$$\boxed{\text{Lecture 8. One-loop renormalization in QED: wave function, mass and charge renormalization}}$$

In the previous Lecture we discussed the 1-loop renormalization of $\lambda\varphi^4$ theory. We will now generalize this onto the case of QED. Recall that the basic idea is to distinguish between "bare" (unphysical) quantities and physical (observable) quantities. The former enter the original Lagrangian. The latter can be introduced there at the price of an additional terms – the counter-terms – in $\mathcal{L}$. The counter-terms appearing are not known a'priory but can be determined by imposing conditions on Green's functions to satisfy and adjusting the counter-terms then.

We ~~repeat~~ now work out the details of this program for QED. The Lagrangian

$$\mathcal{L} = -\frac{1}{4} F^{(0)}_{\mu\nu} F^{(0)\mu\nu} + \bar{\psi}_0 (i\hat{\partial} - m_0)\psi_0 - e_0 \bar{\psi}_0 \gamma^\mu \psi_0 A^{(0)}_\mu \,, \quad F^{(0)}_{\mu\nu} = \partial_\mu A^0_\nu - \partial_\nu A^0_\mu$$

is written in terms of bare fields, bare electric charge and bare mass.

By introducing physical fields for electron and photon through

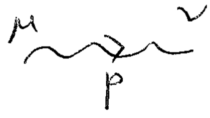
$\psi_0 = Z_2^{1/2}\psi$, $A_\mu^{(0)} = Z_3^{1/3} A_\mu$, we obtain

$$\mathcal{L} = -\frac{1}{4} Z_3 F_{\mu\nu} F^{\mu\nu} + Z_2 \bar{\psi}(i\hat{\partial} - m_0)\psi - e_0 Z_3^{1/2} Z_2 \bar{\psi}\gamma^\mu \psi A_\mu$$

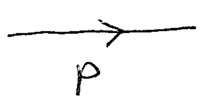
Now, introducing $\delta_3 = Z_3 - 1$, $\delta_2 = Z_2 - 1$, $\delta_m = Z_2 m_0 - m$, and $\delta_1 = \frac{e_0}{e} Z_2 Z_3^{1/2} - 1$, we rewrite $\mathcal{L}$ as

$$\boxed{\begin{aligned}\mathcal{L} = &\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\hat{\partial} - m)\psi - e\bar{\psi}\gamma^\mu\psi A_\mu \\ &- \frac{\delta_3}{4} F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\delta_2\hat{\partial} - \delta_m)\psi - e\delta_1\bar{\psi}\gamma^\mu\psi A_\mu .\end{aligned}} \quad (**)$$

The first line is the Lagrangian proper, the second line is due to counterterms. Electric charge "e" and the mass parameter "m" are physical charge and mass of the electron. Similar to $\lambda\varphi^4$ case, defining A_μ, ψ, e & m precisely is important. We will do this shortly but first we will write down the Feynman rules that follow from Lagrangian $(**)$:



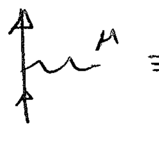
$= \frac{-ig^{\mu\nu}}{p^2 + i0}$ (Photon propagator, Feynman gauge)



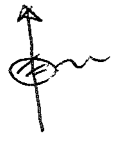
$\equiv \frac{i}{\hat{p} - m + i0}$ (electron propagator)

$\equiv -i(g^{\mu\nu}q^2 - q^\mu q^\nu)\delta_3$

$= i(\not{p}\delta_2 - \delta_m)$



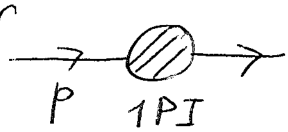
$\equiv -ie\hat{\gamma}^\mu$



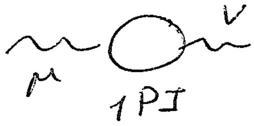
$\equiv -ie\gamma^\mu\delta_1 .$

Definition of physical parameters follows from ~~the~~ conditions of ~~the~~ on appropriate Green's functions. To this end, we require that photon and electron propagators do not receive corrections close to their single-particle poles. A we saw in case of $\lambda\varphi^4$ theory, this condition can be expressed in terms of 1-particle irreducible self-energy contribution. The story in QED is similar.

