag{2.3}
$$

3 Tensor reduction using raising and lowering operators

Tensor integrals can be related to combinations of scalar integrals with higher powers of propagators and/or different values of D [7, 4]. In fact, introducing the Schwinger parameters x_i, we can write the integrand of Eq. (2.1) in the form

$$
\frac{1}{A_1^{\nu_1} \dots A_7^{\nu_7}} = \int \mathcal{D}x \;\exp\left(\sum_{i=1}^{7} x_i A_i\right), \tag{3.1}
$$

where

$$
\int \mathcal{D}x = \prod_{i=1}^{7} \frac{(-1)^{\nu_i}}{\Gamma(\nu_i)} \int_0^\infty dx_i\, x_i^{\nu_i - 1}, \tag{3.2}
$$

$$
\sum_{i=1}^{7} x_i A_i = a\,k^2 + b\,l^2 + 2\,c\,k\cdot l + 2\,d\cdot k + 2\,e\cdot l + f, \tag{3.3}
$$

and

$$
\begin{aligned}
a &= x_1 + x_2 + x_3 + x_6 + x_7 \\
b &= x_1 + x_2 + x_3 + x_5 + x_4 \\
c &= x_1 + x_2 + x_3 \\
d^\mu &= x_1 p_{34}^\mu + x_2 p_{134}^\mu + x_7 p_4^\mu \\
e^\mu &= x_1 p_{34}^\mu + x_2 p_{134}^\mu + x_5 p_3^\mu \\
f &= x_1\, s.
\end{aligned}
\tag{3.4}
$$

We can diagonalize the exponent with the change of variables

$$k^\mu \rightarrow \left(K - \frac{cL}{a} + \mathcal{X}\right)^\mu, \tag{3.5}$$

$$l^\mu \rightarrow (L + \mathcal{Y})^\mu, \tag{3.6}$$

where

$$\mathcal{X}^\mu = \frac{1}{\mathcal{P}}\Big\{ - \left[(x_2 + x_1)(x_7 + x_5 + x_4) + (x_5 + x_4 + x_3)x_7\right] p_4^\mu + \left[x_3x_5 - x_4(x_2 + x_1)\right] p_3^\mu - x_2(x_5 + x_4)p_1^\mu\Big\}, \tag{3.7}$$

$$\mathcal{Y}^\mu = \frac{1}{\mathcal{P}}\Big\{ \left[x_3x_7 - x_6(x_2 + x_1)\right] p_4^\mu - \left[(x_2 + x_1)(x_7 + x_6 + x_5) + (x_7 + x_6 + x_3)x_5\right] p_3^\mu - x_2(x_7 + x_6)p_1^\mu\Big\}, \tag{3.8}$$

and

$$\mathcal{P} = (x_7 + x_6 + x_5 + x_4)\,(x_3 + x_2 + x_1) + (x_5 + x_4)\,(x_7 + x_6)\,. \tag{3.9}$$

The generic tensor integral (dropping all the dependences on ν_i and on the external scales) becomes

$$\begin{aligned}\mathrm{Xbox}^D[k^{\mu_1}\ldots k^{\mu_m} l^{\nu_1}\ldots l^{\nu_n}] &= \int \mathcal{D}x \int \frac{d^DK}{i\pi^{D/2}} \int \frac{d^DL}{i\pi^{D/2}} \\ &\times \left(K - \frac{cL}{a} + \mathcal{X}\right)^{\mu_1} \ldots \left(K - \frac{cL}{a} + \mathcal{X}\right)^{\mu_m} (L+\mathcal{Y})^{\nu_1}\ldots(L+\mathcal{Y})^{\nu_n} \\ &\times \exp\left(aK^2 + \frac{\mathcal{P}}{a}L^2 + \frac{\mathcal{Q}}{\mathcal{P}}\right),\end{aligned} \tag{3.10}$$

where

$$\mathcal{Q} = x_2(x_5x_6 - x_4x_7)t + \left[x_1x_3(x_7 + x_6 + x_5 + x_4) + x_3x_5x_7 - x_2x_4x_7 + x_1x_4x_6\right] s. \tag{3.11}$$

The integration over the loop momenta K and L is now straightforward. For example, the scalar integral and the two one-index tensor integrals are given by

$$\mathrm{Xbox}^D\,(\nu_1,\nu_2,\nu_3,\nu_4,\nu_5,\nu_6,\nu_7;s,t) = \int \mathcal{D}x\ \mathcal{I}, \tag{3.12}$$

$$\mathrm{Xbox}^D\,(\nu_1,\nu_2,\nu_3,\nu_4,\nu_5,\nu_6,\nu_7;s,t)\,[k^\mu] = \int \mathcal{D}x\ \mathcal{X}^\mu\mathcal{I}, \tag{3.13}$$

$$\mathrm{Xbox}^D\,(\nu_1,\nu_2,\nu_3,\nu_4,\nu_5,\nu_6,\nu_7;s,t)\,[l^\mu] = \int \mathcal{D}x\ \mathcal{Y}^\mu\mathcal{I}, \tag{3.14}$$

where

$$\mathcal{I} = \frac{1}{\mathcal{P}^{D/2}}\ \exp\left(\frac{\mathcal{Q}}{\mathcal{P}}\right). \tag{3.15}$$

Recalling the definition (3.2), we see that we can absorb the factors x_i of $\mathcal{X}^\mu$ and $\mathcal{Y}^\mu$ into $\mathcal{D}x$, increasing the power of the i-th propagator by one

$$\frac{(-1)^{\nu_i} x_i^{\nu_i-1}}{\Gamma(\nu_i)}\, x_i \implies -\nu_i \frac{(-1)^{\nu_i+1} x_i^{\nu_i}}{\Gamma(\nu_i+1)} \equiv -\nu_i \mathbf{i}^+, \tag{3.16}$$

where

$$\mathbf{i}^\pm \mathrm{Xbox}^D\left(\ldots,\nu_i,\ldots\right) = \mathrm{Xbox}^D\left(\ldots,\nu_i\pm 1,\ldots\right), \tag{3.17}$$

while the factor $\mathcal{P}$ can be absorbed into $\mathcal{I}$ (see Eq. (3.15))

$$\frac{1}{\mathcal{P}^{D/2}}\,\frac{1}{\mathcal{P}} \implies \frac{1}{\mathcal{P}^{(D+2)/2}}, \tag{3.18}$$

so that $1/\mathcal{P}$ acts as a dimension increaser

$$\frac{1}{\mathcal{P}} \implies \mathbf{d}^+, \qquad \mathcal{P} \implies \mathbf{d}^-, \tag{3.19}$$

where

$$\mathbf{d}^\pm \mathrm{Xbox}^D = \mathrm{Xbox}^{D\pm 2}. \tag{3.20}$$

For example, using the expression for $\mathcal{X}^\mu$ of Eq. (3.7), we can write

$$\begin{aligned}\mathrm{Xbox}^D[k^\mu] \;=\; & \Big\{ \Big[\nu_3\mathbf{3}^+\nu_5\mathbf{5}^+ - \nu_4\mathbf{4}^+\left(\nu_2\mathbf{2}^+ + \nu_1\mathbf{1}^+\right)\Big] p_3^\mu - \nu_2\mathbf{2}^+\left(\nu_5\mathbf{5}^+ + \nu_4\mathbf{4}^+\right) p_1^\mu \\ & - \Big[\left(\nu_2\mathbf{2}^+ + \nu_1\mathbf{1}^+\right)\left(\nu_7\mathbf{7}^+ + \nu_5\mathbf{5}^+ + \nu_4\mathbf{4}^+\right) \\ & + \left(\nu_5\mathbf{5}^+ + \nu_4\mathbf{4}^+ + \nu_3\mathbf{3}^+\right)\nu_7\mathbf{7}^+\Big] p_4^\mu \Big\}\, \mathrm{Xbox}^{D+2}[1]. \end{aligned} \tag{3.21}$$

The task to compute tensor integrals has then been moved to the computation of scalar integrals with higher powers of the propagators in higher dimensions.

In the next sections, we will follow the procedure already used in Ref. [3]:

1) we first reduce the powers of the propagators of the generic scalar diagram and we express it in terms of a finite set (basis) of master diagrams;

2) then we relate the master integrals in higher dimensions to the master integrals in D dimensions (dimensional shift).

3.1 The scalar crossed-box reduction

The strategy to reduce the generic scalar integral to a linear combination of master ones is based on identities that relate scalar integrals with different powers of propagators. Some of these identities can be obtained using the integration-by-parts method [8]. Since there are three external independent momenta and two loop momenta, we can build ten identities, imposing that

$$\int \frac{d^D k}{i\pi^{D/2}} \int \frac{d^D l}{i\pi^{D/2}} \, \frac{\partial}{\partial a^\mu} \left[b^\mu \, f(k,l,p_i) \right] = 0, \tag{3.22}$$

where

$$a^\mu = k^\mu, \, l^\mu \tag{3.23}$$

$$b^\mu = k^\mu, \, l^\mu, \, p_1^\mu, \, p_3^\mu, \, p_4^\mu \tag{3.24}$$

$$f(k,l,p_i) = \frac{1}{A_1^{\nu_1} A_2^{\nu_2} A_3^{\nu_3} A_4^{\nu_4} A_5^{\nu_5} A_6^{\nu_6} A_7^{\nu_7}}. \tag{3.25}$$

Not all the scalar products that appear in the application of the integration-by-parts identities can be written in terms of combinations of propagators: with the two loop momenta and with the three linearly independent external ones, we can form nine scalar products involving k and l. Since we have seven linearly independent propagators, we are left with two irreducible scalar products in the numerator, that we choose to be $(l \cdot p_1)$ and $(l \cdot p_4)$.

Three more identities among the scalar integrals can be derived if we exploit the Lorentz invariance of the Feynman diagrams [9]. In fact, since the Feynman integral is a function only of scalar products of the external momenta, it is invariant under the (infinitesimal) rotation

$$p_i'^\mu = \Lambda^\mu{}_\nu \, p_i^\nu, \qquad \Lambda_{\mu\nu} = g_{\mu\nu} + \delta \, \epsilon_{\mu\nu}, \qquad \epsilon_{\mu\nu} = -\epsilon_{\nu\mu}. \tag{3.26}$$

We can then write

$$\int \frac{d^D k}{i\pi^{D/2}} \int \frac{d^D l}{i\pi^{D/2}} \, f(k,l,p_i) = \int \frac{d^D k}{i\pi^{D/2}} \int \frac{d^D l}{i\pi^{D/2}} \, f(k,l,p_i'). \tag{3.27}$$

Expanding in a Taylor series in δ the right-hand side of Eq. (3.27), we obtain

$$\int \frac{d^D k}{i\pi^{D/2}} \int \frac{d^D l}{i\pi^{D/2}} \sum_{j=1}^{3} \frac{\partial f(k,l,p_i)}{\partial p_j^\mu} \, \epsilon^\mu{}_\nu \, p_j^\nu = 0. \tag{3.28}$$

Using the three independent external momenta, we can build three independent antisymmetric tensors

$$\begin{aligned}
\epsilon_1^{\mu\nu} &= p_1^\mu \, p_2^\nu - p_2^\mu \, p_1^\nu, \\
\epsilon_2^{\mu\nu} &= p_1^\mu \, p_3^\nu - p_3^\mu \, p_1^\nu, \\
\epsilon_3^{\mu\nu} &= p_2^\mu \, p_3^\nu - p_3^\mu \, p_2^\nu,
\end{aligned}$$

that, once inserted into Eq. (3.28), give rise to three more identities.

