Lecture 6: The idea of renormalization, counting divergences in Feynman diagrams

When we discussed perturbative computations in the first two lectures of this semester, we saw that, quite often, we obtain ill-defined expressions. There are different reason why these ill-defined expressions appear and one requires two-steps procedure to deal with them:

1) regularization; which means re-defining original computations in such a way that calculations become formally well-defined. Usually, this requires introduction of some regularization parameter (photon mass, dimensional reg. parameter ε, etc.) which should be considered to be either small or large, to approach an interesting physical regime;

2) understanding of how the dependence of the regularization parameter disappears when physical quantities are computed. In case when the dependence of the regularization parameter is related to behavior of Feynman integrals at very large

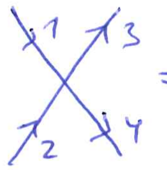
momenta, it is called <u>renormalization</u>.

Before we discuss the details of the renormalization procedure, we will try to <u>understand its meaning</u>. To this end, we will consider an example of a massless scalar field, described by the following Lagrangian

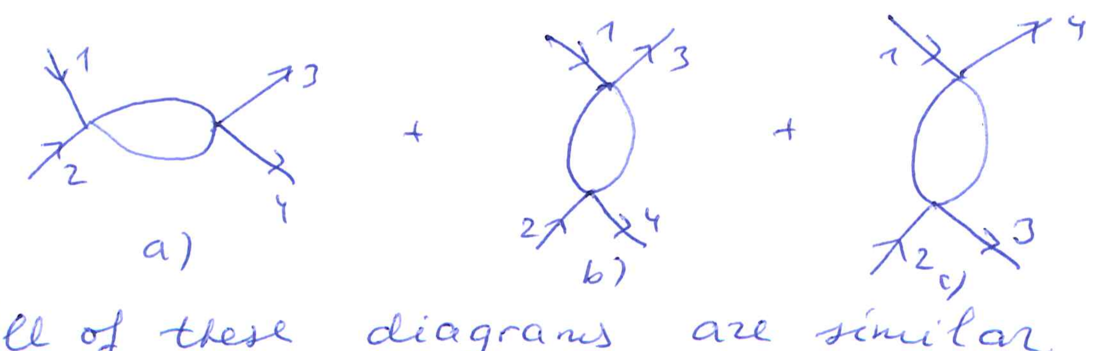
$$\mathcal{L} = -\frac{1}{2}\phi\Box\phi - \frac{\lambda}{4!}\phi^4$$

The constant λ is dimensionless coupling, that is sufficiently small so that the scattering of ϕ- particles can be treated in perturbation theory.

At tree level we have a simple diagram

$iM_1 =$  $= -i\lambda$

At one loop, there are three diagrams that describe the 4-ϕ scattering

$iM_2 =$ 

All of these diagrams are similar, so let's focus on the first one.

The expression for $iM_2^{a)}$ is

$$iM_2^{(a)} = \frac{(-i\lambda)^2}{2} i^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2(p_{12}+k)^2}, \qquad (*)$$

where $p_{12} \equiv p_1 + p_2$. ~~Consider~~ and $1/2$ is a symmetry factor. Next, consider ~~the~~ integral ~~$iM_2^{(a)}$~~ over k in Eq. $(*)$. The integral does not converge at large values of k! Indeed, if $k^\mu \to \infty$, $k^2(p_{12}+k)^2 \underset{\infty}{\approx} k^4$

and $$\int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2(p_{12}+k)^2} \xrightarrow[k_\mu \to \infty]{} \int_{\sqrt{s_{12}}} \frac{d^4k}{(2\pi)^4} \frac{1}{k^4}$$

To understand that last integral, perform the Wick rotation, use the fact that the 4-dim. solid angle is $2\pi^2$ and write $$\int_{\sqrt{s_{12}}}^{\infty} \frac{d^4k}{(2\pi)^4} \frac{1}{k^4} \equiv \frac{i}{8\pi^2} \int_{\sqrt{s_{12}}}^{\infty} \frac{dk_E}{k_E}$$

As we see, the integral doesn't exist; it diverges at $k \equiv \infty$. To make sense out of this integral, we need to avoid reaching $k_E = \infty$. We do that by introducing a cut-off $(\Lambda \gg \sqrt{s_{12}})$

$$\frac{i}{8\pi^2} \int_{\sqrt{s_{12}}}^{\infty} \frac{dk_E}{k_E} \Rightarrow \frac{i}{8\pi^2} \int_{\sqrt{s_{12}}}^{\Lambda} \frac{dk_E}{k_E} = \frac{i}{8\pi^2} \ln\left(\frac{\Lambda}{\sqrt{s_{12}}}\right)$$

For any large, but finite, Λ we can compute the integral. For $\Lambda \to \infty$, we recover the problems of the original integral. This is an example of the regularization procedure introduced earlier.