Define 1PI 2-point functions as



$= -i\hat{\Sigma}(p)$

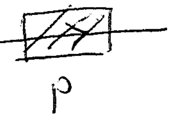


$= i\Pi^{\mu\nu}(q) = i(g^{\mu\nu}q^2 - q^\mu q^\nu)\Pi(q^2)$

The conditions that we impose are

$$\boxed{\begin{aligned} &\hat{\Sigma}(\hat{p} \equiv m) = \phi \quad \text{and} \quad \left.\frac{d\hat{\Sigma}}{d\hat{p}}\right|_{\hat{p}=m} = \phi \\ &\Pi(\phi) \equiv \phi \end{aligned}}$$

To see why these conditions are important, let us explore the first one. ~~Agg~~ Similar to $\lambda\varphi^4$ case, we need to write

$\hat{S}(p) =$ 

$$= \frac{i}{\not{p}-m} + \frac{i}{\not{p}-m}\left(-i\hat{\Sigma}(p)\right)\frac{i}{\not{p}-m} + \cdots$$

$$= \frac{i}{\not{p}-m}\left\{1 \mp i\hat{\Sigma}(p)\left[\frac{i}{\not{p}-m} + \frac{i}{\not{p}-m}(-i\Sigma(p))\frac{i}{\not{p}-m} + \cdots\right]\right\}$$

We realize that expression in square brackets $[\ldots]$ is $\hat{S}(p)$. Therefore

$$\hat{S}(p) = \frac{i}{\hat{p}-m}\left\{1 - i\,\hat{\Sigma}(p)\,\hat{S}(p)\right\}. \qquad (*)$$

Introducing $\frac{i}{\hat{p}-m} \equiv \hat{S}_0(p)$, we rewrite Eq. $(*)$

as $\hat{S}(p) = \hat{S}_0(p)\left[1 - i\,\hat{\Sigma}(p)\,\hat{S}(p)\right] \Rightarrow$

$$\hat{S}_0^{-1}(p) = \hat{S}^{-1}(p) - i\,\hat{\Sigma}(p). \Rightarrow$$

$$\hat{S}^{-1}(p) = \hat{S}_0^{-1}(p) + i\,\hat{\Sigma}(p) = -i\left(\hat{p} - m - \hat{\Sigma}(p)\right)$$

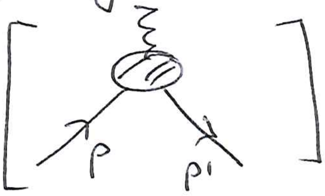
$$\Rightarrow \boxed{\hat{S}(p) = \frac{i}{\hat{p} - m - \hat{\Sigma}(p)}.}$$

Therefore, if $\boxed{\hat{\Sigma}(m) \equiv \emptyset \text{ and } \left.\frac{d\hat{\Sigma}}{d\hat{p}}\right|_{\hat{p}=m} = 0,}$ the propagator $\hat{S}(p)$ describes a particle with the mass ~~mas~~ m (i.e. $p^2 = m^2$ is a pole of the propagator), ~~and~~ that is an excitation of a canonically normalized field.

It is possible to show that the assumed form of the photon polarization ~~assumes~~ implies that the photon is massless and that the photon field is canonically normalized.

Our final condition allows us to fix the physical coupling "e". To this end, we require that the following holds

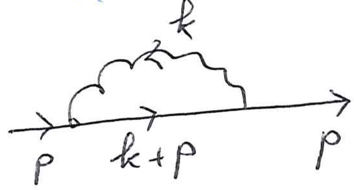
$$\left[\ \right]_{\text{amputated}} \rightarrow -ie\gamma^\mu, \text{ for } p' \to p \text{ and } p'^2 \equiv p^2 \equiv m^2$$



We will now derive one-loop results for QED ~~contributions~~ counterterms.

