Lecture 10 Lie groups & Lie algebras

In this lecture we will discuss some mathematics that will be relevant for the next topic that we will discuss — non-abelian gauge theories.

We will talk about Lie groups and Lie algebras.

We will first define what a group means.

A set of objects with a multiplication operation is called a group G provided that the following conditions are satisfied:

1) for any $a \in G$, $b \in G$, $a * b \in G$
2) for any $a, b, c \in G$, $(a*b)*c = a*(b*c)$
3) there is an element $e \in G$ (unity element) such that $a \cdot e = e \cdot a = a$, for any $a \in G$.
4) for any $a \in G$, there exists $a^{-1} \in G$, such that $a \cdot a^{-1} = e$.

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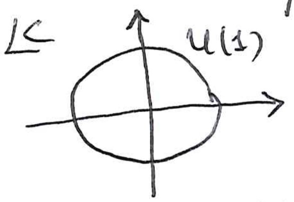
Groups can be both finite-dimensional and continuous. Finite groups are e.g. group of permutations of n objects, or symmetry groups of crystals. We will not talk about those. In particle physics, we are often interested in groups with very large (infinite) number of elements. An example of a continuous group is $U(1)$ → a set of complex number

whose absolute value is 1, $z \in \mathbb{C}$, $|z|=1$.
The multiplication operation is provided by a usual multiplication in $\mathbb{C}$. Then, since $|z_1 z_2| = |z_1||z_2|$, for $z_1, z_2 \in \mathcal{U}(1)$, $z_1 z_2 \in \mathcal{U}(1)$.
Note that $z_\varphi = e^{i\varphi}$, $\varphi \in [0, 2\pi]$, so that the $\mathcal{U}(1)$ group can be represented by a circle of radius 1 in a complex plane.



In fact, for all Lie groups, there is an intimate relation between them and certain geometric objects (surfaces, manifolds) in higher-dimensional vector spaces.

Another example of a group is a set of $n \times n$ unitary matrices, $\mathcal{U}(n)$. Unitary means that any $\mathcal{U}_1 \in \mathcal{U}(n)$ is such that $\mathcal{U}_1^+ \cdot \mathcal{U}_1 = 1$.
Unity element is an identity matrix $\hat{\mathbb{1}}^+ \cdot \hat{\mathbb{1}} = \hat{\mathbb{1}}$.
For $\mathcal{U}_{1,2} \in \mathcal{U}(n)$ $(\mathcal{U}_1 \cdot \mathcal{U}_2)^+ = \mathcal{U}_2^+ \mathcal{U}_1^+$ & so $\mathcal{U}_3 = \mathcal{U}_1 \cdot \mathcal{U}_2$ is unitary. The multiplication procedure is, obviously, just the matrix multiplication.

As we mentioned already, we will be interested in Lie groups. and, generalizing the examples of $U(1)$, we imagine that all these groups can be viewed as related to smooth geometric surfaces so that each group element (point on ~~a~~ surface) can be reached by a smooth path from identity element.

The standard representation of a Lie Group element is $U(\vec{\theta}) = \exp(i\theta^a T^a)\cdot\hat{\mathbb{1}}$, and where $\{\theta^a\}$ is a set of real numbers and T^a are called group (or algebra) generators.

Note that $\hat{\mathbb{1}}$ is an identity element and that $U(\vec{\theta}=0) = \hat{\mathbb{1}}$. Generators $\{T^a\}$ should satisfy the anticommutation relation

$$[T^a, T^b] = i f^{abc}\, T^c, \qquad [T^a, T^b] = T^a T^b - T^b T^a$$

where f^{abc} are called the "structure constants"

A group is called abelian if $U_1 \cdot U_2 = U_2 U_1$.

Take $U_1 = e^{i\theta_1^a T^a}$ $U_2 = e^{i\theta_2^a T^a}$ and consider $\theta_1^a \to 0$, $\theta_2 \to 0$.

$$U_1 U_2 = U_2 U_1 \longrightarrow \sum_{a,b} \theta_1^a \theta_2^b (T^a T^b - T^b T^a) = 0 \Rightarrow$$

$$\boxed{f^{abc} \equiv 0}$$, for ~~any~~ Abelian group.