Taking linear combinations of these thirteen equations, we can build the following system

$$s\nu_1 \mathbf{1}^+ - \nu_7 \mathbf{7}^+ \mathbf{6}^- - \nu_5 \mathbf{5}^+ \mathbf{4}^- - \left(\nu_2 \mathbf{2}^+ + \nu_1 \mathbf{1}^+ \right) \mathbf{3}^- - \nu_{1257} - 2\nu_{346} + 2D = 0 \tag{3.29}$$

$$s\nu_3\mathbf{3}^+ - \nu_6\mathbf{6}^+\mathbf{7}^- - \nu_4\mathbf{4}^+\mathbf{5}^- - \left(\nu_3\mathbf{3}^+ + \nu_2\mathbf{2}^+\right)\mathbf{1}^- - \nu_{2346} - 2\nu_{157} + 2D = 0 \tag{3.30}$$

$$\begin{aligned} &2\left(l\cdot p_4\right)\nu_4\mathbf{4}^+ - \left(\nu_6\mathbf{6}^+ + \nu_5\mathbf{5}^+\right)\mathbf{7}^- + \nu_5\mathbf{5}^+\mathbf{1}^- + \nu_4\mathbf{4}^+\left(\mathbf{3}^- - \mathbf{6}^-\right) \\ &\qquad - \nu_{456} - 2\nu_7 + D = 0 \end{aligned} \tag{3.31}$$

$$\begin{aligned} &2\left(l\cdot p_4\right)\nu_5\mathbf{5}^+ + \nu_7\mathbf{7}^+\mathbf{6}^- + \nu_5\mathbf{5}^+\left(\mathbf{7}^- - \mathbf{1}^- + s\right) + \nu_4\mathbf{4}^+\left(\mathbf{6}^- - \mathbf{3}^-\right) \\ &\qquad + \nu_{457} + 2\nu_6 - D = 0 \end{aligned} \tag{3.32}$$

$$\begin{aligned} &2\left(l\cdot p_4\right)\nu_6\mathbf{6}^+ + \nu_7\mathbf{7}^+\left(\mathbf{5}^- - \mathbf{1}^-\right) + \nu_6\mathbf{6}^+\left(\mathbf{7}^- + \mathbf{5}^- - \mathbf{1}^- + s\right) + \nu_4\mathbf{4}^+\mathbf{5}^- \\ &\qquad + \nu_{47} + 2\nu_5 - D = 0 \end{aligned} \tag{3.33}$$

$$\begin{aligned} &2\left(l\cdot p_4\right)\nu_7\mathbf{7}^+ + \nu_6\mathbf{6}^+\left(\mathbf{3}^- - \mathbf{4}^-\right) - \nu_7\mathbf{7}^+\left(\mathbf{6}^- + \mathbf{4}^- - \mathbf{3}^-\right) - \nu_5\mathbf{5}^+\mathbf{4}^- \\ &\qquad - \nu_{56} - 2\nu_4 + D = 0 \end{aligned} \tag{3.34}$$

$$\begin{aligned} &2s\left(l\cdot p_4\right)\nu_2\mathbf{2}^+ + 4s\left(l\cdot p_4\right)\nu_3\mathbf{3}^+ + \nu_2\mathbf{2}^+\left[(t+s)\left(s + \mathbf{1}^-\right) - t\mathbf{3}^-\right] \\ &\qquad + s\left(2\nu_6\mathbf{6}^+ + 2\nu_3\mathbf{3}^+ + \nu_2\mathbf{2}^+\right)\mathbf{7}^- - s\left(2\nu_3\mathbf{3}^+ + \nu_2\mathbf{2}^+\right)\mathbf{6}^- \\ &\qquad - \left(2D - 2\nu_{13457} - \nu_2\right)s = 0 \end{aligned} \tag{3.35}$$

$$\begin{aligned} &2s\left(l\cdot p_4\right)\nu_3\mathbf{3}^+ - 2s\left(l\cdot p_4\right)\nu_1\mathbf{1}^+ + (t+s)\nu_2\mathbf{2}^+\left(\mathbf{1}^- - \mathbf{3}^-\right) + s\left(\nu_1\mathbf{1}^+ - \nu_3\mathbf{3}^+\right)\mathbf{6}^- \\ &\qquad + s\left(\nu_6\mathbf{6}^+ + \nu_3\mathbf{3}^+ - \nu_1\mathbf{1}^+\right)\mathbf{7}^- - s\nu_5\mathbf{5}^+\mathbf{4}^- - s\nu_1\mathbf{1}^+\mathbf{3}^- \\ &\qquad - \left(\nu_6 - \nu_{15}\right)s = 0 \end{aligned} \tag{3.36}$$

$$\begin{aligned} &2\left(l\cdot p_4\right)\nu_3\mathbf{3}^+ - 2\left(l\cdot p_1\right)\nu_2\mathbf{2}^+ + (t+s)\,\nu_2\mathbf{2}^+ + \left(\nu_6\mathbf{6}^+ + \nu_3\mathbf{3}^+ + \nu_2\mathbf{2}^+ + \nu_1\mathbf{1}^+\right)\mathbf{7}^- \\ &\qquad - \left(\nu_2\mathbf{2}^+ + \nu_1\mathbf{1}^+\right)\mathbf{5}^- - \nu_3\mathbf{3}^+\mathbf{4}^- + \nu_{1236} + 2\nu_7 - D = 0 \end{aligned} \tag{3.37}$$

$$\begin{aligned} &2s\left(l\cdot p_4\right)\nu_3\mathbf{3}^+ - 2s\left(l\cdot p_1\right)\nu_7\mathbf{7}^+ + \nu_7\mathbf{7}^+\left[(t+s)\left(s - \mathbf{1}^-\right) + t\mathbf{5}^- + s\mathbf{2}^-\right] \\ &\qquad + \nu_6\mathbf{6}^+\left(t\mathbf{4}^- - t\mathbf{3}^- + s\mathbf{7}^-\right) + s\nu_3\mathbf{3}^+\left(\mathbf{7}^- - \mathbf{6}^-\right) + (t+s)\nu_2\mathbf{2}^+\mathbf{1}^- + s\nu_1\mathbf{1}^+\mathbf{2}^- \\ &\qquad + s\left(\nu_{126} + 2\nu_{3457} - 2D\right) - t\left(D - \nu_{67} + \nu_2 - 2\nu_{45}\right) = 0 \end{aligned} \tag{3.38}$$

$$\begin{aligned} &2s\left(l\cdot p_4\right)\nu_3\mathbf{3}^+ - 2s\left(l\cdot p_1\right)\nu_5\mathbf{5}^+ + (t+s)\,\nu_5\mathbf{5}^+\left(\mathbf{7}^- - \mathbf{1}^- + s\right) + (t+s)\nu_2\mathbf{2}^+\mathbf{1}^- \\ &\qquad + (t+s)\nu_4\mathbf{4}^+\left(\mathbf{6}^- - \mathbf{3}^-\right) + s\left(\nu_6\mathbf{6}^+ + \nu_3\mathbf{3}^+\right)\mathbf{7}^- - s\nu_3\mathbf{3}^+\mathbf{6}^- - s\nu_1\mathbf{1}^+\mathbf{2}^- \\ &\qquad + s\left(\nu_{1456} - \nu_2 + 2\nu_7 - D\right) + t\left(2\nu_{67} + \nu_{45} - \nu_2 - D\right) = 0 \end{aligned} \tag{3.39}$$

$$\begin{aligned} &2s\left(l\cdot p_4\right)\nu_3\mathbf{3}^+ + 2s\left(l\cdot p_1\right)\nu_6\mathbf{6}^+ + \nu_6\mathbf{6}^+\left[t\left(\mathbf{4}^- - \mathbf{3}^-\right) - s\left(\mathbf{2}^- - \mathbf{1}^- + s\right)\right] - s\nu_4\mathbf{4}^+\mathbf{5}^- \\ &\qquad + s\nu_3\mathbf{3}^+\left(\mathbf{7}^- - \mathbf{6}^- - \mathbf{2}^-\right) + t\nu_7\mathbf{7}^+\left(\mathbf{5}^- - \mathbf{1}^-\right) + (t+s)\nu_2\mathbf{2}^+\mathbf{1}^- \\ &\qquad + s\left(\nu_{34} - \nu_2\right) + t\left(\nu_{67} + 2\nu_{45} - \nu_2 - D\right) = 0 \end{aligned} \tag{3.40}$$

$$2s\left(l\cdot p_4\right)\nu_3\mathbf{3}^+ + 2s\left(l\cdot p_1\right)\nu_4\mathbf{4}^+ + \nu_4\mathbf{4}^+\left[(t+s)\left(\mathbf{6}^- - \mathbf{3}^-\right) + s\mathbf{5}^-\right] + 2s\nu_6\mathbf{6}^+\mathbf{7}^-$$

$$
\begin{aligned}
&+(t+s)\nu_5\mathbf{5}^+\left(\mathbf{7}^--\mathbf{1}^-\right)+(t+s)\nu_2\mathbf{2}^+\mathbf{1}^-+s\nu_3\mathbf{3}^+\left(\mathbf{7}^--\mathbf{6}^-+\mathbf{2}^-\right)\\
&+s\left(\nu_{23}+2\nu_{146}+3\nu_5+4\nu_7-3D\right)+t\left(\nu_{45}-\nu_2+2\nu_{67}-D\right)=0, \qquad (3.41)
\end{aligned}
$$

where we have introduced the shorthand $\nu_{ij}=\nu_i+\nu_j$, $\nu_{ijk}=\nu_i+\nu_j+\nu_k$, etc. Each equation acts on the integrand of the generic crossed box before any loop integration has taken place.

Equations (3.29) and (3.30) of the system, being independent of the two irreducible scalar products, need no further manipulation, and can be rewritten in the form

$$
\begin{aligned}
s\nu_1\mathbf{1}^+ &= \nu_7\mathbf{7}^+\mathbf{6}^-+\nu_5\mathbf{5}^+\mathbf{4}^-+\left(\nu_2\mathbf{2}^++\nu_1\mathbf{1}^+\right)\mathbf{3}^-+\nu_{1257}+2\nu_{346}-2D, \qquad (3.42)\\
s\nu_3\mathbf{3}^+ &= \nu_6\mathbf{6}^+\mathbf{7}^-+\nu_4\mathbf{4}^+\mathbf{5}^-+\left(\nu_3\mathbf{3}^++\nu_2\mathbf{2}^+\right)\mathbf{1}^-+\nu_{2346}+2\nu_{157}-2D. \qquad (3.43)
\end{aligned}
$$

By repeated application of these two identities, we can reduce ν_1 and ν_3 to one. During this process, the generic scalar box is expressed as a linear combination of crossed-box diagrams with $\nu_1=\nu_3=1$ and diagrams belonging to simpler topologies, that originate when powers of propagators are reduced (pinched) to zero by the decreasing operators. We will deal with the pinched diagrams later, concentrating now on the reduction of the remaining propagators.

In order to use the other equations of the system, we have to eliminate the irreducible scalar products in the numerator.

For example, applying the operator $\nu_7\mathbf{7}^+$ to Eq. (3.31) and $\nu_4\mathbf{4}^+$ to Eq. (3.34), and taking the difference, we get

$$
\begin{aligned}
&\left(D-2\nu_7-\nu_{56}-2\right)\nu_7\mathbf{7}^+-\left(D-\nu_{56}-2\nu_4-2\right)\nu_4\mathbf{4}^+-\left(\nu_7-\nu_4\right)\left(\nu_5\mathbf{5}^++\nu_6\mathbf{6}^+\right)\\
&\qquad+\nu_5\mathbf{5}^+\nu_7\mathbf{7}^+\mathbf{1}^--\nu_4\mathbf{4}^+\nu_6\mathbf{6}^+\mathbf{3}^-=0. \qquad (3.44)
\end{aligned}
$$

In the same way, we can apply $\nu_6\mathbf{6}^+$ to Eq. (3.32) and $\nu_5\mathbf{5}^+$ to Eq. (3.33) and take the difference, to obtain

$$
\begin{aligned}
&\left(D-\nu_{47}-2\nu_5-2\right)\nu_5\mathbf{5}^+-\left(D-\nu_{47}-2\nu_6-2\right)\nu_6\mathbf{6}^++\left(\nu_6-\nu_5\right)\left(\nu_4\mathbf{4}^++\nu_7\mathbf{7}^+\right)\\
&\qquad+\nu_5\mathbf{5}^+\nu_7\mathbf{7}^+\mathbf{1}^--\nu_4\mathbf{4}^+\nu_6\mathbf{6}^+\mathbf{3}^-=0. \qquad (3.45)
\end{aligned}
$$

Combining Eq. (3.44) and (3.45) to eliminate $\nu_5\mathbf{5}^+$, we have

$$
\begin{aligned}
\nu_4\mathbf{4}^+ &= \frac{(D-2\nu_{57}-2)}{(D-2\nu_{45}-2)}\nu_7\mathbf{7}^+-2\frac{(\nu_7-\nu_4)}{(D-2\nu_{45}-2)}\nu_6\mathbf{6}^+\\
&\quad+\frac{1}{(D-\nu_{4567}-2)}\left(\nu_5\mathbf{5}^+\nu_7\mathbf{7}^+\mathbf{1}^--\nu_4\mathbf{4}^+\nu_6\mathbf{6}^+\mathbf{3}^-\right), \qquad (3.46)
\end{aligned}
$$

that can be used to reduce ν_4 to one, at the expense of increasing ν_6 and ν_7. If, on the other hand, we eliminate $\nu_4\mathbf{4}^+$, we obtain the symmetric equation that can reduce ν_5 to one. At this point, all the powers of the propagators except ν_2, ν_6 and ν_7 have been reduced to one.