We begin with the fermion self-energy

$$\boxed{-i\,\Sigma(\hat{p}) = -i\,\hat{\Sigma}_A(p) - i\,\hat{\Sigma}_{ct}}$$



$$= -i\,\Sigma_A(p)\,; \quad \rightarrow\otimes\rightarrow = -i\,\Sigma_{ct}(p) = = i(\hat{p}\,\delta_2 - \delta_m)$$

As usual, counterterms are not known; we need to use physical conditions for $\Sigma(\hat{p})$ to find δ_2 and δ_m.

The loop integral $\hat{\Sigma}_A(p)$ reads

$$-i\,\Sigma_A(p) = (-ie)^2 \int \frac{d^d k}{(2\pi)^d} \frac{\gamma^\alpha(\hat{p}+\hat{k}+m)\gamma_\alpha}{k^2[(k+p)^2-m^2]}$$

We can use standard dim.reg ~~results~~ and familiar rules for Feynman integral calculus:

$$\gamma^\alpha(\hat{p}+\hat{k}+m)\gamma_\alpha = (2-d)(\hat{p}+\hat{k}) + d\cdot m\,;$$

$$\frac{1}{k^2[(k+p)^2-m^2]} \equiv \int_0^1 dx \frac{1}{[(k+px)^2+p^2x(1-x)-m^2x]^2}$$

Shifting the loop momentum and integrating over it, we find

$$-i\Sigma_A(p) = -ie^2\int_0^1 dx \, \frac{\Gamma(1+\varepsilon)}{(4\pi)^{d/2}} \otimes \frac{1}{\epsilon} \frac{\left[(2-d)\hat{p}(1-x)+dm\right]}{\left[m^2x-p^2x(1-x)\right]^{2-\frac{d}{2}}}$$

This is a generic expression that is valid for arbitrary p. However, for $\boxed{\hat{p} \equiv m}$, we have a simpler result

$$-i\Sigma_A(m) = -ie^2\frac{\Gamma(1+\varepsilon)}{(4\pi)^{d/2}}\frac{1}{\epsilon}\int_0^1 dx \, \frac{m\left[(2-d)(1-x)+d\right]}{(m^2x^2)^{\epsilon}} =$$

$$= \frac{-ie^2\Gamma(1+\varepsilon)}{(4\pi)^{d/2}}\frac{1}{\epsilon}\, m^{1-2\varepsilon}\int_0^1 \frac{dx}{x^{2\varepsilon}}\left(2+x\cdot 2(1-\varepsilon)\right)$$

$$= \frac{-ie^2\Gamma(1+\varepsilon)}{(4\pi)^{d/2}}\frac{1}{\epsilon}\, m^{1-2\varepsilon}\left\{^{0}\frac{1}{1-2\epsilon}+\frac{1-\varepsilon}{2-2\varepsilon}\right\}\cdot 2$$

$$= \frac{-ie^2\Gamma(1+\varepsilon)}{(4\pi)^{d/2}}\frac{1}{\varepsilon}\, m^{1-2\varepsilon}\left\{\frac{2}{1-2\varepsilon}+1\right\} = \frac{-ie^2\Gamma(1+\varepsilon)}{(4\pi)^{d/2}}\frac{1}{\epsilon}\, m^{1-2\varepsilon}\frac{3-2\varepsilon}{1-2\varepsilon}$$

The value of $-i\Sigma(\hat{p})$, for $\hat{p}=m$, becomes

$$-i\Sigma(m) = \frac{-ie^2\Gamma(1+\varepsilon)}{(4\pi)^{d/2}}\frac{1}{\epsilon}\, m^{1-2\varepsilon}\left(\frac{3-2\varepsilon}{1-2\varepsilon}\right) + i\left(\hat{p}\,\delta_2-\delta_m\right)\Big|_{\hat{p}=m}$$