The structure constants f^{abc} satisfy a relation that follows from the commutation relation. Consider the sum of 3 commutators. The result is

$$[T^a,[T^b,T^c]] + [T^b,[T^c,T^a]] + [T^c,[T^a,T^b]] = 0$$

Writing these commutators through structure constants, we obtain

$$\boxed{f^{abd} f^{dce} + f^{bcd} f^{dae} + f^{cad} f^{dbe} = 0}$$

This is called the Jacobi identity and the structure ~~functions~~ constants of any Lie algebra should satisfy it.

Note that, because of the relation between a group element and generators $U = e^{i\theta^a T^a}$, generators follow from $\frac{1}{i}\frac{\partial U}{\partial \theta^a}\Big|_{\vec{\theta}\to 0} = T^a$.

A Linear space spanned by ~~linear~~ generators T^a is called Lie algebra. An ideal of an algebra G is a subalgebra $I \in G$ such that for any $g \in G$ and $i \in I$, $[g,i] \in I$.
A simple Lie algebra has no ideals.
An example of simple algebras are $SU(N)$ (group of unitary matrices with determinant 1) and $SO(N)$ (group of orthogonal matrices with determinant 1).

The importance of simple algebras in physics is related to the fact that their finite-dimensional representations are Hermitian. For this reason they are useful for constructing unitary theories.

Representations. Let us now talk about "representations": The "representation" of a group G in a linear space V is a mapping $g \to T(g)$, where $T(g)$ is a linear operator, acting in V. The mapping should ~~have~~ satisfy obvious requirements

$$T(g_1 \cdot g_2) = T(g_1) \cdot T(g_2)$$
$$T(g^{-1}) = [T(g)]^{-1}.$$

The representation of a Lie algebra G is defined similarly, i.e for A & $B \in G$,

$$T(A+B) = T(A) + T(B)$$
$$T([A,B]) = [T(A), T(B)]$$

If we fix the basic of the linear space V $\{e_i\}$, action of $T(g)$ on V is described by acting of $T(g)$ on e_i. Let's write the result of this operation in terms of basis vector and we obtain

$$\boxed{T(g)\, e_i = e_j\, T_{ji}(g)}$$

This implies that we can think about representation as a procedure that gives a mapping ~~from~~ of group elements to $n \times n$ matrices, where n is the dimensionality of vector space V.

Let us consider a group $SU(\underline{N})$. Elements of this group can be represented by $\underline{N \times N}$ unitary (special unitary) matrices. Consider a vector space of N-dimensional vectors ψ.

For $U \in SU(N)$. $(T(U)\psi)_i = U_{ij}\psi_j$

Such a representation is called fundamental

To construct representation of the algebra, we take $U = \hat{\mathbb{1}} + i\theta^a T^a$, where T^a are hermitian, traceless matrices and we have a representation of Lie algebra elements provided that
$[T^a, T^b] = if^{abc}T^c$.

Anti-fundamental representation for an element $U \in SU(N)$ is given by $(T_{AF}(U)\psi)_i = \overset{*}{U}_{ij}\psi_j$

$$U_{ij} \simeq 1_{ij} + i\theta^a T^a_{ij} \Rightarrow \overset{*}{U}_{ij} \simeq 1_{ij} - i\theta^a \overset{*}{T}{}^a_{ij} = $$
$$= 1_{ij} - i\theta^a T^a_{ji}$$

Hence $[T_{AF}(U)\psi]_i \to \psi_i - i\theta^a \psi_j T^a_{ji}$

So the anti-fundamental representation operates on columns.

One can explicitely write down algebra generators ~~in~~ of the fundamental representations of simple Lie algebras.

For $SU(2)$, we have $\tau^a = \frac{\sigma^a}{2}$, where

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are the Pauli matrices.

For $SU(3)$, we have ~~Pauli~~ Gell-mann matrices as Lie algebra generators.

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda^4 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \lambda^5 = \begin{pmatrix} 0 & & -i \\ 0 & 0 & 0 \\ i & & 0 \end{pmatrix} \quad \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

~~The generators of the Lie~~

Another popular representation is called adjoint.