In the same spirit we can derive

$$
\begin{aligned}
st\nu_2\mathbf{2}^+\nu_6\mathbf{6}^+ \;=\;& (2\nu_{1567}+\nu_2+2-2D)\,s\nu_6\mathbf{6}^+ + (\nu_{467}+2\nu_5-D)\,s\nu_2\mathbf{2}^+ \\
&- 2\,(D-\nu_{467}-2\nu_5)\left(\nu_6\mathbf{6}^+\mathbf{7}^- + \nu_4\mathbf{4}^+\mathbf{5}^- + \nu_3\mathbf{3}^+\mathbf{1}^- + \nu_2\mathbf{2}^+\mathbf{1}^-\right) \\
&+ s\left(2\nu_3\mathbf{3}^+ + \nu_2\mathbf{2}^+\right)\left[\left(\nu_7\mathbf{7}^+ + \nu_6\mathbf{6}^+ + \nu_4\mathbf{4}^+\right)\mathbf{5}^- - \nu_7\mathbf{7}^+\mathbf{1}^-\right] \\
&+ t\nu_2\mathbf{2}^+\nu_6\mathbf{6}^+\left(\mathbf{3}^- - \mathbf{1}^-\right) + 2s\nu_4\mathbf{4}^+\nu_6\mathbf{6}^+\mathbf{5}^- \\
&+ 2\,(D-\nu_{467}-2\nu_5)\,(2D-2\nu_{157}-\nu_{2346})\,, \qquad (3.47)
\end{aligned}
$$

that, together with the symmetric one for $\nu_2\mathbf{2}^+\nu_7\mathbf{7}^+$ and with

$$
\begin{aligned}
s\nu_6\mathbf{6}^+\nu_7\mathbf{7}^+ \;=\;& (D-\nu_{567}-2\nu_4-1)\,\nu_6\mathbf{6}^+ + (D-\nu_{467}-2\nu_5-1)\,\nu_7\mathbf{7}^+ \\
&+ \nu_6\mathbf{6}^+\nu_7\mathbf{7}^+\left(\mathbf{3}^- + \mathbf{1}^- - \mathbf{5}^- - \mathbf{4}^-\right) - \nu_4\mathbf{4}^+\nu_7\mathbf{7}^+\mathbf{5}^- - \nu_5\mathbf{5}^+\nu_6\mathbf{6}^+\mathbf{4}^- \\
&+ \nu_7(\nu_7+1)\mathbf{7}^{++}\left(\mathbf{1}^- - \mathbf{5}^-\right) + \nu_6(\nu_6+1)\mathbf{6}^{++}\left(\mathbf{3}^- - \mathbf{4}^-\right), \qquad (3.48)
\end{aligned}
$$

reduces all powers except one (ν_2 or ν_6 or ν_7) to unity.

We can decrease ν_2 at the expense of increasing ν_6 and ν_7 using

$$
\begin{aligned}
&\left[(\nu_4-\nu_7+2\nu_{23}+2-D)\,s + (\nu_{45}-\nu_{67})\,t\right]\nu_2\mathbf{2}^+ = (\nu_5-\nu_3)\,s\nu_4\mathbf{4}^+ + (\nu_7-\nu_3)\,s\nu_6\mathbf{6}^+ \\
&\quad - (D-2\nu_7-\nu_{16}-2)\,s\nu_7\mathbf{7}^+ - (D-2\nu_5-\nu_{14}-2)\,s\nu_5\mathbf{5}^+ \\
&\quad + (t+s)\nu_2\mathbf{2}^+\left[\nu_7\mathbf{7}^+\left(\mathbf{1}^- - \mathbf{5}^-\right) + \nu_4\mathbf{4}^+\left(\mathbf{6}^- - \mathbf{3}^-\right)\right] \\
&\quad + t\nu_2\mathbf{2}^+\left[\nu_5\mathbf{5}^+\left(\mathbf{7}^- - \mathbf{1}^-\right) + \nu_6\mathbf{6}^+\left(\mathbf{3}^- - \mathbf{4}^-\right)\right] \\
&\quad - s\nu_1\mathbf{1}^+\left(\nu_7\mathbf{7}^+\mathbf{5}^- + \nu_5\mathbf{5}^+\mathbf{7}^-\right) \\
&\quad + s\nu_3\mathbf{3}^+\left[\left(2\nu_7\mathbf{7}^+ + \nu_4\mathbf{4}^+\right)\mathbf{6}^- + \left(2\nu_5\mathbf{5}^+ + \nu_6\mathbf{6}^+\right)\mathbf{4}^-\right] \\
&\quad - (2D-2\nu_{57}-3\nu_{46})\left[\nu_6\mathbf{6}^+\mathbf{7}^- + \nu_4\mathbf{4}^+\mathbf{5}^- + \left(\nu_3\mathbf{3}^+ + \nu_2\mathbf{2}^+\right)\mathbf{1}^-\right] \\
&\quad + (\nu_{57}-2\nu_2)\left[\nu_7\mathbf{7}^+\mathbf{6}^- + \nu_5\mathbf{5}^+\mathbf{4}^- + \left(\nu_2\mathbf{2}^+ + \nu_1\mathbf{1}^+\right)\mathbf{3}^-\right] \\
&\quad + 4D^2 - 2\,(5\nu_{57}+4\nu_{46}+\nu_3-\nu_2+2\nu_1)\,D + \nu_7\,(5\nu_{17}+10\nu_{456}+4\nu_3+\nu_2) \\
&\quad + \nu_6\,(3\nu_6+10\nu_5+6\nu_{14}+3\nu_3-\nu_2) + \nu_5\,(5\nu_5+10\nu_4+4\nu_3+\nu_2+5\nu_1) \\
&\quad + \nu_4\,(3\nu_4+3\nu_3-\nu_2+6\nu_1) - 2\nu_2\,(2\nu_3+\nu_{12})\,. \qquad (3.49)
\end{aligned}
$$

The power of the seventh propagator can be reduced with

$$
s(t+s)\nu_7(\nu_7+1)\mathbf{7}^{++} = \sigma\nu_7\mathbf{7}^+ - (\nu_7-1)\frac{(D-6)\,t + (5D-2\nu_7-26)\,s}{D-2\nu_7-6}\mathbf{6}^+
$$

$$
\begin{aligned}
&+\rho\frac{(2D-\nu_7-11)\,t+(3D-2\nu_7-15)\,s}{D-\nu_7-5}\mathbf{5}^+\nu_7\mathbf{7}^+\mathbf{1}^-\\
&+\rho\frac{(D-2\nu_7-4)\,t+(5-D)\,s}{D-\nu_7-5}\mathbf{4}^+\mathbf{6}^+\mathbf{3}^- +\rho(t+s)\Big\{-\nu_7(\nu_7+1)\mathbf{7}^{++}\mathbf{6}^-\\
&+\mathbf{5}^+\nu_7\mathbf{7}^+\left(\mathbf{3}^- -\mathbf{4}^- -\mathbf{6}^-\right)+\mathbf{4}^+\nu_7\mathbf{7}^+\left[2\left(\mathbf{3}^- -\mathbf{6}^-\right)+\mathbf{1}^-\right]\\
&-\mathbf{2}^+\left(\nu_7\mathbf{7}^+ +\mathbf{4}^+\right)\mathbf{1}^- +\mathbf{5}^+\mathbf{6}^+\mathbf{3}^- -2\,\mathbf{5}^{++}\mathbf{4}^-\Big\}\\
&-2\nu_7\frac{t+s}{D-2\nu_7-6}\left(\mathbf{4}^+\nu_7\mathbf{7}^+\mathbf{5}^- +\mathbf{5}^+\mathbf{6}^+\mathbf{4}^-\right)+(t+s)\Big[2\,\mathbf{6}^{++}\left(\mathbf{4}^- -\mathbf{3}^-\right)\\
&+\nu_7(\nu_7+1)\mathbf{7}^{++}\left(\mathbf{5}^- -\mathbf{1}^-\right)+\mathbf{6}^+\nu_7\mathbf{7}^+\left(\mathbf{5}^- +\mathbf{4}^- -\mathbf{3}^- -\mathbf{1}^-\right)\Big]\\
&+\rho s\Big[\mathbf{3}^+\mathbf{5}^+\left(\mathbf{1}^- -\mathbf{4}^- -\mathbf{7}^-\right)-\mathbf{3}^+\mathbf{6}^+\left(\mathbf{4}^- +\mathbf{7}^-\right)-\mathbf{3}^+\nu_7\mathbf{7}^+\left(\mathbf{4}^- +\mathbf{2}^-\right)\\
&-\mathbf{4}^+\left(\nu_7\mathbf{7}^+\mathbf{2}^- +\mathbf{6}^+\mathbf{7}^- +\mathbf{1}^+\mathbf{2}^- +\mathbf{3}^+\mathbf{7}^-\right)\Big]\\
&+(2D-3\nu_7-7)\rho\Big[\mathbf{6}^+\mathbf{7}^- +\mathbf{4}^+\mathbf{5}^- +\left(\mathbf{3}^+ +\mathbf{2}^+\right)\mathbf{1}^- -2(D-\nu_7-4)\Big], \qquad (3.50)
\end{aligned}
$$

where we have introduced the shorthands

$$
\begin{aligned}
\rho &= \frac{D-6}{D-2\nu_7-6}\\
\sigma &= \frac{(5D^2-8\nu_7 D-50D+2\nu_7^2+42\nu_7+124)\,s+(2D^2-3\nu_7 D-21D+18\nu_7+54)\,t}{D-2\nu_7-6}.
\end{aligned}
$$

Equation (3.50) is not as general as the previous ones since we have set all the powers of the other propagators to unity. In addition, since this equation contains $\mathbf{7}^{++}$, we cannot always reduce ν_7 to one, but we also have integrals where $\nu_7=2$. A similar identity can be obtained by symmetry for $\mathbf{6}^{++}$, so that we are left with three integrals: $\mathrm{Xbox}^D(1,1,1,1,1,1,1;s,t)$, $\mathrm{Xbox}^D(1,1,1,1,1,1,2;s,t)$ and $\mathrm{Xbox}^D(1,1,1,1,1,2,1;s,t)$.

The last step is to write the integral with $\nu_6=2$ as a combination of the other two. This can be done with the identity that links $\mathbf{6}^+$ with $\mathbf{7}^+$. We derived such an identity, equating the expressions obtained by acting with $\nu_7\mathbf{7}^+$ on $\nu_2\mathbf{2}^+\nu_4\mathbf{4}^+$ and by acting with $\nu_2\mathbf{2}^+$ on $\nu_4\mathbf{4}^+\nu_7\mathbf{7}^+$