Since $-i\Sigma(m) \equiv 0$, we find

$$\boxed{m\,\delta_2-\delta_m \equiv \frac{e^2\Gamma(1+\varepsilon)}{(4\pi)^{d/2}}\frac{m^{1-2\varepsilon}}{\epsilon}\frac{3-2\varepsilon}{1-2\varepsilon}}$$

The second condition requires calculation of $\left.\frac{\partial \Sigma(\hat{p})}{\partial \hat{p}}\right|_{\hat{p}=m}$. To compute this derivative, the only thing you need to remember is that $\hat{p}\cdot\hat{p} \equiv p^2$. Therefore

$\frac{\partial p^2}{\partial \hat{p}} \equiv 2\hat{p}$. Therefore

$$\frac{\partial}{\partial \hat{p}}\left(-i\hat{\Sigma}_A(\hat{p})\right) = ie^2 \int_0^1 dx \frac{\Gamma(1+\varepsilon)}{(4\pi)^{d/2}} \frac{1}{\epsilon} \left\{ \frac{(2-d)(1-x)}{[m^2x - p^2x(1-x)]^\epsilon} - \frac{\epsilon\left[(2-d)\hat{p}(1-x) + dm\right]}{(m^2x - p^2x(1-x))^{1+\varepsilon}}\left(-2\hat{p}x(1-x)\right)\right\}$$

We set now $\hat{p}$ to m, to obtain the derivative at the pole. Then, the expression in the curly brackets turns to

$$\{\dots\} \longrightarrow \frac{(2-d)(1-x)}{(\cancel{m\,x}\; m^{2\varepsilon}x^{2\varepsilon}} + \frac{2\varepsilon\left[2+x(2-2\varepsilon)\right](1-x)}{m^{2\varepsilon}\,x^{1+2\varepsilon}}$$

The condition $\boxed{\left.\frac{d\Sigma(\hat{p})}{d\hat{p}}\right|_{p=m} = 0}$ involves derivative of the counterterm; $\frac{d}{d\hat{p}}\left(i(\hat{p}\delta_2 - \delta_m)\right) = i\delta_2$.

We obtain, after

$$0 = \delta_2 + \frac{e^2\Gamma(1+\varepsilon)}{(4\pi)^{d/2}}\frac{m^{-2\varepsilon}}{\epsilon}\left\{\frac{(2-d)}{(1-2\varepsilon)(2-2\varepsilon)} + 2\varepsilon\left[\frac{2}{-2\varepsilon} - \frac{(4-d)}{1-2\varepsilon} + \frac{(2-d)}{2-2\varepsilon}\right]\right\}.$$

Using $d = 4-2\varepsilon$ & expanding in ε, we obtain

$$\boxed{\delta_2 = -\frac{e^2\Gamma(1+\varepsilon)}{(4\pi)^{d/2}}\, m^{-2\varepsilon}\left(\frac{3}{\varepsilon} + 4\right)}$$

Using Equation for $m\delta_2 - \delta_m$ on page 6, we find

$$\boxed{\delta_m = \frac{-2e^2\Gamma(1+\varepsilon)}{(4\pi)^{d/2}} m^{1-2\varepsilon}\left(\frac{3}{\epsilon} + 4\right)}$$

We will now turn δ_2 & δ_m into expressions for the renormalization constants Z_2 and Z_m.

For Z_2, we get $\boxed{Z_2 = 1 + \delta_2 = 1 - \frac{e^2\Gamma(1+\varepsilon)}{(4\pi)^{d/2}}\left(\frac{3}{\epsilon} + 4\right)}$

For ~~Z_m~~ the mass renormalization constant, we define $m_0 = Z_m \cdot m$ and then rewrite $\delta_m = Z_2 m_0 - m = (Z_2 Z_m - 1)\, m \Rightarrow$

$$Z_m = \frac{1}{Z_2}\left[1 + \frac{\delta_m}{m}\right] \approx 1 + \frac{\delta_m}{m} - \delta_2 \Rightarrow$$

