Lecture 7: Renormalized perturbation theory at one-loop in φ^4-theory

The idea behind renormalization is that all ultraviolet divergences in a theory can be removed by expressing observables through other observables (or pseudoobservables).

We have seen in the previous lecture that this procedure removes parameters of the Lagrangian from ~~exactly~~ relations between observables and substitutes them by "physical" or "measurable" parameters.

One way to make this procedure explicit is to ~~introduce~~ ~~counter-terms~~ rewrite the Lagrange density through "physical" parameters ~~to~~ from the beginning.

We will consider how this can be done in φ^4 theory. The Lagrangian reads

$$\boxed{\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_0)(\partial^\mu \phi_0) - \frac{1}{2} m_0^2 \phi_0^2 - \frac{\lambda_0}{4!}\phi_0^4}$$

Note that every quantity in that formula carries an index "0" which is supposed to emphasize that these quantities are unphysical or "bare".

Unphysical and physical quantities are related to each other in an unknown way. We will parametrize these relations by writing $\phi_0 = \sqrt{Z}\,\phi$, where ϕ is the "physical" field (to be explained later), so that

$$\mathcal{L} = \frac{1}{2} Z (\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m_0^2 Z \phi^2 - \frac{\lambda_0 Z^2}{4!} \phi^4$$

$$= \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!}\phi^4$$

$$+ \frac{1}{2}\delta_Z (\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}\delta_{m^2}\phi^2 - \frac{1}{4!}\delta_\lambda \phi^4,$$

where $\delta_Z = Z - 1$, $\delta_m = m_0^2 Z - m^2$, $\delta_\lambda = \lambda_0 Z^2 - \lambda$

The Lagrangian $\boxed{\mathcal{L} \equiv \mathcal{L}_{phys} + \mathcal{L}_{ct}}$, where

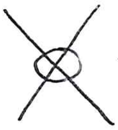
$$\boxed{\begin{aligned} \mathcal{L}_{phys} &= \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!}\phi^4; \\ \mathcal{L}_{ct} &= \frac{1}{2}\delta_Z (\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}\delta_{m^2}\phi^2 - \frac{1}{4!}\delta_\lambda \phi^4 \end{aligned}}$$

$\mathcal{L}_{ct}$ is usually refered to as the "counterterm" Lagrangian.

We treat quadratic part of $\mathcal{L}_{phys}$ as a free Lagrangian and all other terms, including every term in $\mathcal{L}_{ct}$, as a perturbation.

Therefore, our set of Feynman rules is

 $= \dfrac{i}{p^2 - m^2}$  $= -i\lambda$

 $= -i\delta_\lambda$ and  $= i(p^2\delta_Z - \delta_{m^2})$

We also assume that δ_λ, δ_Z and δ_{m^2} are ~~gave~~ functions of λ and that

$$\delta_\lambda = \sum_{n=2}^{\infty} a_n^{(\lambda)} \lambda^n \;, \quad \delta_Z = \sum_{n=1}^{\infty} a_n^{(Z)} \lambda^n \;;$$

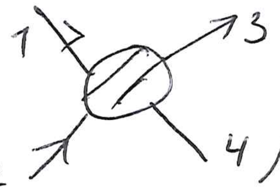
$$\delta_{m^2} = \sum_{n=1}^{\infty} a_n^{(m)} \lambda^n .$$

The coefficients $\{a_n^{(\lambda)}, a_n^{(Z)}, a_n^{(m)}\}$ are not known a'priori and should be determined order-by-order in perturbation theory by requiring that certain "physical" conditions are satisfied.

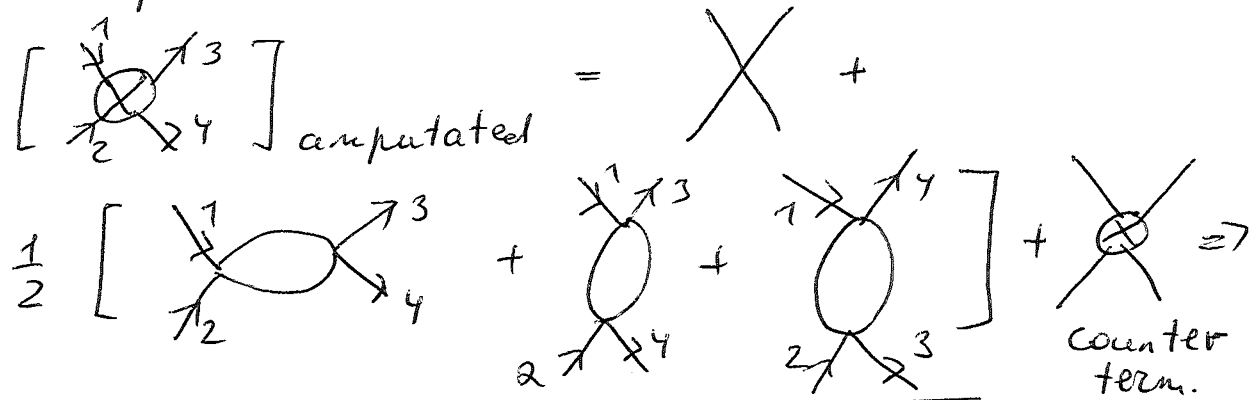
As such conditions we will choose

1) a two-point function  $\simeq \dfrac{i}{p^2 - m^2} + \text{finite}$ at $p^2 \to m^2$