$$
\begin{aligned}
(D-6)(D-5)\frac{t}{t+s}\mathbf{6}^+ &= (D-6)(D-5)\mathbf{7}^+ -4\frac{(D-5)^3}{t+s}\\
&-\left(\mathbf{5}^+\mathbf{7}^+ +\mathbf{1}^+\mathbf{7}^+ +\mathbf{1}^+\mathbf{5}^+ +\mathbf{1}^+\mathbf{4}^+\right)\mathbf{6}^- -\left(\mathbf{3}^+\mathbf{7}^+ +\mathbf{4}^+\mathbf{6}^+ +\mathbf{3}^+\mathbf{6}^+ +\mathbf{3}^+\mathbf{4}^+\right)\mathbf{5}^-\\
&+\left(\mathbf{3}^+\mathbf{4}^+ +\mathbf{4}^+\mathbf{6}^+ +\mathbf{5}^+\mathbf{6}^+ +\mathbf{3}^+\mathbf{6}^+\right)\mathbf{1}^- -(D-7)\mathbf{4}^+\mathbf{7}^+\mathbf{1}^-\\
&-\frac{1}{2}\left(\mathbf{2}^+\mathbf{7}^+\mathbf{6}^- +\mathbf{2}^+\mathbf{5}^+\mathbf{6}^- +\mathbf{4}^+\mathbf{7}^+\mathbf{5}^- +\mathbf{2}^+\mathbf{6}^+\mathbf{5}^- -\mathbf{1}^+\mathbf{4}^+\mathbf{3}^-\right)\\
&-\frac{t+s}{2}\left[\mathbf{2}^+\mathbf{4}^+\mathbf{7}^+\left(\mathbf{6}^- +\mathbf{5}^-\right)+2\left(\mathbf{2}^{++}\mathbf{7}^+ -\mathbf{2}^+\mathbf{4}^+\mathbf{7}^+ +\mathbf{2}^{++}\mathbf{4}^+\right)\mathbf{1}^-\right]
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{s}{2}\Big\{\mathbf{2}^+\Big[\mathbf{3}^+\left(\mathbf{6}^+ + \mathbf{5}^+ + \mathbf{4}^+\right)\mathbf{7}^- + \mathbf{4}^+\mathbf{6}^+\mathbf{7}^- + \mathbf{3}^+\left(\mathbf{7}^+ + \mathbf{6}^+ + \mathbf{5}^+\right)\mathbf{4}^-\Big]\\
&+\mathbf{3}^+\mathbf{5}^+\left(2\,\mathbf{7}^+ - \mathbf{2}^+\right)\mathbf{1}^-\Big\} - (D-6)\frac{(t+2s)}{2(t+s)}\Big[\mathbf{2}^+\left(\mathbf{7}^+\mathbf{4}^- + \mathbf{4}^+\mathbf{7}^- + \mathbf{5}^+\mathbf{4}^-\right)\\
&+2\Big(\mathbf{1}^+\mathbf{7}^+ + \mathbf{5}^+\mathbf{7}^+ + \mathbf{1}^+\mathbf{6}^+ + \mathbf{1}^+\mathbf{5}^+\Big)\mathbf{4}^- - 2\,\mathbf{1}^+\mathbf{6}^+\mathbf{3}^-\Big] - \frac{2(D-5)t+s}{2s}\mathbf{2}^+\mathbf{7}^+\mathbf{5}^-\\
&-(D-6)\Big[\mathbf{4}^+\mathbf{7}^+\mathbf{3}^- + \Big(\mathbf{3}^+\mathbf{4}^+ + \mathbf{4}^+\mathbf{6}^+ + \mathbf{3}^+\mathbf{6}^+ + \mathbf{3}^+\mathbf{5}^+\Big)\mathbf{7}^-\Big]\\
&+(D-6)\frac{s}{t+s}\Big[\mathbf{7}^+\mathbf{3}^+\mathbf{4}^- + \Big(\mathbf{7}^+\mathbf{3}^+ + \mathbf{7}^+\mathbf{4}^+ + \mathbf{1}^+\mathbf{4}^+\Big)\mathbf{2}^-\Big] + \frac{2D-13}{2}\mathbf{4}^+\mathbf{7}^+\mathbf{6}^-\\
&+(D-5)\frac{(D-5)t+(2D-11)s}{s(t+s)}\Big[\Big(\mathbf{2}^+ + \mathbf{1}^+\Big)\mathbf{3}^- + \mathbf{7}^+\mathbf{6}^-\Big]\\
&+(D-5)\frac{t}{t+s}\Big[2\,\mathbf{3}^+\mathbf{7}^+\mathbf{6}^- + \mathbf{3}^+\mathbf{4}^+\mathbf{6}^- - \mathbf{1}^+\mathbf{7}^+\mathbf{5}^- - \mathbf{1}^+\mathbf{5}^+\mathbf{7}^-\Big] + \frac{t}{2s}\mathbf{2}^+\mathbf{5}^+\mathbf{3}^-\\
&-(D-5)\frac{(D-5)t-s}{s(t+s)}\Big[\mathbf{4}^+\mathbf{5}^- + \mathbf{2}^+\mathbf{1}^- + \mathbf{6}^+\mathbf{7}^- + \mathbf{3}^+\mathbf{1}^-\Big] - \frac{2D-15}{2}\mathbf{3}^+\mathbf{7}^+\mathbf{1}^-\\
&+(D-5)\Big[\mathbf{1}^-\Big(\mathbf{3}^+\mathbf{5}^+ + 2\,\mathbf{2}^{++} + \mathbf{2}^+\mathbf{3}^+\Big)\Big] + \frac{t}{2}\Big[\mathbf{2}^+\Big(\mathbf{5}^+\mathbf{7}^+\mathbf{1}^- + \mathbf{4}^+\mathbf{6}^+\mathbf{3}^-\Big)\Big]\\
&+\frac{5t-2(D-7)s}{2(t+s)}\mathbf{5}^+\mathbf{7}^+\mathbf{1}^- - \frac{(2D-9)t^2-(D-5)st-2(D-5)s^2}{2s(t+s)}\mathbf{2}^+\mathbf{5}^+\mathbf{1}^-\\
&-(D-6)\frac{t+2s}{2s}\mathbf{2}^+\mathbf{7}^+\mathbf{3}^- - \frac{2(D-5)t^2+(D-6)st+2(D-6)s^2}{2s(t+s)}\mathbf{2}^+\mathbf{6}^+\mathbf{4}^-\\
&+\frac{(3D-16)t+2(D-5)s}{2s}\mathbf{2}^+\mathbf{7}^+\mathbf{1}^- + \frac{(D-4)t+2s}{2(t+s)}\mathbf{2}^+\mathbf{6}^+\mathbf{7}^-\\
&+\frac{2(D-5)t^2-(D-6)st-2(D-6)s^2}{2s(t+s)}\mathbf{2}^+\mathbf{5}^+\mathbf{7}^- - (4D-21)\frac{t}{2(t+s)}\mathbf{4}^+\mathbf{6}^+\mathbf{3}^-\\
&+\frac{(D-5)t+(D-6)s}{t+s}\mathbf{3}^+\mathbf{6}^+\mathbf{4}^- - \frac{(D-4)t-2(D-6)s}{2s}\mathbf{2}^+\mathbf{4}^+\mathbf{3}^-\\
&+\frac{(2D-11)t^2+(D-7)st+2(D-6)s^2}{2s(t+s)}\mathbf{2}^+\mathbf{6}^+\mathbf{3}^- + \frac{t+2s}{2s}\mathbf{2}^+\mathbf{6}^+\mathbf{1}^-\\
&+\frac{2(D-5)t+(D-6)s}{t+s}\mathbf{3}^+\mathbf{5}^+\mathbf{4}^- + (D-5)\frac{(D-5)t+(2D-11)s}{s(t+s)}\mathbf{5}^+\mathbf{4}^-\\
&+\frac{2D-11}{2}\mathbf{2}^+\mathbf{4}^+\mathbf{5}^- - \frac{(D-6)t-2s}{2s}\mathbf{2}^+\mathbf{4}^+\mathbf{1}^- + \frac{2(D-5)t-s}{2s}\mathbf{2}^+\mathbf{4}^+\mathbf{6}^-, \qquad (3.51)
\end{aligned}
$$

where we have set all the powers of the propagators to unity.

At the end of this reduction program, we are left with the following two crossed-box integrals: $\mathrm{Xbox}^D(1,1,1,1,1,1,1;s,t)$ and $\mathrm{Xbox}^D(1,1,1,1,1,1,2;s,t)$, plus simpler diagrams that can always be expressed as a combination of:

- the master crossed triangle of Fig. 2 (a)

$$\text{Xtri}^{D}\left(s\right)=\text{Xbox}^{D}\left(1,0,1,1,1,1,1;s,t\right), \tag{3.52}$$

- the master diagonal box of Fig. 2 (b), produced by

$$\text{Dbox}^{D}\left(s,t\right)=\text{Xbox}^{D}\left(0,1,1,0,1,1,1;s,t\right)=\text{Xbox}^{D}\left(1,1,0,1,1,1,0;s,t\right), \tag{3.53}$$

  together with

$$\text{Dbox}^{D}\left(s,u\right)=\text{Xbox}^{D}\left(1,1,0,1,0,1,1;s,t\right)=\text{Xbox}^{D}\left(0,1,1,1,1,0,1;s,t\right), \tag{3.54}$$

  and

$$\text{Dbox}^{D}\left(t,u\right)=\text{Xbox}^{D}\left(0,1,0,1,1,1,1;s,t\right), \tag{3.55}$$

- the master box with a bubble insertion of Fig. 2 (c), produced by

$$\text{Bbox}^{D}\left(s,t\right)=\text{Xbox}^{D}\left(1,1,1,0,1,1,0;s,t\right), \tag{3.56}$$

  together with

$$\text{Bbox}^{D}\left(s,u\right)=\text{Xbox}^{D}\left(1,1,1,1,0,0,1;s,t\right), \tag{3.57}$$

- the master triangle with a bubble insertion of Fig. 2 (d), produced by

$$\text{Btri}^{D}\left(s\right)=\text{Xbox}^{D}\left(1,0,1,0,1,1,0;s,t\right)=\text{Xbox}^{D}\left(1,0,1,1,0,0,1;s,t\right), \tag{3.58}$$

- the master sunset diagram of Fig. 2 (e), produced by

$$\text{Sset}^{D}\left(s\right)=\text{Xbox}^{D}\left(0,0,1,0,1,0,1;s,t\right)=\text{Xbox}^{D}\left(1,0,0,1,0,1,0;s,t\right), \tag{3.59}$$

  together with

$$\text{Sset}^{D}\left(t\right)=\text{Xbox}^{D}\left(0,1,0,0,1,1,0;s,t\right), \tag{3.60}$$

  and

$$\text{Sset}^{D}\left(u\right)=\text{Xbox}^{D}\left(0,1,0,1,0,0,1;s,t\right), \tag{3.61}$$

where we explicitly used the symmetry of the crossed box

$$\text{Xbox}^{D}\left(\nu_1,\nu_2,\nu_3,\nu_4,\nu_5,\nu_6,\nu_7;s,t\right)=\text{Xbox}^{D}\left(\nu_3,\nu_2,\nu_1,\nu_7,\nu_6,\nu_5,\nu_4;s,t\right). \tag{3.62}$$

The reduction and dimensional shift of the generic scalar diagonal box and of the box with a bubble insertion have been treated in Ref. [4]. The tensor reduction and dimensional shift of the crossed triangle is treated in Appendix A. All the diagrams obtained from the pinchings

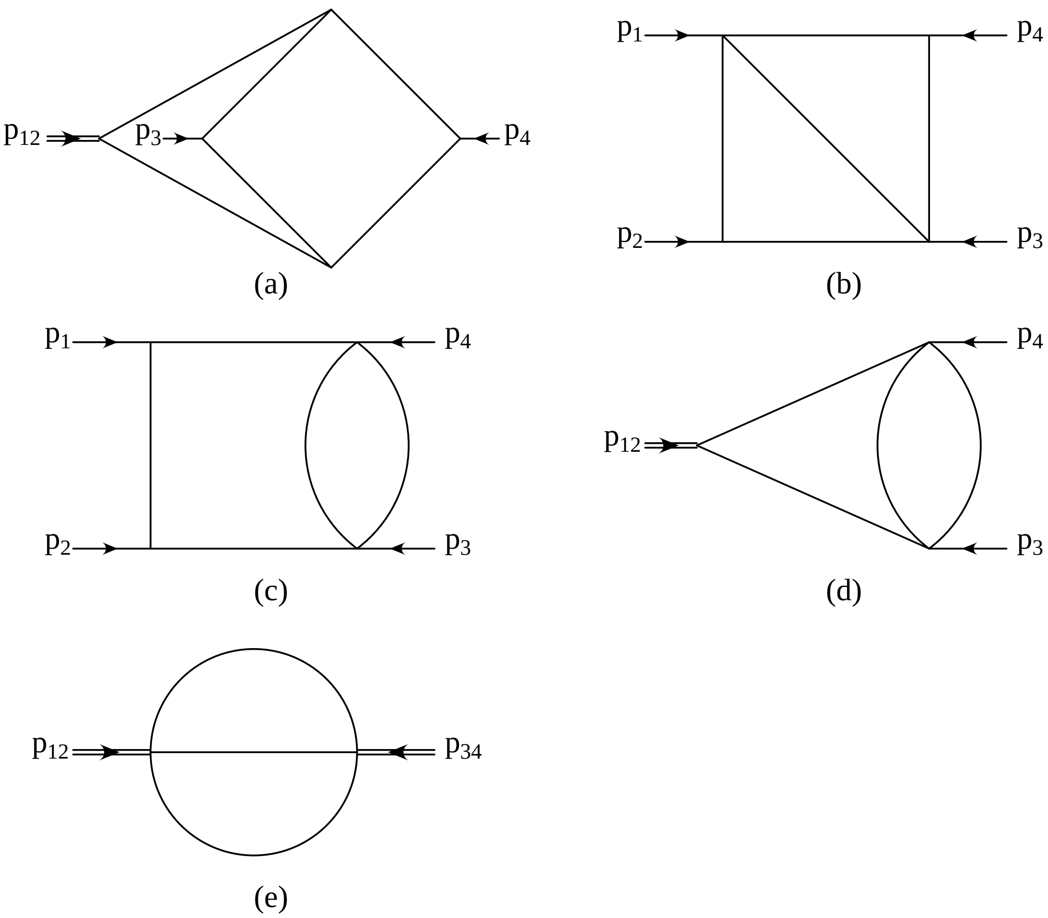

Figure 2: *The master integral topologies of the pinchings coming from the reduction of the scalar crossed two-loop box.*

of the other propagators can be easily reduced to the previous ones using the integration-by-parts identities for the crossed box and for the crossed triangle, with the corresponding powers of propagators set to zero.