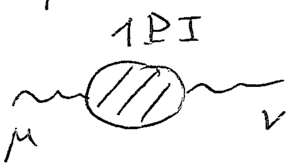
$$Z_m = 1 - \frac{2e^2\Gamma(1+\varepsilon)}{(4\pi)^{d/2}} m^{-2\varepsilon}\left(\frac{3}{\varepsilon} + 4\right) + \frac{e^2\Gamma(1+\varepsilon)}{(4\pi)^{d/2}}\left(\frac{3}{\varepsilon} + 4\right) \Rightarrow$$

$$\boxed{Z_m = 1 - \frac{e^2\Gamma(1+\varepsilon)}{(4\pi)^{d/2}}\left(\frac{3}{\varepsilon} + 4\right)}$$

We see that $Z_m = Z_2$ at one-loop. This equality is an accident and it is violated if higher-order radiative corrections are taken into account.

Next, we discuss calculation of the photon field renormalization δ_3.

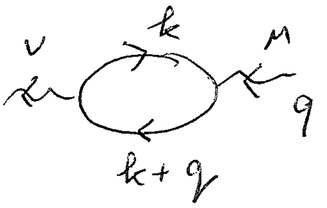
We obtain δ_3 from the on-shell condition for the photon vacuum polarization



$$\equiv i\Pi^{\mu\nu}(q) = i(g^{\mu\nu}q^2 - q^\mu q^\nu)\,\Pi(q^2)$$

The condition is $\Pi(\phi) = 0$, including, of course, the counterterm contribution.

The one-loop contribution to $i\Pi^{\mu\nu}(q)$ reads



$$i\Pi^{\mu\nu}_{loop}(q) = \underbrace{(-1)}_{\text{fermion loop}}(e^2)\int\frac{d^dk}{(2\pi)^d} \times$$

$$\times\ \frac{\mathrm{Tr}\left[\gamma^\mu(\hat{k}+m)\gamma^\nu(\hat{k}+\hat{q}+m)\right]}{(k^2-m^2)\left[(k+q)^2-m^2\right]}$$

Calculating trace, combining propagators using Feynman parameters, shifting the loop momentum and integrating over it, we arrive at

$$\boxed{\begin{aligned} i\Pi^{\mu\nu}_{loop}(q) &= -8e^2(g_{\mu\nu}q^2 - q_\mu q_\nu)\int_0^1 \frac{dx\, i\,\Gamma(\epsilon)\, x(1-x)}{(4\pi)^{d/2}\left[m^2 - q^2x(1-x)\right]^\epsilon} \\ &= -8e^2(g_{\mu\nu}q^2 - q_\mu q_\nu)\frac{i\Gamma(1+\epsilon)}{(4\pi)^{d/2}\epsilon}\int_0^1 \frac{dx\, x(1-x)}{\left[m^2-q^2x(1-x)\right]^\epsilon}\end{aligned}}$$

The counter-term contribution is

$$= -i(g^{\mu\nu}q^2 - q^\mu q^\nu)\delta_3 = i\Pi^{\mu\nu}_{ct}$$

When loop & counterterm terms are

combined we get the full self-energy correction for the photon:

$$i\,\Pi^{\mu\nu}(q) = i\left(g_{\mu\nu}q^2 - q_\mu q_\nu\right)\Pi(q^2) =$$

$$\equiv i\left(g_{\mu\nu}q^2 - q_\mu q_\nu\right)\left[\frac{-8e^2 i\,\Gamma(1+\varepsilon)}{(4\pi)^{d/2}\varepsilon}\int_0^1 \frac{dx\; x(1-x)}{\left[m^2 - q^2x(1-x)\right]^{\varepsilon}}\;\cancel{\equiv}\;\delta_3\right].$$