2) a four-point function, amputated,

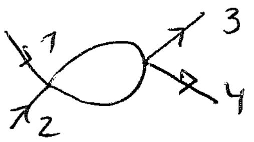
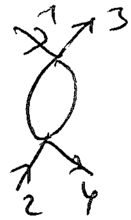
$\left(\right.$  $\left. \right)_{\text{amputated}} = -i\lambda$, at $s = (p_1 + p_2)^2 = 4m^2$, $t = 0$, $u = 0$ and $p_i^2 = m^2$, $i \in 1..4$

Let us now see how these conditions and the counterterms can be used together in perturbative calculations. We will start with a 4-point function at one-loop. It reads:

$iM =$ 

$$\boxed{iM = -i\lambda - i\lambda^2\left(V(s)+V(t)+V(u)\right) - i\delta_\lambda},$$

where we defined

$-i\lambda^2 V(s_{ij}) \equiv$  $;\ -i\lambda^2 V(t) \equiv$ , etc.

According to our ~~defenition~~ definition of $\underline{\lambda}$,

$$iM(s=4m^2, t=0, u=0) \equiv -i\lambda \Rightarrow$$

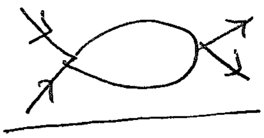
$$-i\lambda = -i\lambda - i\lambda^2\left(V(4m^2)+2V(0)\right) - i\delta_\lambda \Rightarrow$$

$$\boxed{\delta_\lambda = -\lambda^2\left[V(4m^2)+2V(0)\right].}$$

Hence,

$$\boxed{iM = -i\lambda - i\lambda^2\left[V(s)-V(4m^2)+V(t)-V(0)+V(u)-V(0)\right].}$$

We would like to check that ~~(Eq)~~ iM is finite. To this end, we ~~realize~~ need to ~~that~~ calculate $V(p^2)$.

$V(p^2) =$  $\Big/ (-i\lambda^2)$

$$= \frac{i}{2}\int \frac{d^dk}{(2\pi)^d} \frac{1}{[k^2-m^2][(k+p)^2-m^2]}$$

This integral can be easily computed by introducing Feynman parameters.

The result is

$$V(p^2) = -\frac{1}{2}\frac{\Gamma(\varepsilon)}{(4\pi)^{d/2}}\int_0^1 dx \frac{1}{[m^2-x(1-x)p^2]^{\varepsilon}} =$$

$$= -\frac{1}{2}\frac{\Gamma(1+\varepsilon)}{(4\pi)^{d/2}}\left[\frac{1}{\epsilon} - \int_0^1 dx \log\left(m^2-x(1-x)p^2\right)\right]$$

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iM receives contributions from $V(s)-V(4m^2)$, $V(t)-V(0)$ and $V(u)-V(0)$ and each of these terms is finite because divergence of $V(p^2)$ is independent of p^2. Hence,

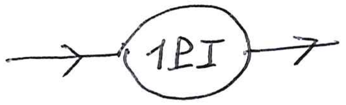
$$iM = -i\lambda - i\lambda^2\big(V(s)-V(4m^2)+V(t)-V(0)+V(u)-V(0)\big) \text{ is finite.}$$

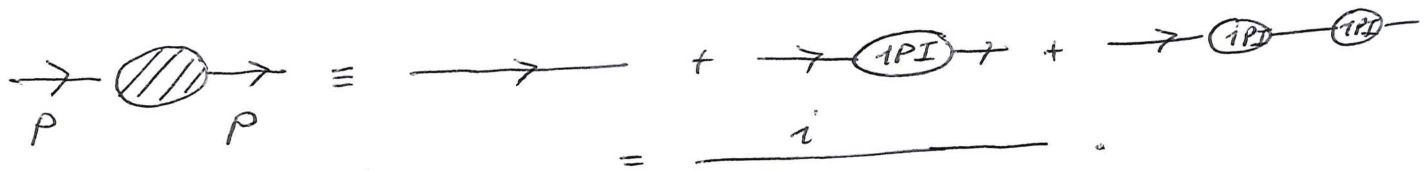
The counterterm $\delta_\lambda \equiv -\lambda^2[V(4m^2)+2V(0)]$ is computed through $O(\lambda^2)$ by this procedure.