Instead of keeping $\mathrm{Xbox}^D(1,1,1,1,1,1,2;s,t)$ as one of the two members of the basis, we prefer to switch to a more symmetric integral: $\mathrm{Xbox}^D(1,2,1,1,1,1,1;s,t)$. The expression for this one, in terms of the other two, can be easily obtained through the application of the reduction formalism outlined above

$$\begin{aligned}\mathrm{Xbox}^D(1,2,1,1,1,1,1;s,t) &= c_1\,\mathrm{Xbox}^D(1,1,1,1,1,1,1;s,t)\\ &+ c_2\,\mathrm{Xbox}^D(1,1,1,1,1,1,2;s,t) + c_3\,\mathrm{Xtri}^D\,(s) + c_4\,\mathrm{Dbox}^D\,(s,t)\\ &+ c_5\,\mathrm{Dbox}^D\,(s,u) + c_6\,\mathrm{Dbox}^D\,(t,u) + c_7\,\mathrm{Bbox}^D\,(s,t) + c_8\,\mathrm{Bbox}^D\,(s,u)\\ &+ c_9\,\mathrm{Btri}^D\,(s) + c_{10}\,\mathrm{Sset}^D\,(s) + c_{11}\,\mathrm{Sset}^D\,(t) + c_{12}\,\mathrm{Sset}^D\,(u)\,, \end{aligned} \tag{3.63}$$

where the coefficients c_j are collected in Appendix B. This allows us to define as master integrals

$$\mathrm{Xbox}_1^D(s,t) = \mathrm{Xbox}^D(1,1,1,1,1,1,1;s,t) \tag{3.64}$$
$$\mathrm{Xbox}_2^D(s,t) = \mathrm{Xbox}^D(1,2,1,1,1,1,1;s,t), \tag{3.65}$$

which are symmetric under the exchange $t \leftrightarrow u$. This will produce more compact results in the rest of the paper.

3.2 The dimensional shift for the two master integrals

Exploiting the equivalence between $\mathbf{d}^-$ and $\mathcal{P}$ of Eq. (3.19), we can rewrite Eq. (3.20) using (3.9)

$$\begin{aligned}\mathrm{Xbox}^D&(\nu_1,\nu_2,\nu_3,\nu_4,\nu_5,\nu_6,\nu_7;s,t)\\ &= \Big[\left(\nu_7\mathbf{7}^+ + \nu_6\mathbf{6}^+ + \nu_5\mathbf{5}^+ + \nu_4\mathbf{4}^+\right)\left(\nu_3\mathbf{3}^+ + \nu_2\mathbf{2}^+ + \nu_1\mathbf{1}^+\right)\\ &+ \left(\nu_5\mathbf{5}^+ + \nu_4\mathbf{4}^+\right)\left(\nu_7\mathbf{7}^+ + \nu_6\mathbf{6}^+\right)\Big]\mathrm{Xbox}^{D+2}(\nu_1,\nu_2,\nu_3,\nu_4,\nu_5,\nu_6,\nu_7;s,t)\,,\end{aligned} \tag{3.66}$$

and more specifically

$$\begin{aligned}\mathrm{Xbox}_1^D(s,t) &= \Big[\left(\mathbf{7}^+ + \mathbf{6}^+ + \mathbf{5}^+ + \mathbf{4}^+\right)\left(\mathbf{3}^+ + \mathbf{2}^+ + \mathbf{1}^+\right)\\ &\quad + \left(\mathbf{5}^+ + \mathbf{4}^+\right)\left(\mathbf{7}^+ + \mathbf{6}^+\right)\Big]\mathrm{Xbox}_1^{D+2}(s,t)\,, \end{aligned}\tag{3.67}$$
$$\begin{aligned}\mathrm{Xbox}_2^D(s,t) &= \Big[\left(\mathbf{7}^+ + \mathbf{6}^+ + \mathbf{5}^+ + \mathbf{4}^+\right)\left(\mathbf{3}^+ + 2\,\mathbf{2}^+ + \mathbf{1}^+\right)\\ &\quad + \left(\mathbf{5}^+ + \mathbf{4}^+\right)\left(\mathbf{7}^+ + \mathbf{6}^+\right)\Big]\mathrm{Xbox}_2^{D+2}(s,t)\,. \end{aligned}\tag{3.68}$$

The right-hand side of the two equations can then be reduced to a linear combination of master integrals in $D+2$ dimensions, and the system can then be inverted to give $\mathrm{Xbox}_1^{D+2}(s,t)$ and $\mathrm{Xbox}_2^{D+2}(s,t)$ as a function of master integrals in D dimensions

$$\mathrm{Xbox}_1^{D+2}(s,t) = A(t,u) + A(u,t), \tag{3.69}$$
$$\mathrm{Xbox}_2^{D+2}(s,t) = B(t,u) + B(u,t), \tag{3.70}$$

where

$$\begin{aligned}A(t,u) &= a_1\,\mathrm{Xbox}_1^D(s,t) + a_2\,\mathrm{Xbox}_2^D(s,t) + a_3\,\mathrm{Xtri}^D(s) + a_4\,\mathrm{Dbox}^D(s,t)\\ &\quad + a_5\,\mathrm{Dbox}^D(t,u) + a_6\,\mathrm{Bbox}^D(s,t) + a_7\,\mathrm{Btri}^D(s)\\ &\quad + a_8\,\mathrm{Sset}^D(s) + a_9\,\mathrm{Sset}^D(t)\,,\end{aligned}$$

$$
\begin{aligned}
B(t,u) &= b_1\,\mathrm{Xbox}_1^D\,(s,t) + b_2\,\mathrm{Xbox}_2^D\,(s,t) + b_3\,\mathrm{Xtri}^D\,(s) + b_4\,\mathrm{Dbox}^D\,(s,t) \\
&\quad + b_5\,\mathrm{Dbox}^D\,(t,u) + b_6\,\mathrm{Bbox}^D\,(s,t) + b_7\,\mathrm{Btri}^D\,(s) \\
&\quad + b_8\,\mathrm{Sset}^D\,(s) + b_9\,\mathrm{Sset}^D\,(t)\,. \qquad (3.71)
\end{aligned}
$$

Some of the coefficients a_i and b_i of the system are collected in Appendix C.

4 Differential equations for the two master integrals

In the previous sections we showed how tensor integrals can be expressed in terms of the two master crossed boxes, Xbox_1^D and Xbox_2^D plus simpler master diagrams. The analytic expansion in $\epsilon = (4-D)/2$ of the first master integral was computed in Ref. [6]. We can obtain the analytic form for the second one by writing the derivative of Xbox_1^D with respect to one of the two independent physical scales (that we choose to be t), as a combination of master integrals, and solving the equation for Xbox_2^D.

Moreover we can verify the correctness of both the expressions of $\mathrm{Xbox}_1^D\,(s,t)$ and $\mathrm{Xbox}_2^D\,(s,t)$, by deriving an analogous differential equation for Xbox_2^D, and checking that the obtained identity is satisfied.

Using the procedure outlined in Section 3, we differentiate with respect to t the two master integrals written in the form of Eq. (3.12). In this way, the only t-dependence comes from $\mathcal{Q}$ of Eq. (3.15) and the result is

$$
\begin{aligned}
\frac{\partial}{\partial t}\mathrm{Xbox}_1^D\,(s,t) &= \mathrm{Xbox}^{D+2}\,(1,2,1,2,1,1,2;s,t) - \mathrm{Xbox}^{D+2}\,(1,2,1,1,2,2,1;s,t)\,, \\
\frac{\partial}{\partial t}\mathrm{Xbox}_2^D\,(s,t) &= 2\,\mathrm{Xbox}^{D+2}\,(1,3,1,2,1,1,2;s,t) - 2\,\mathrm{Xbox}^{D+2}\,(1,3,1,1,2,2,1;s,t)\,.
\end{aligned}
$$

Applying the reduction formulae of Section 3.1 and the dimensional-shift of Section 3.2 to the right-hand sides of the system, we can rewrite them as a combination of master crossed boxes and pinchings in D dimensions.

The final system of differential equations in arbitrary D is given by

$$
\frac{\partial}{\partial t}\mathrm{Xbox}_1^D\,(s,t) = \frac{1}{t-u}\left[H(t,u) + H(u,t),\right] \qquad (4.1)
$$

$$
\frac{\partial}{\partial t}\mathrm{Xbox}_2^D\,(s,t) = \frac{1}{t-u}\left[K(t,u) + K(u,t),\right] \qquad (4.2)
$$

where

$$
H(t,u) = h_1\,\mathrm{Xbox}_1^D\,(s,t) + h_2\,\mathrm{Xbox}_2^D\,(s,t) + h_3\,\mathrm{Xtri}^D\,(s) + h_4\,\mathrm{Dbox}^D\,(s,t)
$$

$$
\begin{aligned}
& + h_5 \,\mathrm{Dbox}^D\,(t,u) + h_6 \,\mathrm{Bbox}^D\,(s,t) + h_7 \,\mathrm{Btri}^D\,(s) \\
& + h_8 \,\mathrm{Sset}^D\,(s) + h_9 \,\mathrm{Sset}^D\,(t)\,, \qquad (4.3)
\end{aligned}
$$

$$
\begin{aligned}
K(t,u) \;=\; & k_1 \,\mathrm{Xbox}_1^D\,(s,t) + k_2 \,\mathrm{Xbox}_2^D\,(s,t) + k_3 \,\mathrm{Xtri}^D\,(s) + k_4 \,\mathrm{Dbox}^D\,(s,t) \\
& + k_5 \,\mathrm{Dbox}^D\,(t,u) + k_6 \,\mathrm{Bbox}^D\,(s,t) + k_7 \,\mathrm{Btri}^D\,(s) \\
& + k_8 \,\mathrm{Sset}^D\,(s) + k_9 \,\mathrm{Sset}^D\,(t)\,, \qquad (4.4)
\end{aligned}
$$

and the coefficients h_i and k_i are listed in Appendix D.

From the symmetry $t \leftrightarrow u$ of the two master integrals, we expect the two derivatives to be anti-symmetric with respect to the exchange $t \leftrightarrow u$, and this is a further check of the correctness of the system.

5 The off-shell method

An alternative way to derive the system of equations (4.1) and (4.2) is based on the construction of a set of differential equations for the crossed-box integrals with the momentum p_1 taken off-shell. This is done using the algorithms and computer programs described in [9], by means of which the differential equations for any massless two-loop four-point function with three light-like and one off-shell leg can be obtained. We then take the on-shell limit $p_1^2 \to 0$.

To derive the differential equations for the diagrams with one leg off-shell, we take p_2, p_3 and p_4, all massless and on-shell, as independent momenta, and form the derivatives with respect to the external invariants s_{23}, s_{24}, s_{34}, where $s_{ij} = p_{ij}^2$

$$
\begin{aligned}
s_{23}\frac{\partial}{\partial s_{23}} &= \frac{1}{2}\left(+p_2^\mu \frac{\partial}{\partial p_2^\mu} + p_3^\mu \frac{\partial}{\partial p_3^\mu} - p_4^\mu \frac{\partial}{\partial p_4^\mu} \right), \\
s_{24}\frac{\partial}{\partial s_{24}} &= \frac{1}{2}\left(+p_2^\mu \frac{\partial}{\partial p_2^\mu} - p_3^\mu \frac{\partial}{\partial p_3^\mu} + p_4^\mu \frac{\partial}{\partial p_4^\mu} \right), \qquad (5.1) \\
s_{34}\frac{\partial}{\partial s_{34}} &= \frac{1}{2}\left(-p_2^\mu \frac{\partial}{\partial p_2^\mu} + p_3^\mu \frac{\partial}{\partial p_3^\mu} + p_4^\mu \frac{\partial}{\partial p_4^\mu} \right).
\end{aligned}
$$

The derivatives with respect to p_i^μ act on the propagators in the Feynman integrand of Eq. (2.1) and give rise to a number of different integrals with increased powers of the propagators. Using integration-by-parts and Lorentz-invariance identities, all these different integrals can be reduced to a small number of master integrals. In particular, any crossed-box topology integral can be reduced to a linear combination of two master crossed-box integrals, which we can choose to be the off-shell versions of Xbox_1^D and Xbox_2^D, plus a number of pinched master integrals. Thus, we obtain a system of two coupled linear first-order differential equations for the two master crossed-box integrals.