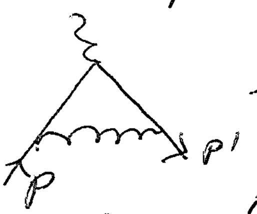
Since $\Pi(0) = 0$, we find

$$\delta_3 = \frac{-8e^2\Gamma(1+\varepsilon)}{(4\pi)^{d/2}\varepsilon}\int_0^1 \frac{dx\; x(1-x)}{\left[m^2 - q^2x(1-x)\right]^{\varepsilon}}\Bigg|_{q^2=0} = -\frac{4}{3\epsilon}\,\frac{e^2\Gamma(1+\varepsilon)}{(4\pi)^{d/2}}\,m^{-2\varepsilon}$$

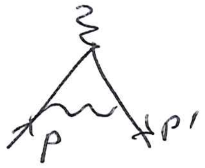
$$\boxed{\delta_3 = -\frac{4}{3\epsilon}\,\frac{e^2\Gamma(1+\varepsilon)}{(4\pi)^{d/2}}\,m^{-2\varepsilon}} \Rightarrow$$

$$\boxed{Z_3 = 1 + \delta_3 = 1 - \frac{4}{3\varepsilon}\,\frac{e^2\Gamma(1+\varepsilon)}{(4\pi)^{d/2}}\,m^{-2\varepsilon}}$$

The final renormalization constant that we need to compute is Z_1; it fixes the amputated electron-photon interaction vertex to be $-ie\gamma^\mu$, to all orders of perturbative expansion. To obtain Z_1, we need to compute the one-loop vertex correction



in the approximation $p^2 = p'^2 = m^2$ and $p \equiv p'$.

Let us write  $\equiv -ie\Lambda^\mu(p',p)$

Using standard technique for computing Feynman integrals, we can write $\Lambda^\mu(p',p)$ as

$$\Lambda^\mu(p',p) = -ie^2 \cdot 2 \int (dx)_3 \int \frac{d^d\ell}{(2\pi)^d} \frac{Num(\ell)}{[\ell^2-\mathcal{P}^2]^3} \quad (*)$$

where $\Big[Num(\ell) = \frac{(2-d)^2}{d} \bar{u}_{p'} \gamma^\mu u_p \ell^2$

$+ \bar{u}_{p'} \Big[\Big(-2q^2(1-x_2)(1-x_3) - 2m^2(1-4x_1+x_1^2) \Big) \gamma^\mu$

$-2mi\sigma^{\mu\nu} q_\nu x_1(1-x_1) \Big] u_p + \underline{(4-d)} Num_1(\ell), \Big]$

with $Num_1(\ell) = \bar{u}_{p'} \{ (\hat{p}'-\hat{\mathcal{P}})\gamma^\mu(\hat{p}-\hat{\mathcal{P}}) -$

$-(\hat{p}'-\hat{\mathcal{P}})\gamma^\mu - \gamma^\mu(\hat{p}-\hat{\mathcal{P}}) + m^2\gamma^\mu \} u_p$ and

$\mathcal{P}^\mu = p^\mu x_2 + p'^\mu x_3$ and $\mathcal{P}^2 = x^2[m^2 - q^2 y(1-y)]$

where $x_2 = xy$ $x_3 = x(1-y)$, and $x_1 = 1-x$.

In the integral $(*)$ $\int (dx)_3 = \int_0^1 dx_1 dx_2 dx_3\, \delta(1-\sum_{i=1}^3 x_i)$, after the change of variables $x_1, x_2, x_3 \to x, y$,

$$\int (dx)_3 = \int_0^1 dx\, dy\, x$$

Also, $\bar{u}_{p'}$ and u_p are spinors that satisfy the ~~one~~ Dirac equation

$(\hat{p}-m)u_p = 0$ and $(\hat{p}'-m)u_{p'} = 0$.

These equations are used to simplify $Num(\ell)$.

The above expressions are general. We now simplify them for the calculation of Z_1 since in that case we require $p' \to p$. Then, $q \to 0$ and $q^2 = 0$. Note that we kept $(d-4)\, Num_1(\ell)$ in the expression for ~~de~~ $Num(\ell)$. This is done because integral over Feynman parameters may diverge. ~~However each~~ ~~one may check that~~ We will check if it is true in a second.