With the cut-off regularization, we obtain

$$i M_2^{(a)} = \frac{\lambda^2}{2} \frac{i}{8\pi^2} \ln \frac{\Lambda}{\sqrt{s_{12}}} = \frac{i\lambda^2}{32\pi^2} \ln\left(\frac{\Lambda^2}{s_{12}}\right)$$

It is clear that a similar calculation for b) & c) gives similar results resp to $s_{12} \to -t$ for b) & $s_{12} \to -u$ for c)

Therefore

$$i M_2^{t} = \frac{i\lambda^2}{32\pi^2}\left(\ln \frac{\Lambda^2}{s} + \ln \frac{\Lambda^2}{-t} + \ln \frac{\Lambda^2}{-u}\right).$$

The full amplitude is

$$\boxed{i M = i M_1 + i M_2 = -i\lambda + \frac{i\lambda^2}{32\pi^2}\left[\ln \frac{\Lambda^2}{s} + \ln \frac{\Lambda^2}{-t} + \ln \frac{\Lambda^2}{-u}\right]}$$

Let's try to understand what this expression means. First point to note is that if we take $\Lambda \to \infty$ keeping λ fixed, iM blows up. So this is not a meaningfull procedure.

Second point is that if we calculate the difference of the two scattering amplitudes at two values of s, t & u, the result is Λ-independent

$$iM(s,t,u) - iM(\bar{s},\bar{t},\bar{u}) = -\frac{i\lambda^2}{32\pi^2}\left(\ln\frac{s}{\bar{s}} + \ln\frac{-t}{-\bar{t}} + \ln\frac{-u}{-\bar{u}}\right).$$

At this point, we can ask the following question — how do we know what λ is?

True, λ is a parameter in the Lagrangian but for us to actually know what it is it must be measured. We can only measure it from the scattering of 4 ϕ-particles but then, according to our results, we will never get λ, but rather $\lambda - \frac{\lambda^2}{32\pi^2}(3\ln\Lambda^2 + \dots)$.

Therefore, the only quantity that we can determine experimentally is a strange combination of λ and $\ln\Lambda^2$ and it is this combination that we would have to call a physical coupling.

To make this explicit, let us declare, that a physical coupling λ is measured at a ~~point~~ kinematic point (s_0, t_0, u_0); i.e.

$$\boxed{iM(s_0, t_0, u_0) = -i\lambda_{phys}.}$$

According to our calculation

$$\lambda_{phys} = \lambda - \frac{\lambda^2}{32\pi^2}\left(\ln\frac{\Lambda^2}{s_0} + \ln\frac{\Lambda^2}{-t_0} + \ln\frac{\Lambda^2}{-u_0}\right).$$

It follows that

$$\lambda = \lambda_{phys} + \frac{\lambda_{phys}^2}{32\pi^2}\left(\ln\frac{\Lambda^2}{s_0} + \ln\frac{\Lambda^2}{-t_0} + \ln\frac{\Lambda^2}{-u_0}\right) + O(\lambda_{phys}^3).$$

and, if we use this formula to re-express the scattering amplitude in terms of physical coupling λ_{phys}, we find

$$\boxed{iM = -i\lambda_{PHYS} + \frac{i\lambda_{PHYS}^2}{32\pi^2}\left(\ln\left(\frac{s_0}{s}\right) + \ln\left(\frac{t_0}{t}\right) + \ln\left(\frac{u_0}{u}\right)\right)}$$

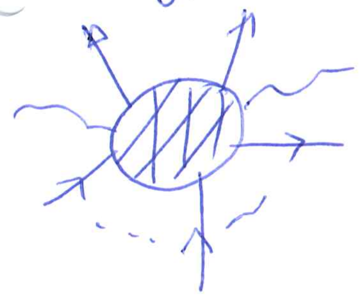
The scattering amplitude become independent of Λ^2 and all divergences are absorbed into λ_{PHYS}. Because of our definition, λ_{PHYS} is directly measurable and we can predict iM in terms of λ_{PHYS} without experiencing any problems with divergences. This is the program of the renormalization in a nutshell.

> The key message is that divergences that are encountered in perturbation theory disappear if observables are written in terms of observables.

There are subtleties associated with the above statement but it conveys the general idea. We will now turn to more systematic studies of divergencies in perturbative QFT.

To this end, we consider Quantum Electrodynamics (QED)
A typical diagram in QED is characterized by : 1) N_e, the # of external lepton lines;
2) N_γ, the # of external photon lines;
3) $P_{e,\gamma}$, the # of internal (i.e. loop-momentum dependent) propagators for electrons and photons
and
4) V – the # of vertices that involve internal lines and 5) L, the # of loops.