Here, the generators are represented by the structure constants $(T_A{}^a)_{bc} = -if^{abc}$

$$(T_A^{\dagger a})_{bc} = (T^a_{cb})^* = if^{acb} = -if^{abc} = (T^a)_{BC}$$

Also, the commutation relation follows from Jacobi identity since ~~adjoint~~ the structure constants are real the vector space V_A for the adjoint repr. can be choosen to be real-valued.

~~For SU(2),~~ Note that the dimensionality of the adjoint repr. coincides with the dimensionality of Lie algebra (# of independent generators) which, for $SU(N)$, is N^2-1. As an example, we can write explicitely the generators in adjoint representation of $SU(2)$:

$$T_A^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \qquad T_A^2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} \qquad T_A^3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Note that these are 3×3 matrices that represent the same generators as what Pauli matrices do in the fund. rep.

So far we have distinguished different representations by their dimensionality and by ~~[illegible]~~ explicitely writing the generators in the matrix form. It would be useful if we could avoid the need to do that. Note that in Quantum mechanics different representations (irreducible) of the rotation group are ~~represented~~ states with different value of the total spin $\hat{J}$ and all states in the corresponding irreducible representation have the same value of spin, meaning that they are eigenstates of the operator $\vec{J}^2 = \sum_{i=1}^{3} J_i^2$.

Note that $\vec{J}^2$ commutes with all generators of the rotation group J_i and this is the reason why it can be used to classify irreducible representations.

We would like to do ~~the~~ something similar for the case of $su(N)$. Consider the operator $\vec{T}^2 \equiv \sum_{a=1}^{N^2-1} T^a \cdot T^a$ and compute

$$[\vec{T}^2, T^b] = \sum_{a=1}^{N^2-1} [T^a \cdot T^a, T^b] =$$

$$= \sum_{a=1}^{N^2-1} \left(T^a [T^a, T^b] + [T^a, T^b] \cdot T^a\right) =$$

$$= \sum_{\substack{a=1 \\ c=1}}^{N^2-1} \left[T^a \cdot i f^{abc} T^c + i f^{abc} T^c T^a\right]$$

= (change dummy indices $a \leftrightarrow c$ in the second term)

$$= \sum_{\substack{a=1 \\ c=1}}^{N^2-1} \left[T^a T^c (i f^{abc} + i f^{cba})\right] = 0, \text{ since}$$

$$f^{abc} = - f^{cba}.$$

Therefore, $[\vec{T}^2, T^a] = \emptyset$. According to Schur's lemma, ~~in~~ in any irreducible representation, an operator that commutes with all generators of a Lie algebra is proportional to an identity operator

$$\boxed{\sum_{a=1}^{N^2-1} T^a \cdot T^a \equiv C_2(R) \cdot \mathbb{1}}$$

$C_2(R)$ here is the representation-dependent constant — the Casimir operator.

This constant assumes different values in different representations and, therefore, can be used to distinguish them.

To evaluate $C_2(R)$, we write down the normalization conditions for the generators

$$tr\left[T_R^a T_R^b\right] = T(R)\,\delta^{ab}.$$

Contract it with δ^{ab}. $\delta^{ab}\cdot\delta_{ab} \equiv d(G)$, the dimension of the Lie algebra (# of generators)

$$tr\left(T_R^a \cdot T_R^b\right)\delta_{ab} = tr(\hat{\mathbb{1}})\cdot C_2(R) \equiv d(R)\cdot C_2(R),$$

where $d(R)$ is the dimension of the representation.

Hence, we find

$$\boxed{C_2(R)\cdot d(R) = T(R)\cdot d(G)},$$

so that

$$\boxed{C_2(R) = \frac{T(R)\cdot d(G)}{d(R)}}$$

Let us calculate the Casimir for two popular choices of representations:

1) fundamental: $Tr\left(T_F^a \cdot T_F^b\right) \equiv \frac{1}{2}\delta^{ab}$ is the standard choice; $\Rightarrow T(F) = 1/2$.

$d(G) = N^2-1 \rightarrow$ independent of rep.