Since, in the on-shell limit, we are interested in the derivatives at fixed $p_1^2 = 0$, we rewrite the system of equations in terms of the variables $p_1^2 = s_{23} + s_{24} + s_{34}$, $s = s_{34}$, $t = s_{23}$, and the corresponding derivative operators:

$$
\begin{aligned}
\frac{\partial}{\partial s} &= \frac{\partial}{\partial s_{34}} - \frac{\partial}{\partial s_{24}}, \\
\frac{\partial}{\partial t} &= \frac{\partial}{\partial s_{23}} - \frac{\partial}{\partial s_{24}}, \\
\frac{\partial}{\partial p_1^2} &= \frac{\partial}{\partial s_{24}}.
\end{aligned} \tag{5.2}
$$

The next step is to take the on-shell limit $p_1^2 \to 0$. A complication here is the fact that factors of $1/p_1^2$ and $1/(p_1^2)^2$ appear in some of the coefficients of the pinched master integrals. It is also important to realize that several of the off-shell master integrals with less than seven propagators become reducible in the on-shell limit (an example of such a case will be given below). When these reductions are taken into account, all the terms proportional to $1/p_1^2$ and $1/(p_1^2)^2$ cancel out, so that the on-shell limit of the system of equations for the two master crossed-box integrals is indeed well defined.

However, because of these factors of p_1^2 in the denominator, it is not sufficient merely to replace all pinched master integrals by their limit $p_1^2 \to 0$: subleading terms in p_1^2 also have to be included. The Taylor series around $p_1^2 = 0$ for the pinched diagrams can be obtained in a straightforward manner by using the differential equation in p_1^2. As an example, we consider the propagator diagram $\mathrm{Sset}^D\left(p_1^2 - s - t\right)$, which fulfills the homogeneous differential equation

$$
\frac{\partial}{\partial p_1^2} \mathrm{Sset}^D\left(p_1^2 - s - t\right) = \frac{D-3}{p_1^2 - s - t} \mathrm{Sset}^D\left(p_1^2 - s - t\right). \tag{5.3}
$$

Iterating this equation, we obtain

$$
\begin{aligned}
\mathrm{Sset}^D\left(p_1^2 - s - t\right) &= \mathrm{Sset}^D\left(-s-t\right) + p_1^2 \frac{D-3}{-s-t} \mathrm{Sset}^D\left(-s-t\right) \\
&\quad + \frac{1}{2} p_1^4 \frac{(D-3)(D-4)}{(-s-t)^2} \mathrm{Sset}^D\left(-s-t\right) + \mathcal{O}\left(p_1^6\right).
\end{aligned} \tag{5.4}
$$

If a factor of $1/p_1^2$ appears in the homogeneous term of the differential equations for an off-shell master integral, this integral becomes reducible in the limit $p_1^2 \to 0$. An example of a reducible master integral is the vertex diagram $\mathrm{Rtri}\left(p_1^2, s\right)$ of Fig. 3, which fulfils the differential equation

$$
\frac{\partial}{\partial p_1^2} \mathrm{Rtri}\left(p_1^2, s\right) = \frac{D-4}{2} \frac{2p_1^2 - s}{p_1^2\left(p_1^2 - s\right)} \mathrm{Rtri}\left(p_1^2, s\right) - \frac{3D-8}{2} \frac{1}{p_1^2\left(p_1^2 - s\right)} \mathrm{Sset}^D(s). \tag{5.5}
$$

Multiplying by p_1^2, one can immediately read off the limit

$$
\mathrm{Rtri}\left(p_1^2 \to 0, s\right) = -\frac{3D-8}{D-4} \frac{1}{s} \mathrm{Sset}^D(s) + \mathcal{O}\left(p_1^2\right). \tag{5.6}
$$

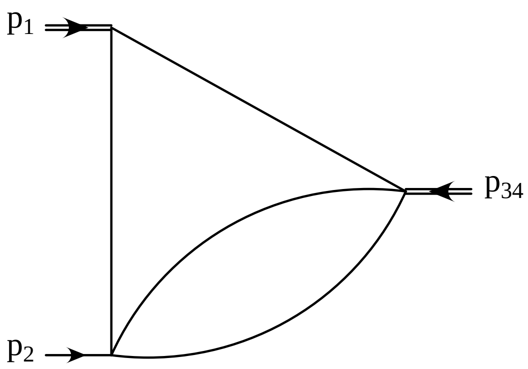

Figure 3: *The master integral* Rtri (p_1^2, s), *which becomes reducible when* p_1 *is on-shell.*

Subleading terms in p_1^2 are again obtained by successive iteration of the differential equation (5.5).

All the information required to derive the massless limit of the system of equations for the master crossed-box diagrams is contained in the differential equations for the pinched master integrals.

The differential equations obtained with this method are in perfect agreement with (4.1) and (4.2), giving an independent confirmation of the correctness of the two methods.

In addition, we checked that the two methods give the same results for the crossed boxes with the powers of propagators equal to unity and with one scalar product in the numerator, of the type $p_i \cdot k$ or $p_i \cdot l$. This was done computing Eq.(3.21) with the method described in Section 3 and then contracting the result with one of the external momenta.

6 Analytic expansion of the second master integral

Inserting the ϵ expansion of $\mathrm{Xbox}_1^D(s,t)$ computed in Ref. [6] and the ϵ expansions of the sub-topologies listed in Refs. [3, 4, 11, 12] into Eq. (4.1), and solving it with respect to $\mathrm{Xbox}_2^D(s,t)$, we obtain, in the physical region $s > 0$, $t, u < 0$,

$$\mathrm{Xbox}_2^D(s,t) = \Gamma^2(1+\epsilon)\left\{\frac{G_1(t,u)}{s^3 t} + \frac{G_2(t,u)}{s^2 t^2} + \frac{G_1(u,t)}{s^3 u} + \frac{G_2(u,t)}{s^2 u^2}\right\}, \qquad (6.1)$$

where

$$G_1(t,u) = s^{-2\epsilon}\left\{\frac{6}{\epsilon^3} + \frac{1}{\epsilon^2}(32 - 6T - 6U)\right.$$

$$+\frac{1}{\epsilon}\left(1-12\,\pi^2-24\,T+T^2-24\,U+16\,T\,U+U^2\right)-43-18\,T+13\,T^2+\frac{8}{3}\,T^3$$
$$-18\,U+16\,T\,U+11\,T^2\,U+13\,U^2-20\,T\,U^2+\frac{8}{3}\,U^3+\pi^2\left(17\,T+17\,U-\frac{112}{3}\right)$$
$$-122\,\zeta(3)+62\,T\,\mathrm{Li}_2\left(-\frac{t}{s}\right)-62\,\mathrm{Li}_3\left(-\frac{t}{s}\right)+62\,\mathrm{S}_{1,2}\left(-\frac{t}{s}\right)$$
$$+i\pi\left[\frac{1}{\epsilon}\left(16+6\,T+6\,U\right)-34-9\,\pi^2-6\,T-10\,T^2-6\,U+14\,T\,U-10\,U^2\right]\Bigg\}, \tag{6.2}$$

$$G_2(t,u)=s^{-2\epsilon}\Bigg\{-\frac{2}{\epsilon^4}+\frac{1}{\epsilon^3}\left(-8+\frac{5}{2}\,T+\frac{7}{2}\,U\right)$$
$$+\frac{1}{\epsilon^2}\left(-\frac{29}{2}-\frac{5}{12}\,\pi^2+7\,T-T^2+20\,U-4\,T\,U-U^2\right)$$
$$+\frac{1}{\epsilon}\Bigg[-\frac{1}{2}+17\,T+2\,T^2-\frac{T^3}{3}+\frac{\pi^2}{6}\left(14+5\,T-29\,U\right)+13\,U-28\,T\,U-4\,U^2$$
$$+3\,T\,U^2-U^3+\frac{19}{2}\,\zeta(3)-2\,T\,\mathrm{Li}_2\left(-\frac{t}{s}\right)+2\,\mathrm{Li}_3\left(-\frac{t}{s}\right)-2\,\mathrm{S}_{1,2}\left(-\frac{t}{s}\right)\Bigg]$$
$$+\frac{37}{2}+\frac{37}{40}\,\pi^4+7\,T-5\,T^2-\frac{22}{3}\,T^3+\frac{2}{3}\,T^4+5\,U-20\,T\,U+\frac{8}{3}\,T^3\,U-2\,U^2$$
$$+24\,T\,U^2-T^2\,U^2-8\,U^3-\frac{4}{3}\,T\,U^3+\frac{4}{3}\,U^4$$
$$+\frac{\pi^2}{6}\left(79-22\,T-5\,T^2-200\,U+76\,T\,U+25\,U^2\right)+\left(68-13\,T-33\,U\right)\,\zeta(3)$$
$$+\left(10\,\pi^2-32\,T+17\,T^2+12\,T\,U\right)\,\mathrm{Li}_2\left(-\frac{t}{s}\right)+\left(32-60\,T-12\,U\right)\,\mathrm{Li}_3\left(-\frac{t}{s}\right)$$
$$+\left(28\,T-6\,U-32\right)\,\mathrm{S}_{1,2}\left(-\frac{t}{s}\right)-26\,\mathrm{S}_{1,3}\left(-\frac{t}{s}\right)-36\,\mathrm{S}_{2,2}\left(-\frac{t}{s}\right)+86\,\mathrm{Li}_4\left(-\frac{t}{s}\right)$$
$$+i\pi\Bigg[\frac{2}{\epsilon^3}+\frac{1}{\epsilon^2}\left(11-T+U\right)+\frac{1}{\epsilon}\left(1-\frac{31}{6}\,\pi^2-10\,T-2\,T^2+4\,U-2\,T\,U-2\,U^2\right)$$
$$+11+4\,T-2\,T^2+\frac{10}{3}\,T^3+\frac{\pi^2}{3}\left(-65+28\,T-U\right)+2\,U-8\,T\,U-8\,U^2$$
$$+2\,U^3-89\,\zeta(3)+\left(14\,T+18\,U\right)\,\mathrm{Li}_2\left(-\frac{t}{s}\right)-32\,\mathrm{Li}_3\left(-\frac{t}{s}\right)+44\,\mathrm{S}_{1,2}\left(-\frac{t}{s}\right)\Bigg]\Bigg\}, \tag{6.3}$$

and $T=\log(-t/s)$, $U=\log(-u/s)$. We used Nielsen's generalized polylogarithms $\mathrm{S}_{n,p}$ [13, 14], defined by

$$\mathrm{S}_{n,p}\left(x\right)=\frac{(-1)^{n+p-1}}{(n-1)!\,p!}\int_0^1 \mathrm{d}t\;\frac{\log^{n-1}(t)\log^p(1-xt)}{t},\qquad n,p\geq 1,\quad x\leq 1, \tag{6.4}$$

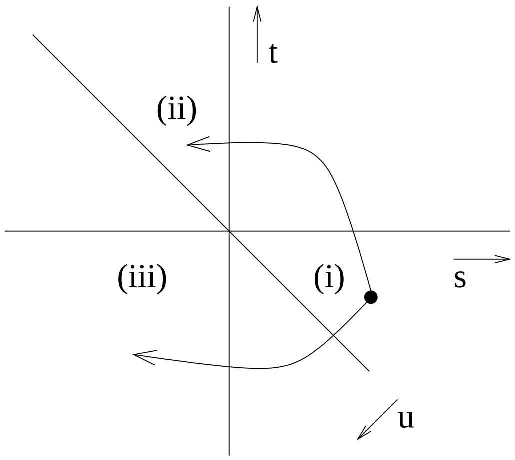

Figure 4: *The physical regions* (i), (ii) *and* (iii) *in the* (s,t,u)*-plane.*

where the usual polylogarithms are given by

$$\mathrm{Li}_n(x) = \mathrm{S}_{n-1,1}\,(x)\,. \tag{6.5}$$

The three kinematically accessible regions of the phase-space are depicted in Fig. 4.

(i) $\boldsymbol{s > 0,\ t, u < 0}$. All logarithms and polylogarithms occurring in Eqs. (6.2) and (6.3) are real in this region.

Formulae for the other two regions, (ii) and (iii), can be derived by analytic continuation, starting from region (i) and following the paths indicated in the figure.

The analytic continuation can be performed through a few simple steps.

(ii) $\boldsymbol{t > 0,\ s, u < 0}$. Going from region (i) to region (ii), we have to pass through two branches: $t = 0$ and $s = 0$. We can then split the analytic continuation into two steps:

- we first split the logarithm $T = \log(-t) - \log(s)$. At $t = 0$, nothing happens to the polylogarithms $\mathrm{S}_{n,p}\,(-t/s)$, but $\log(-t)$ gets an imaginary part: $\log(-t) \to \log(t) - i\pi$.

  We are now in an unphysical region, where both s and t are positive and u is negative. Using the transformation formulae for $x \to 1/x$ (see, eg. Refs. [14, 4]), we can express $\mathrm{S}_{n,p}\,(-t/s)$ in terms of $\mathrm{S}_{n,p}\,(-s/t)$ and $\log(t/s)$.