A typical diagram looks like



$$\sim \int d^4k_1 \dots d^4k_L \frac{1}{\hat{k}_1 - m} \cdots \frac{1}{k_j^2} \quad (**)$$

We would like to know what happens to this expression if all loop momenta are scaled by uniform (large) factor:

$$k_{i\in 1..L} \to \lambda k_{i\in 1\dots L}, \quad \lambda \to \infty.$$

If we assume that the L-loop integral in Eq.(**) contains P_e external electron lines and P_γ external photon lines, the scaling is easily computed

$$\boxed{I_{**} \sim \lambda^{4L - P_e - 2P_\gamma}.}$$

We call the exponent of λ $\underline{D = 4L - P_e - 2P_\gamma}$ as the <u>superficial degree of divergence</u>

The meaning of D is clear. It tells us ~~too~~ about the contribution to I_{**} ~~that~~ from the integration region where all loop momenta become very large. If $D<0$, the integral over that region converges; if $D=0$, the integral diverges logarithmically; if $D>0$, the integral diverges as power...

It is important to have in mind that even diagrams with $D<0$ can have divergent subdiagrams whose presence isn't captured by D. This isn't a problem sine D is not supposed to identify all divergences. It is designed to capture particular ones.

Our next step is to express $D = 4L - P_e - 2P_\gamma$ through quantities that are easier to assess than P_e & P_γ. As the first step, note that in QED $\boxed{L = P_e + P_\gamma - V + 1}$

This relation is valid because the number of loop momenta that are independent follows from the number of original # of independent momenta ($P_e + P_\gamma$, i.e. one per propagator) reduced by the number of equations ($V-1$) that force energy-momentum conservation.

The number of vertices can also be constrained. Indeed each vertex contains 1 photon line and <u>2 electron lines</u>.

Hence, the # of vertices is

$$V = 2P_\gamma + N_\gamma \quad \text{or} \quad V = \frac{1}{2}(2P_e + N_e)$$

We can use these relations to write $\mathcal{D}$ as:

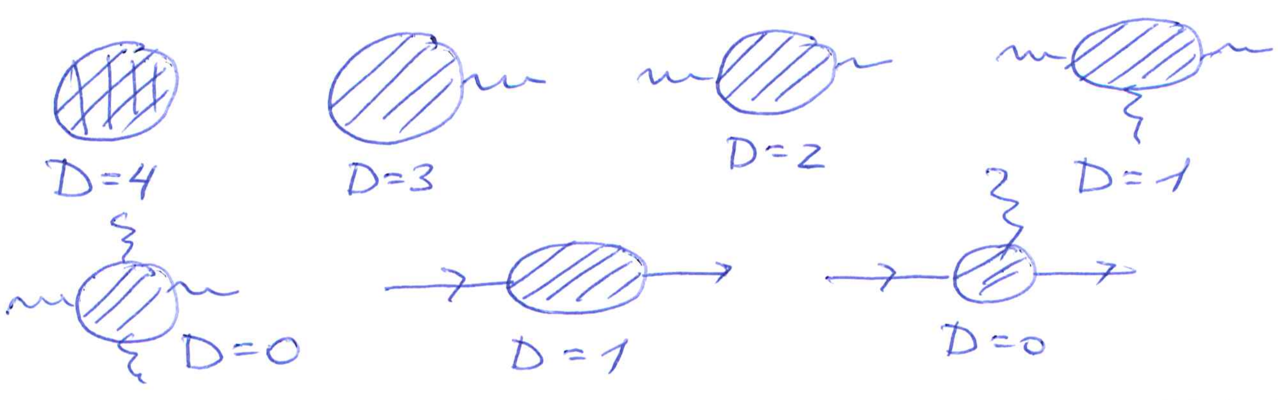
$$\mathcal{D} = 4L - P_e - 2P_\gamma = 4(P_e + P_\gamma - V + 1) - P_e - 2P_\gamma$$

$$= 3P_e + 2P_\gamma - 4V + 4 = 3(P_e - V) + 2P_\gamma - V + 4$$

$$= 3\left(-\frac{N_e}{2}\right) - N_\gamma + 4 = 4 - \frac{3N_e}{2} - N_\gamma.$$

$$\boxed{D = 4 - \frac{3N_e}{2} - N_\gamma}$$

We see that the degree of divergence in QED only depends on the number of external legs in a particular diagram. This allows us to easily find all Green's functions that have $D \geq 0$. This is so, because D <u>decreases</u> when the number of external legs <u>increases</u>; so there is countable number of cases: We find