$d(F) = N \Rightarrow$

$$\boxed{C_2(F) = \frac{N^2-1}{2N}}$$

2) <u>adjoint</u> $\mathrm{Tr}(T_A^a T_A^b) =$

$$= -(f^a)_{km}(f^b)_{mk} = -f^{akm} f^{bmk} =$$

$$= \underline{f^{akm} f^{bkm} = N \cdot \delta^{ab}}$$ (this ~~choice~~ result

is consistent with the choice of generators in the fundamental repr. since $[T^a, T^b] = i f^{abc} T^c$ is <u>not</u> invariant under the change in the normalization of operators.)

So $C_2(A) = \dfrac{N \cdot (N^2-1)}{(N^2-1)}$, since $T(A) = N$, $d(G) = d(A) = N^2-1$

$$\Rightarrow \boxed{C_2(A) = N}$$

In practical applications, as we will see shortly, we will require calculations of "color factors" to describe scattering amplitudes. These can be e.g. traces of products of $SU(N)$ generators in the fundamental repr. or contractions of the structure constants.

One can compute those things in different ways using various results discussed in these lectures.

For example, $f_{abc}\,\mathrm{Tr}[T^a T^b T^c] =$

$$= \frac{1}{2} f_{abc}\,\mathrm{Tr}[T^a, [T^b, T^c]] = \frac{1}{2} f_{abc}\, i f^{bck} \times$$

$$\times\, \mathrm{Tr}[T^a T^k] = \frac{i}{2} f_{abc} f^{bck} \frac{1}{2} \delta^{ak} =$$

$$= \frac{i}{4} f^{abc} f^{kbc} \delta^{ak} = \frac{i}{4} \cdot N \delta^{ak} \cdot \delta^{ak} = \frac{i}{4} N(N^2-1).$$

$$\Rightarrow f_{abc} \, \mathrm{Tr}\left[T^a T^b T^c\right] = \frac{i}{4} N(N^2-1).$$

Another useful relation is the analog of the Fiertz identity for $SU(N)$ generators in the fundamental representation.

Consider ~~Write~~ $\sum_a T^a_{ij} \cdot T^a_{mk}$. This quantity can be written as

$$\boxed{\sum_a T^a_{ij} \cdot T^a_{mk} = A \delta_{ik} \delta_{mj} + B \delta_{ij} \delta_{mk}}, \quad (**)$$

since only in this case ~~for~~ $SU(N)$ transformation properties of the l.h.s. and the r.h.s. should be the same.

~~As ...~~ To find A and B, we contract Eq. (**) with δ_{ij}. Since T^a is trace-less, we find $0 = \delta_{ij} T^a_{ij} T^a_{mk} = A \delta_{km} + BN \delta_{mk}$

$$\Rightarrow \boxed{B = -\frac{A}{N}}$$

To find A, we contract with δ_{jn}:

$$T^a_{ij} \cdot T^a_{mk} \delta_{jn} = \left(T^a T^a\right)_{ik} = \frac{N^2-1}{2N} \delta_{ik}.$$

At the same time,

$$\delta_{jn} A\left(\delta_{ik}\delta_{mj} - \frac{1}{N}\delta_{ij}\delta_{mk}\right) = A \delta_{ik}\left(N - \frac{1}{N}\right) =$$

$$= A \frac{N^2-1}{N}. \Rightarrow \boxed{A = \frac{1}{2}} \Rightarrow$$

The Fiertz identity, therefore, becomes

$$\boxed{\sum_a T^a_{ij} \cdot T^a_{mk} = \frac{1}{2}\left(\delta_{ik}\,\delta_{mj} - \frac{1}{N}\,\delta_{ij}\,\delta_{mk}\right)}$$

Finally since in any representation

$$\mathrm{tr}\left([T^a_R, T^b_R]\cdot T^c_R\right) = i f^{abd}\,\mathrm{tr}\left(T^d_R T^c_R\right) = i f^{abc}\,T(R),$$

it is always possible to ~~..~~ remove structure constants in favor of traces of generators. For example, for $R = F$, we find

$$\boxed{f^{abc} = \frac{-i}{T_F}\,\mathrm{tr}\left([T^a, T^b]\,T^c\right)}$$