- To enter region (ii), we have to pass now the branch point at $s = 0$. We split $\log(t/s) = \log(t) - \log(s)$ and $U = \log(s+t) - \log(s)$ and we analytically continue $\log(s) \to \log(-s) + i\pi$.

In this way, for example, the logarithms in Eqs. (6.2) and (6.3) undergo the transformation

$$\log\left(-\frac{t}{s}\right) \;\to\; \log\left(-\frac{t}{s}\right) - 2\,i\,\pi, \tag{6.6}$$

$$\log\left(-\frac{u}{s}\right) \;\to\; \log\left(\frac{u}{s}\right) - i\,\pi. \tag{6.7}$$

(iii) $\boldsymbol{u > 0,\ s, t < 0}$. The procedure to go from region (i) to region (iii) is similar to the previous one, but it requires an additional step.

- We rewrite $\mathrm{S}_{n,p}\,(-t/s)$ in terms of $\mathrm{S}_{n,p}\,((s+t)/s)$, $\log(-t/s)$ and $\log((s+t)/s)$, using the transformation $x \to 1-x$, and we split the logarithms as before. In passing the first branch point at $u=0$, the polylogarithms are well defined while $\log(s+t) \to \log(-s-t) - i\pi$.
- We invert now the argument of the polylogarithms, expressing $\mathrm{S}_{n,p}\,((s+t)/s)$ in terms of $\mathrm{S}_{n,p}\,(s/(s+t))$ and $\log((-s-t)/s) = \log(-s-t) - \log(s)$. Finally, $\log(s) \to \log(-s) + i\pi$, as we pass the branch point at $s=0$ and enter region (iii).

The logarithms in Eqs. (6.2) and (6.3) undergo the transformation

$$\log\left(-\frac{t}{s}\right) \;\to\; \log\left(\frac{t}{s}\right) - i\,\pi, \tag{6.8}$$

$$\log\left(-\frac{u}{s}\right) \;\to\; \log\left(-\frac{u}{s}\right) - 2\,i\,\pi. \tag{6.9}$$

The expression for $G_1(t,u)$ and $G_2(t,u)$ in this region can also be obtained directly from the expressions in region (ii), using the symmetry $t \leftrightarrow u$.

A non-trivial check of the correctness of the expressions of $\mathrm{Xbox}_1^D\,(s,t)$ and $\mathrm{Xbox}_2^D\,(s,t)$ comes from Eq. (4.2), that must be identically satisfied, once the respective ϵ expansions are used.

7 Conclusions

High-energy scattering processes are one of the most important sources of information on short-distance physics. Recent improvements of experimental measurements demand nowadays the knowledge of $2 \to 2$ scattering rates at two-loop order.

Two non-trivial topologies characterize these processes: planar two-loop boxes [2, 3], and crossed two-loop boxes.

The crossed boxes constituted the last barrier towards the completion of this goal. In this paper we presented an algorithm to reduce tensor two-loop massless crossed boxes with light-like external legs to two master crossed boxes (the first one with all powers of propagators equal to one, and the second one with the power of the second propagator of Fig. 1 equal to two), plus simpler diagrams.

We derived the equations that connect the two master integrals in dimensions D and $D+2$ and the system of first-order coupled differential equations satisfied by the two master integrals.

This last part was done following two different methods: using the raising and lowering operators, and using the on-shell limit of the differential equations for the crossed box with one off-shell leg [9]. The agreement between the results of the two methods is a strong support for the validity of the two different procedures.

The differential equation for the first master integral allowed us to derive an ϵ expansion for the second master integral, once the known [6] expression of the first master integral is inserted. The differential equation for the second master integral then provided us a non-trivial way to check the obtained expansion.

Acknowledgement

We thank E.W.N. Glover, M.E. Tejeda-Yeomans and J.J. van der Bij for assistance and useful suggestions. C.A. acknowledges the financial support of the Greek Government, C.O. acknowledges the financial support of the INFN, E.R. thanks the Alexander-von-Humboldt Stiftung for supporting his stay at the Institut für Theoretische Teilchenphysik of the University of Karlsruhe and J.B.T. acknowledges the financial support of the DFG-Forschergruppe “Quantenfeldtheorie, Computeralgebra und Monte-Carlo-Simulation”. We gratefully acknowledge the support of the British Council and German Academic Exchange Service under ARC project 1050.

A Tensor reduction and dimensional shift of the crossed two-loop triangle

For completeness, we give here the reduction formulae for the crossed two-loop triangle that appears as a sub-topology in the scalar reduction of the crossed box. We make use of integration-by-parts and Lorentz-invariance identities to reduce the generic scalar triangle to one master integral which has all powers of propagators equal to one [11, 12], plus pinchings. For an alternative solution to this reduction problem, exploiting a connection with massless

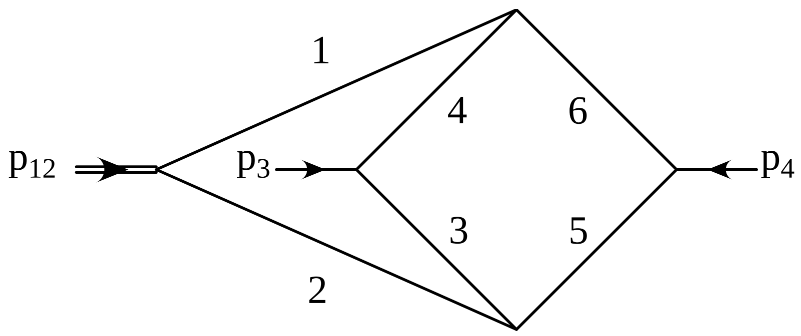

Figure 5: *The generic two-loop crossed triangle.*

three-loop propagator integrals, see Ref. [15].

The generic two-loop scalar crossed triangle of Fig. 5 is given by

$$\mathrm{Xtri}^{D}\left(\nu_1,\nu_2,\nu_3,\nu_4,\nu_5,\nu_6;s\right)=\int\frac{d^Dk}{i\pi^{D/2}}\int\frac{d^Dl}{i\pi^{D/2}}\,\frac{1}{A_1^{\nu_1}A_2^{\nu_2}A_3^{\nu_3}A_4^{\nu_4}A_5^{\nu_5}A_6^{\nu_6}},\tag{A.1}$$

where the propagators are

$$\begin{aligned}
A_1 &= (k+l+p_3+p_4)^2+i0, \qquad (A.2)\\
A_2 &= (k+l)^2+i0,\\
A_3 &= l^2+i0,\\
A_4 &= (l+p_3)^2+i0,\\
A_5 &= k^2+i0,\\
A_6 &= (k+p_4)^2+i0.
\end{aligned}$$

The external momenta are in-going, two of them being light-like, $p_3^2=p_4^2=0$, while $p_{12}^2=s$ is the only physical scale.

Repeating the reasoning of Section 3.1, we can build eight integration-by-parts identities and a single Lorentz-invariance identity. They depend upon one irreducible scalar product in the numerator, that we choose to be $(l\cdot p_4)$:

$$s\nu_1\mathbf{1}^+ +(2D-2\nu_{235}-\nu_{146})-\nu_4\mathbf{4}^+\mathbf{3}^- -\nu_1\mathbf{1}^+\mathbf{2}^- -\nu_6\mathbf{6}^+\mathbf{5}^- =0\tag{A.3}$$

$$s\nu_2\mathbf{2}^+ +(2D-2\nu_{146}-\nu_{235})-\nu_3\mathbf{3}^+\mathbf{4}^- -\nu_2\mathbf{2}^+\mathbf{1}^- -\nu_5\mathbf{5}^+\mathbf{6}^- =0\tag{A.4}$$

$$\begin{aligned}
&2\,(l\cdot p_4)\,\nu_1\mathbf{1}^+ -(D-\nu_{24}-2\nu_3)+\nu_1\mathbf{1}^+\left(\mathbf{2}^-+\mathbf{4}^--\mathbf{5}^-\right)\\
&\qquad+\nu_2\mathbf{2}^+\left(\mathbf{3}^--\mathbf{5}^-\right)+\nu_4\mathbf{4}^+\mathbf{3}^- =0 \qquad (A.5)
\end{aligned}$$

$$2\,(l\cdot p_4)\,\nu_2\mathbf{2}^+ -(D-\nu_{125}-2\nu_6)+\nu_2\mathbf{2}^+\left(\mathbf{6}^--\mathbf{3}^-\right)$$

$$- \nu_1 \mathbf{1}^+ \left(\mathbf{6}^- - \mathbf{4}^-\right) - \nu_5 \mathbf{5}^+ \mathbf{6}^- = 0 \tag{A.6}$$

$$2\left(l \cdot p_4\right) \nu_3 \mathbf{3}^+ + \left(D - \nu_{345} - \nu_6\right) + \nu_3 \mathbf{3}^+ \left(\mathbf{2}^- - \mathbf{5}^-\right) + \nu_4 \mathbf{4}^+ \left(\mathbf{1}^- - \mathbf{6}^-\right)$$
$$- \nu_5 \mathbf{5}^+ \mathbf{6}^- = 0 \tag{A.7}$$

$$2\left(l \cdot p_4\right) \nu_4 \mathbf{4}^+ + s\nu_4 \mathbf{4}^+ - \left(D - \nu_{346} - 2\nu_5\right) + \nu_3 \mathbf{3}^+ \left(\mathbf{5}^- - \mathbf{2}^-\right)$$
$$+ \nu_4 \mathbf{4}^+ \left(\mathbf{6}^- - \mathbf{1}^-\right) + \nu_6 \mathbf{6}^+ \mathbf{5}^- = 0 \tag{A.8}$$

$$2\left(l \cdot p_4\right) \nu_5 \mathbf{5}^+ + s\nu_5 \mathbf{5}^+ - \left(D - \nu_{36} - 2\nu_4\right) + \nu_5 \mathbf{5}^+ \left(\mathbf{4}^- + \mathbf{6}^- - \mathbf{1}^-\right)$$
$$+ \nu_6 \mathbf{6}^+ \left(\mathbf{4}^- - \mathbf{1}^-\right) + \nu_3 \mathbf{3}^+ \mathbf{4}^- = 0 \tag{A.9}$$

$$2\left(l \cdot p_4\right) \nu_6 \mathbf{6}^+ - \left(D - \nu_{45} - 2\nu_3\right) - \nu_6 \mathbf{6}^+ \left(\mathbf{3}^- + \mathbf{5}^- - \mathbf{2}^-\right)$$
$$+ \nu_5 \mathbf{5}^+ \left(\mathbf{2}^- - \mathbf{3}^-\right) + \nu_4 \mathbf{4}^+ \mathbf{3}^- = 0 \tag{A.10}$$

$$2\left(l \cdot p_4\right) \nu_1 \mathbf{1}^+ - \left(D - \nu_{2356}\right) + \nu_1 \mathbf{1}^+ \left(\mathbf{2}^- + \mathbf{6}^- - \mathbf{5}^-\right) + \nu_4 \mathbf{4}^+ \mathbf{3}^- = 0. \tag{A.11}$$