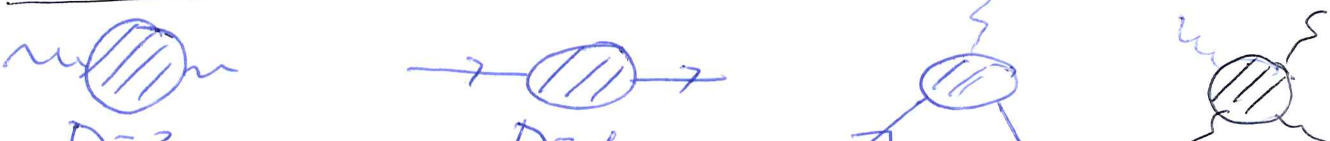
and all other Green's functions have $D < 0$. Next, we can argue 3 divergent Green's functions away:

$D = 4$

is the vacuum fluctuations, do not have external legs, do not needed to construct scattering amplitudes, $\Rightarrow$ irrelevant

in QED all Green's functions with odd number of photons vanish. (Furry's theorem)

This leaves us with 4 ~~diagrams~~ divergent Green's functions



$D=2$ $D=1$ $D=0$ $D=0$

to understand.

Let us generalize these considerations. Consider QED but in d, rather than 4 dimens. Repeating the above derivation, we find

$$D = d - L - P_e - 2P_\gamma \Rightarrow$$

$$D = d + \left(\frac{d-4}{2}\right)V - \left(\frac{d-2}{2}\right)N_\gamma + \left(\frac{d-1}{2}\right)N_e$$

This result differs from $d=4$ one in that D depends on the number of vertices. If $d < 4$, D becomes <u>smaller</u> for larger V, for fixed N_γ & N_e.

Therefore, for $d < 4$, the number of divergent -11- diagrams in QED$_d$ (not Green's functions but diagrams) <u>is finite</u>. For $d = 4$, the number of divergent Green's functions is finite. For $d > 4$, for any N_e and N_γ, there exists an order in perturbation theory where contributions to <u>any</u> Green's function become divergent.

The three different situations that we described are refered to as "super-renormalizable" ($d < 4$), "renormalizable" ($d = 4$) and "not renormalizable" ($d > 4$).

We have seen that large-momentum (ultraviolet) properties of the theory can be changed by changing dimensionality of space time. But this is not the only way. To show this, consider the Lagrange density of the form

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \varphi \partial^\mu \varphi\right) - \frac{1}{2} m^2 \varphi^2 - \frac{1}{n!} \lambda \varphi^n,$$

in the space-time of dimensionality d.

Degree of divergence is computed as in QED example, but some formulas need to be modified.

Here is the derivation:

$$D = dL - 2P_\varphi \qquad D = (d-2)P_\varphi - dV + d$$

$$L = P_\varphi - V + 1 \quad \Rightarrow \quad = \frac{(d-2)}{2}(V\cdot n - N_\varphi) - dV + d$$

$$Vn = 2P_\varphi + N_\varphi \qquad = d + \left(\frac{(d-2)}{2}n - d\right)V - \frac{(d-2)}{2}N.$$

$$\Rightarrow \boxed{D = d + \left[\frac{(d-2)}{2}n - d\right]V - \frac{(d-2)}{2}N_\varphi}$$

It is clear from that formula that D becomes worse as "n" increases. If $d=4$,

$$D = 4 + (n-4)V - N_\varphi,$$ which means

that interactions $\lambda\varphi^n$ are

* non-renormalizable, $n>4$
* renormalizable, $n=4$
* super-renormalizable, $n<4$.

We can notice that this feature is related to the mass dimension of the coupling constant λ. Since $\dim[\varphi] \sim$ mass, $\dim(\lambda) \sim \text{mass}^{4-n}$, since $\dim[\lambda\varphi^n] \sim \text{mass}^4$.

Therefore, in <u>renormalizable theory</u> couplings are <u>dimensionless</u>, in super-renormalizable – couplings have positive mass dimensions and in <u>non-renormalizable</u> – negative mass dimensions.

As we will see, in renormalizable theories, it will be possible to make all Green's function finite (i.e. not divergent) by expressing them through a finite number of observables. For non-renormalizable theories the number of observables required to make ~~the~~ theory prediction independent of the UV cut-off to all orders in perturbation theory, is infinite.

Let us also comment about the fact that we only considered overall divergences of Feynman graphs and Green's functions, in spite of the fact that subdivergences can exist as well. The reason is simple: subdivergences, if exist, will be related to Green's functions with lower number of loops. We imagine that, eventually, all divergences in Green's functions are treated recursively; therefore, when we discuss divergences of Green's functions at L-loops, we assume that all Green's functions at L-1 loops – and therefore all subdivergences of L-loop Green's functions, are already understood and removed by some procedure.