Gauge-invariance, Ward identities and all that

A recap of gauge invariance in QED.

$$\mathcal{L}_{QED} = \bar{\psi}\,[i\gamma^\mu D_\mu - m]\,\psi - \frac{1}{4} F_{\mu\nu}F^{\mu\nu}, \quad D_\mu = \partial_\mu - ieA_\mu.$$

This Lagrangian is invariant under

$$\psi \to e^{i\theta(x)}\psi(x) \quad \& \quad A_\mu \to A_\mu + \frac{1}{e}\partial_\mu\theta(x),$$

the "gauge transformations". This freedom has important consequences: suppose we want to compute the photon propagator. We need to take the $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$, find the equations of motion

$$\partial_\mu F^{\mu\nu} = 0$$

and solve it for the δ-like sourse on the R.H.S:

$$\partial_\mu F^{\mu\nu} = J^\nu$$

$$[g^{\mu\nu}\Box - \partial^\mu\partial^\nu]\,A^\mu = J^\nu \quad \Rightarrow$$

$$A^\mu(x) = \int \Pi^{\mu\nu}(x-y)\,J^\nu(y)\,d^4y \quad \Rightarrow$$

$$\boxed{[g^{\mu\nu}\Box_x - \partial^\mu_x\partial^\nu_x]\,\Pi^{\mu\alpha}(x) = i\delta^{(4)}(x)\,g^{\nu\alpha}}$$

Solving this equation will give us photon propagator (rep to a causal prescription).

We know that this modification of the photon propagator does not change the theory because of the conservation of the electromagnetic current $J^\mu = \bar{\psi}\gamma^\mu\psi$.

Indeed: the ~~coupling~~ interaction term is $\sim A_\mu J^\mu$

$$A_\mu \to p_\mu \quad , \quad A_\mu J^\mu \to p_\mu \cdot J^\mu \to \partial_\mu J^\mu \equiv \phi.$$

That is the reason why the modification of the QED Lagrangian with $-\frac{1}{2\xi}(\partial_\mu A^\mu)^2$ term does not change the theory.

– Let's make the discussion ~~also~~ of the role of the current conservation for Green's functions more precise.

To this end, consider a simple Green's function $\Pi^{\mu\nu}(x) = \langle 0 | T J^\mu(x) J^\nu(0) | 0 \rangle$

Compute $\partial_\mu \Pi^{\mu\nu}(x)$.

$$\partial_\mu \Pi^{\mu\nu}(x) = \partial_\mu \{ \theta(x_0) J^\mu(x) J^\nu(0) + \theta(-x_0) J^\nu(0) J^\mu(x) \}$$

$$= \delta(x_0) \left(J^0(x) J^\nu(0) - J^\nu(0) J^0(x) \right) \Rightarrow$$

$$\partial_\mu \Pi^{\mu\nu}(x) = \delta(x_0) \left[J^{(0)}(x), J^\nu(0) \right]$$

The $\delta(x_0)$ ensures that the commutator is <u>equal-time</u> & equal-time commutators

As we know, this condition has important consequences for the renormalizability of QED $\Rightarrow$ it ~~automatically~~ ensures that the photon mass ~~always~~ <u>automatically remains</u> <u>zero</u> and the only adjustment that we need to do is the photon field renormalization.

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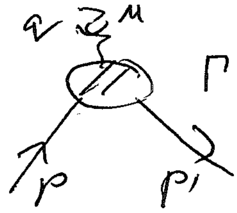
There is another ~~the~~ example where gauge-invariance enforces relations between Green's functions and those relations lead to constraints on the renormalization constants.

There is a relation between the 1PI three-point function



Γ^μ and the propagators of the two electrons

$$q_\mu \hat{\Gamma}^\mu = i\,\hat{S}_F^{-1}(p') - i\,\hat{S}_F^{-1}(p),$$

where $\hat{S}_F(p) = \dfrac{i}{\not{p} - m - \hat{\Sigma}(p)}$. } This is Ward identity.

From the proof of this equation, it follows that it applies to ~~bare~~ unrenormalized quantities ~~without~~ ~~counterterms~~

Now, consider the limit $q \to 0$, $p'^2 \to m^2$, $p^2 \to m^2$.

Then $q_\mu \hat{\Gamma}^\mu \to \hat{q}\, F_1(0)$, where $F_1(q^2)$ is the Dirac form-factor. Also, $\hat{p}' - m - \hat{\Sigma}(p') - (\hat{p} - m - \hat{\Sigma}(p))$

$$= \hat{q} - (\hat{\Sigma}(\hat{p}+\hat{q}) - \hat{\Sigma}(\hat{p})) \simeq \hat{q} - \frac{\partial \hat{\Sigma}}{\partial \hat{p}}\hat{q} \simeq$$

$$\simeq \hat{q}\left(1 - \frac{\partial \hat{\Sigma}}{\partial \hat{p}}\right).$$

Now, considering Ward identity, we find

$$\boxed{F_1(0) = \left(1 - \left.\frac{\partial \hat{\Sigma}}{\partial \hat{p}}\right|_{\hat{p}=m}\right).}$$

We now consider this equation at 1-loop.