By eliminating the irreducible scalar product in the numerator, we obtain

$$s\nu_1 \mathbf{1}^+ = -\left(2D - 2\nu_{235} - \nu_{146}\right) + \nu_4 \mathbf{4}^+ \mathbf{3}^- + \nu_1 \mathbf{1}^+ \mathbf{2}^- + \nu_6 \mathbf{6}^+ \mathbf{5}^-, \tag{A.12}$$

which, together with the symmetric one for $\nu_2 \mathbf{2}^+$, can reduce ν_1 and ν_2 to unity. Using

$$\begin{aligned} \nu_3 \mathbf{3}^+ \quad &= \quad \frac{1}{D - 2 - \nu_{3456}} \left(\nu_4 \mathbf{4}^+ \nu_6 \mathbf{6}^+ \mathbf{1}^- - \nu_3 \mathbf{3}^+ \nu_5 \mathbf{5}^+ \mathbf{2}^-\right) \\ &\quad + \frac{1}{D - 2 - 2\nu_{34}} \left[\left(D - 2 - 2\nu_{46}\right) \nu_6 \mathbf{6}^+ + 2\left(\nu_3 - \nu_6\right) \nu_5 \mathbf{5}^+\right], \end{aligned} \tag{A.13}$$

and the symmetric one for $\nu_4 \mathbf{4}^+$, we can reduce ν_3 and ν_4 to one. To complete the reduction, we use

$$\left(\nu_{56} - \nu_{34}\right) = \nu_2 \mathbf{2}^+ \left(\mathbf{3}^- - \mathbf{5}^-\right) + \nu_1 \mathbf{1}^+ \left(\mathbf{4}^- - \mathbf{6}^-\right), \tag{A.14}$$

that can be re-iterated until $\left(\nu_{56} - \nu_{34}\right) = 0$. Since we are applying this identity to scalar integrals where ν_3 and ν_4 have already been reduced to one, the reduction procedure will stop when $\nu_5 = \nu_6 = 1$. This integral cannot be reduced any further, and we choose the crossed master triangle to be

$$\mathrm{Xtri}^D(s) = \mathrm{Xtri}^D\left(1,1,1,1,1,1;s\right). \tag{A.15}$$

The application of the identity

$$\mathbf{d}^- = \left(\nu_6 \mathbf{6}^+ + \nu_5 \mathbf{5}^+ + \nu_4 \mathbf{4}^+ + \nu_3 \mathbf{3}^+\right) \left(\nu_2 \mathbf{2}^+ + \nu_1 \mathbf{1}^+\right) + \left(\nu_4 \mathbf{4}^+ + \nu_3 \mathbf{3}^+\right) \left(\nu_6 \mathbf{6}^+ + \nu_5 \mathbf{5}^+\right) \tag{A.16}$$

to $\mathrm{Xtri}^{D+2}(s)$ and its further reduction, gives rise to the dimensional-shift formula

$$\begin{aligned}
\mathrm{Xtri}^{D+2}(s) &= -\frac{(D-4)s^2}{4(D-2)(2D-7)(2D-5)}\,\mathrm{Xtri}^{D}(s) \\
&\quad - \frac{37D^3-313D^2+858D-752}{2(D-4)(D-2)(2D-7)(2D-5)(3D-8)}\,\mathrm{Btri}^{D}(s) \\
&\quad + \frac{43D^4-478D^3+1963D^2-3530D+2352}{2(D-4)^2(D-3)(D-2)(2D-7)(2D-5)s}\,\mathrm{Sset}^{D}(s)\,. \qquad \text{(A.17)}
\end{aligned}$$

The expression of the master integral of Eq. (A.15) has been computed in Refs. [11, 12].

B Coefficients c_i

We collect here the coefficients c_i of Eq. (3.63).

$$\begin{aligned}
c_1 &= 4\frac{(D-5)^2u}{(D-6)st} \\
c_2 &= \frac{t-u}{t} \\
c_3 &= -2\frac{(D-4)(2D-9)}{(D-6)st} \\
c_4 &= -3\frac{(D-4)(3D-14)u^2}{(D-6)^2(2D-11)s^3t^3}\left[2\left(7D^2-76D+206\right)t+(D-5)(5D-28)s\right] \\
c_5 &= -3\frac{(D-4)(3D-14)t}{(D-6)^2(2D-11)s^3u^2}\left[2\left(7D^2-76D+206\right)t+(D-5)(5D-28)s\right] \\
c_6 &= 3\frac{(D-5)(D-4)(3D-14)s\left[2(D-5)t-(D-6)s\right]}{(D-6)^2(2D-11)t^3u^2} \\
c_7 &= 6\frac{(D-3)(3D-14)\left[(5D-26)t+2(D-5)s\right]}{(D-6)^2st^3} \\
c_8 &= 6\frac{(D-3)(3D-14)\left[(5D-26)t+2(D-5)s\right]}{(D-6)^2stu^2} \\
c_9 &= -\frac{3}{2}\frac{(D-3)(3D-10)}{(D-6)^2(D-5)(D-4)s^3t^3u^2}\Big[-\left(41D^3-620D^2+3124D-5248\right)st^3 \\
&\quad +2(D-6)^2(D-4)t^4-\left(61D^3-922D^2+4640D-7776\right)s^2t^2 \\
&\quad -4(D-5)(3D-14)(4D-21)s^3t-4(D-5)^2(3D-14)s^4\Big] \\
c_{10} &= -3\frac{(D-3)(3D-10)(3D-8)}{(D-6)^2(D-5)(D-4)^2(2D-11)s^4t^3u^2}
\end{aligned}$$

$$
\begin{aligned}
&\times \Big[2(D-6)(D-4)(2D-11)(5D-22)t^4 \\
&- 2\left(64D^4 - 1305D^3 + 9978D^2 - 33902D + 43180\right) st^3 \\
&- \left(208D^4 - 4209D^3 + 31899D^2 - 107314D + 135216\right) s^2t^2 \\
&- (D-5)(3D-14)\left(34D^2 - 359D + 944\right) s^3t - (D-5)^2(3D-14)(5D-28)s^4\Big]
\end{aligned}
$$

$$
\begin{aligned}
c_{11} &= 3\frac{(D-3)(3D-14)(3D-10)(3D-8)}{(D-6)^2(D-5)(D-4)^2(2D-11)s^3t^4u^2} \\
&\times \Big[2(D-5)\left(7D^2 - 76D + 206\right) t^4 + \left(61D^3 - 952D^2 + 4941D - 8528\right) st^3 \\
&+ \left(92D^3 - 1417D^2 + 7253D - 12336\right) s^2t^2 \\
&+ \left(56D^3 - 853D^2 + 4317D - 7258\right) s^3t + 2(D-5)(2D-11)(2D-9)s^4\Big] \\
c_{12} &= -3\frac{(D-3)(3D-14)(3D-10)(3D-8)}{(D-6)^2(D-5)(D-4)^2(2D-11)s^3t^3u^3}\Big[2(D-5)\left(7D^2 - 76D + 206\right) t^4 \\
&- \left(9D^3 - 130D^2 + 615D - 948\right) st^3 - 3(D-6)(D-5)(D-4)s^2t^2 \\
&- (D-5)\left(3D^2 - 30D + 74\right) s^3t + (D-6)(D-5)^2s^4\Big] \qquad \text{(B.1)}
\end{aligned}
$$

C Coefficients of the dimensional-shift system

We give here only the first three coefficients of the system (3.71). The whole list can be obtained from the authors (C.A. and C.O.).

$$
\begin{aligned}
a_1 &= \frac{3}{512}\frac{(D-4)s^2\left[(3D-14)(3D-10)s^2 - 4(5D^2 - 39D + 74)tu\right]}{(D-3)^3(2D-9)(2D-7)tu} \\
a_2 &= \frac{(D-6)(D-4)s\left[4(D-3)tu - 3(3D-10)s^2\right]}{512(D-5)(D-3)^3(2D-9)(2D-7)} \\
a_3 &= \frac{(D-4)s\left[4(D-6)(D-3)tu + 3(3D-14)(3D-10)s^2\right]}{512(D-5)(D-3)^3(2D-7)tu} \\
b_1 &= \frac{s\left[3(D-4)(3D-14)s^2 - 4(D-6)(D-5)tu\right]}{128(D-3)(2D-9)(2D-7)tu} \\
b_2 &= -\frac{(D-6)\left[4(D-3)tu + 3(D-4)s^2\right]}{128(D-5)(D-3)(2D-9)(2D-7)} \\
b_3 &= \frac{3(D-4)(3D-14)s^2 - 4(D-6)(D-3)tu}{128(D-5)(D-3)(2D-7)tu} \qquad \text{(C.1)}
\end{aligned}
$$

D Coefficients of the differential equations for the two master integrals

The coefficients of the system of differential equations (4.1) and (4.2) for the two master integrals are given by

$$
\begin{aligned}
h_1 &= \frac{(D-4)s^2 - 4tu}{4tu} \\
h_2 &= -\frac{(D-6)s}{4(D-5)} \\
h_3 &= \frac{(D-4)(2D-9)s}{4(D-5)tu} \\
h_4 &= \frac{3}{2}\frac{(D-4)(3D-14)u}{(D-5)st^2} \\
h_5 &= \frac{3}{2}\frac{(D-4)(3D-14)s^2}{(D-6)t^2u^2} \\
h_6 &= 3\frac{(D-3)(3D-14)}{(D-5)t^2} \\
h_7 &= \frac{3}{4}\frac{(D-3)(3D-10)\left[(3D-14)\left(u^2+t^2\right)+2(D-4)tu\right]}{(D-5)(D-4)st^2u^2} \\
h_8 &= \frac{3}{4}\frac{(D-3)(3D-10)(3D-8)\left[(D-5)(3D-14)\left(u^2+t^2\right)-(D-6)(D-4)tu\right]}{(D-5)^2(D-4)^2s^2t^2u^2} \\
h_9 &= \frac{3}{2}\frac{(D-3)(3D-14)(3D-10)(3D-8)}{(D-6)(D-5)^2(D-4)^2st^3u^2}\Big[(2D-9)(3D-16)u^2 \\
&\quad + \left(7D^2-68D+164\right)tu + 2(D-5)^2t^2\Big],
\end{aligned}
\tag{D.1}
$$

and by

$$
\begin{aligned}
k_1 &= \frac{(D-5)^2 s}{tu} \\
k_2 &= -\frac{(D-6)(u^2+t^2)}{2tu} \\
k_3 &= \frac{(D-4)(2D-9)}{tu} \\
k_4 &= 6\frac{(D-4)(3D-14)u\left[(5-D)u+(2D-11)t\right]}{(D-6)s^2t^3}
\end{aligned}
$$

$$
\begin{aligned}
k_5 &= \frac{3}{2}\frac{(D-5)(D-4)(3D-14)s^3}{(D-6)t^3u^3} \\
k_6 &= 3\frac{(D-3)(3D-14)}{(D-6)st^3u}\Big[(5D-28)tu+(D-6)t^2-2(D-5)u^2\Big] \\
k_7 &= \frac{3}{2}\frac{(D-3)(3D-10)}{(D-6)(D-4)s^2t^3u^3}\Big[2(D-6)(3D-14)tu\left(u^2+t^2\right) \\
&\quad -(D-5)(3D-14)\left(u^4+t^4\right)+2\left(5D^2-49D+118\right)t^2u^2\Big] \\
k_8 &= 3\frac{(D-3)(3D-10)(3D-8)}{(D-6)(D-5)(D-4)^2s^3t^3u^3}\Big[3(D-5)^2(3D-14)tu\left(u^2+t^2\right) \\
&\quad -(D-5)^2(3D-14)\left(u^4+t^4\right)-(D-4)\left(7D^2-70D+176\right)t^2u^2\Big] \\
k_9 &= 3\frac{(D-3)(3D-14)(3D-10)(3D-8)}{(D-6)^2(D-5)(D-4)^2s^2t^4u^3}\Big[(D-5)^2(D-2)u^4 \\
&\quad +(D-6)\left(13D^2-129D+318\right)tu^3+2\left(5D^3-80D^2+422D-734\right)t^2u^2 \\
&\quad +(D-6)(D-5)(5D-24)t^3u+(D-6)(D-5)^2t^4\Big].
\end{aligned}
\tag{D.2}
$$

References

[1] Z. Bern, L. Dixon, D.A. Kosower, *J. High Energy Phys.* **0001** (2000) 027.

[2] V.A. Smirnov, *Phys. Lett.* **B460** (1999) 397.

[3] V.A. Smirnov and O.L. Veretin, *Nucl. Phys.* **B566** (2000) 469.

[4] C. Anastasiou, E.W.N. Glover and C. Oleari, "The two-loop scalar and tensor pentabox graph with light-like legs", `hep-ph/9912251`. To appear in *Nucl. Phys.* **B**.

[5] C. Anastasiou, E.W.N. Glover and C. Oleari, *Nucl. Phys.* **B565** (2000) 445.

[6] J.B. Tausk, *Phys. Lett.* **B469** (1999) 225.

[7] O.V. Tarasov, *Phys. Rev.* **D54** (1996) 6479, *Nucl. Phys.* **B502** (1997) 455.

[8] F.V. Tkachov, *Phys. Lett.* **100B** (1981) 65;
K.G. Chetyrkin and F.V. Tkachov, *Nucl. Phys.* **B192** (1981) 159.

[9] T. Gehrmann and E. Remiddi, "Differential equations for two-loop four-point functions", Karlsruhe preprint TTP99-49, `hep-ph/9912329`.

[10] E. Remiddi, *Nuovo Cim.* **110A** (1997) 1435, `hep-th/9711188`.

[11] R.J. Gonsalves, *Phys. Rev.* **D28** (1983) 1542.

[12] G. Kramer and B. Lampe, *J. Math. Phys.* **28** (1987) 945.

[13] N. Nielsen, *Nova Acta Leopoldina* **90** (1909) 123.

[14] K.S. Kölbig, J.A. Mignaco and E. Remiddi, *B.I.T.* **10** (1970) 38.

[15] P.A. Baikov and V.A. Smirnov, "Equivalence of recurrence relations for Feynman integrals with the same total number of external and loop momenta", `hep-ph/0001192`.