So, integrating over ℓ and taking the limit $q \to 0$, we find

$$\Lambda^{\mu}_{q\to 0} = -ie^2 \cdot 2 \frac{i}{(4\pi)^{d/2}} \frac{1}{2} \int (dx)_3 \Big\{ \bar{u}_{p'} \gamma^{\mu} u_p \frac{\Gamma(\varepsilon)}{[\varphi^2]^{\varepsilon}_{q=0}} \frac{(2-d)^2}{d}$$

$$+ \frac{\Gamma(1+\varepsilon)}{[\varphi^2]^{1+\varepsilon}_{q=0}} \Big[\bar{u}_{p'} \big[2m^2 (1 - 4x_1 + x_1^2) \gamma^{\mu} \big] u_p - (4-d) Num_1(\ell) \Big|_{q\to 0} \Big] \Big\}$$

Since $(dx)_3 = dy\, x\, dx$ and $\varphi^2_{q=0} = m^2 x^2$, if we consider

$$\frac{(dx)_3}{[\varphi^2]^{1+\varepsilon}} \equiv \frac{x\, dx\, dy}{(m^2)^{1+\varepsilon} x^{2+2\varepsilon}} = \frac{dx\, dy}{(m^2)^{1+\varepsilon} x^{1+2\varepsilon}},$$

we find, potentially, a divergence at $x=0$.

As we mentioned, this is the reason we need to keep the $O(4-d)$ term in the numerator function. Next step that we make is to check what is $Num_1(\ell)$ at $x=0$, since if it is zero, we can drop it.

$Num_1(\ell)$ is defined on page 11 and it is easy to see that $x \to 0$, $q \to 0$ limit, leads to $Num_1(\ell) = 0$. Hence, we actually do not need it. The calculation of $\Lambda^{\mu}_{q\to 0}$ can be easily completed by performing x integration and expanding in ε. We find

$$\Lambda^{\mu}_{q\to 0} = \frac{e^2\Gamma(1+\varepsilon)}{(4\pi)^{d/2}} m^{-2\varepsilon} \bar{u}_{p'}\gamma^{\mu}u_p \times \left\{ \frac{3}{\varepsilon} + 4 \right\}$$

The correction (one-loop) to electron-photon vertex (amputated) comes, as usual, from two sources: the vertex proper and the counterterm.

$-ie\Lambda^{\mu}(p', p)\big|_{\cancel{\text{fey}}} + (-ie\gamma^{\mu})\delta_1$. Taking $q \to 0$, this correction must vanish. We therefore find

$$\delta_1 = \frac{-e^2\Gamma(1+\varepsilon)}{(4\pi)^{d/2}} \left\{ \frac{3}{\varepsilon} + 4 \right\} m^{-2\varepsilon}$$

We therefore find that $\delta_1 = \delta_2$

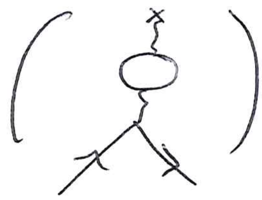
or $Z_1\big|_{1\text{-loop}} = Z_2\big|_{1\text{-loop}}$. or that the electron wave function (Z_2) is renormalized in the same way as ~~bare~~ amputated $ee\gamma$ vertex.

This implies that the relation between physical (e) and bare (e_0) electric charges that we introduced at the beginning $e(1+\delta_1) = e_0 Z_2 Z_3^{1/2}$

$$e Z_1 = e_0 Z_2 Z_3^{1/2}$$

implies

$$\boxed{e = e_0 Z_3^{1/2}}.$$

In other words, the difference in the 2 charges is only due to vacuum polarization of the photon. 

We showed the equality of Z_1 and Z_2 at one-loop. However, this equality is more general; it is valid to all orders in perturbation theory. The reason is the gauge invariance of QED, as we will discuss in the next lecture.