We find $F_1(0) = 1 + \delta F_1(0)$ & $\delta F_1(0) = -\delta_1$.

Hence $F_1(0) \simeq \dfrac{1}{Z_1}$. Similarly, $1 - \left.\dfrac{\partial \hat{\Sigma}}{\partial \hat{p}}\right|_{\hat{p}=m} \simeq \dfrac{1}{Z_2}$

Then, it follows from the Ward-identity that $Z_1^{-1} = Z_2^{-1} \Rightarrow Z_1 = Z_2$.

This is an exact relation between the two renormalization constants (valid to all orders in pert. theory) that is a direct consequence of gauge invariance.

can be computed using canonical quantization conditions:

$$\{ \psi^\dagger_\alpha(t,\vec{x}), \psi_\beta(t,\vec{y}) \} = \delta^{(3)}(\vec{x}-\vec{y})\, \delta_{\alpha\beta}$$

$$\{ \psi^\dagger_\alpha(t,\vec{x}), \psi^\dagger_\beta(t,\vec{y}) \} = \{ \psi_\alpha(t,\vec{x}), \psi_\beta(t,\vec{y}) \} \equiv \phi$$

Now, we have $J^0(x) = \psi^\dagger_\alpha (\gamma^0\gamma^0)_{\alpha\beta} \psi_\beta \big|_x = \psi^\dagger_\alpha \psi_\alpha \big|_x$

$$J^\nu(0) = \psi^\dagger_\alpha (\gamma^0\gamma^\nu)_{\alpha\beta} \psi_\beta = (\gamma^0\gamma^\nu)_{\alpha\beta}\, \psi^\dagger_\alpha \psi_\beta \big|_0$$

$$\Rightarrow \left[J^{(0)}(x), J^\nu(0) \right]\Big|_{x_0=0} = (\gamma^0\gamma^\nu)_{\rho\sigma} \quad \otimes$$

$$\left[\psi^\dagger_\alpha(x)\, \psi_\alpha(x), \psi^\dagger_\rho(0)\, \psi_\sigma(0) \right] =$$

$$= (\gamma^0\gamma^\nu)_{\rho\sigma} \Big(\psi^\dagger_\alpha(x) \{ \psi_\alpha(x), \psi^\dagger_\rho(0) \} \psi_\sigma(0) - \psi^\dagger_\rho(0) \{ \psi^\dagger_\alpha(x), \psi_\sigma(0) \} \psi_\alpha(x) \Big)$$

$$= (\gamma^0\gamma^\nu)_{\rho\sigma} \Big[\psi^\dagger_\alpha(x)\, \psi_\sigma(0)\, \delta_{\alpha\rho}\, \delta^{(3)}(\vec{x}) - \psi^\dagger_\rho(0)\, \delta_{\alpha\sigma}\, \delta^{(3)}(\vec{x})\, \psi_\alpha(x) \Big] \equiv 0 \Rightarrow$$

$$\partial_\mu \Pi^{\mu\nu}(x) = \partial_\mu \langle 0 | T J^\mu(x) J^\nu(0) | 0 \rangle \equiv 0.$$

Now, take the Fourier transform of both sides, to obtain

$$\boxed{0 = \int d^4x\, e^{iq\cdot x}\, q_\mu \langle 0 | T J^\mu(x) J^\nu(0) | 0 \rangle}$$

$$\Rightarrow \quad q_\mu \Pi^{\mu\nu}(q) = 0 \quad \Rightarrow$$

$$\Pi^{\mu\nu}(q) = (q^2 g^{\mu\nu} - q^\mu q^\nu)\, \Pi(q^2).$$

To solve it, write $\Pi^{\mu\alpha}(x) = \int d^4p\, e^{ipx}\, \Pi^{\mu\alpha}(p)$ so that $\left[-p^2 g^{\mu\nu} + p^\mu p^\nu\right] \Pi^{\mu\alpha}(p) = i g^{\nu\alpha}$, which implies that $\Pi^{\mu\alpha}(p)$ is the <u>inverse matrix</u> of $-p^2 g^{\mu\nu} + p^\mu p^\nu$.

It is easy to see, however that the <u>inverse does not exist</u>, since $\det(-p^2 g^{\mu\nu} + p^\mu p^\nu) = 0$ (indeed, $(-p^2 g^{\mu\nu} + p^\mu p^\nu)\, p^\nu = 0 \Rightarrow$ there is a vanishing eigenvalue).

To take care of this problem, we change the QED Lagrangian:

$$\boxed{\mathcal{L}_{QED} \to \mathcal{L}_{QED} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2},$$

where ξ is a gauge parameter. The new Lagrangian <u>is not</u> invariant under gauge transform. anymore. However, a similar calculation as above leads to

$$\left[-p^2 g_{\mu\nu} + \left(1 - \frac{1}{\xi}\right) p_\mu p_\nu\right] A_\mu = J_\nu \quad \Rightarrow$$

$$\Pi_{\mu\nu}(p) = \frac{-i}{p^2 + i\varepsilon}\left[g_{\mu\nu} - (1-\xi)\frac{p_\mu p_\nu}{p^2}\right],$$

a "photon propagator".