We will next discuss the renormalization condition for the 2-point function

 $\simeq \dfrac{i}{p^2-m^2}$ + finite, at $p^2 \to m^2$.

To understand how to use it, we introduce a <u>1-particle irreducible self-energy</u> function

 $\equiv -i\,\Sigma(p^2)$. In terms of 1PI contribution, the 2-point function reads



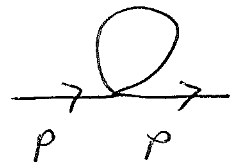
$+ \ldots = \dfrac{i}{p^2-m^2-\Sigma(p^2)}$.

Our renormalization condition ~~is~~ requires that $\Sigma(m^2)=0$ and $\left.\dfrac{d\Sigma}{dp^2}\right|_{p^2=m^2} = \phi$

(to see this, Taylor expand $\Sigma(p^2)$ around $p^2=m^2$). As the result, the condition for the 2-point function ~~is~~ leads to two "physical" conditions:

$$\begin{cases} \Sigma(m^2) = 0\,; \\ \left.\dfrac{d\Sigma}{dp^2}\right|_{p^2=m^2} = \phi. \end{cases}$$

To compute $\Sigma(p^2)$, we need to consider to contributions

$-i\Sigma(p^2) =$  $+$  $=$

counterterm

$$= -\frac{i\lambda}{2}\int\frac{d^dk}{(2\pi)^d}\frac{i}{k^2-m^2} + i(p^2\delta_Z - \delta m^2)$$

The first term is independent of $p^2 \Rightarrow$

$$\frac{d\Sigma(p^2)}{dp^2} = -\delta_Z \qquad \left.\frac{d\Sigma(p^2)}{dp^2}\right|_{m^2=p^2} = 0 \Rightarrow \boxed{\delta_Z = \phi}$$

The calculation of $\int\frac{d^dk}{(2\pi)^d}\frac{i}{k^2-m^2}$ is straightforward and leads to

$$\boxed{\delta_{m^2} = -\frac{\lambda}{2}\,\frac{i\,\Gamma(1+\varepsilon)\,m^{2-2\varepsilon}}{(4\pi)^{d/2}\,\varepsilon(1-\varepsilon)}}$$

As we see from these examples, by adjusting counter-terms, we can force Green's functions to satisfy physical conditions at certain kinematic points. Once this is accomplished, Green's functions become free of ultraviolet singularities.

In higher orders of perturbation theory counter-terms should be adjusted again, etc.

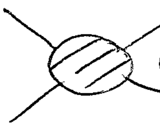
Before we move on, let us make a few comments concerning this construction. <u>First</u> point concerns <u>divergent Green's functions</u> in <u>$\lambda\varphi^4$ theory</u>. Those can be determined with the help of the artificial degree of divergence introduced in the previous lecture.

We find $D = 4 - N_\varphi$. This means that all ~~external~~ Green's functions with ~~more~~ less then 5 external legs diverge. However since $\mathcal{L}$ is symmetric under $\varphi \leftrightarrow -\varphi$, it is not possible to generate Green's functions with <u>odd number</u> of ~~effec~~ external φ legs.

Hence, neglecting also the vacuum bubble, we conclude that only 2 Green's functions


$(D=2)$

&


$(D=4)$

have $D \geq 0$ ~~and~~ in φ^4 theory

We have shown that at one loop both of these functions can be made finite using counter-terms

The <u>second point</u> concerns the assumed relation of physical and "bare" fields ($\phi_0 = \sqrt{Z}\,\phi$). To motivate this relation, recall that Kallen representation of the 2-point function implies

$$\int d^4x\, e^{ipx} \langle \Omega | T\, \varphi_0(x)\, \varphi_0(0) | \Omega \rangle \simeq \frac{iZ}{p^2 - m^2} + \text{non-pole}$$

("non-pole" means "non-singular" at $p^2 = m^2$)

With the change $\phi_0 = \sqrt{Z}\,\phi$, we get

$$\int d^4x\, e^{ipx} \langle \Omega | T\, \varphi(x)\, \phi(0) | \Omega \rangle \simeq \left. \frac{i}{p^2 - m^2} \right|_{p^2 \to m^2},$$

so the field φ has canonical normalization restored. It is "physical" in this sense.

<u>The mass parameter</u> m in $i/p^2 - m^2$ corresponds to the pole of the propagator and, therefore, is the mass of ~~the~~ the physical particle that is the excitation of the φ-field. Again, to an extent that the mass of a particle can be measured, this is a physical parameter.