Lecture 5 Lehmann - Symanzik - Zimmerman formula and Lehmann - Källen representation of the 2-point function

In this lecture we will discuss some <u>exact</u> results in quantum field theory. They will concern a relation between <u>Green's function</u> and <u>scattering amplitudes</u> and (somewhat) related to this topic about the exact representation of a 2-point function in a particular way.

Let's start with the first point. To describe scattering experiments, we need to know S-matrix elements $S_{fi} = \langle f | i \rangle =$

$$= \sqrt{2\omega_1 2\omega_2 2\omega_3 \ldots 2\omega_n} \, \langle \Omega | a_{p_3}(\infty) \ldots a_{p_n}(\infty) \, a^+_{p_1}(-\infty) \, a^+_{p_2}(-\infty) | \Omega \rangle$$

where $|\Omega\rangle$ is the exact vacuum state,

$a^+_{p_i}(-\infty)$ are single-particle creation ~~states~~ operators at $t = -\infty$ and

$a_{p_3}(+\infty) \ldots a_{p_n}(+\infty)$ are single-particle annihilation ~~states~~ operators at $t = +\infty$.

Hence S_{fi} above corresponds to the process $p_1 + p_2 \to p_3 + \ldots + p_n$.

We will assume that the interaction in our theory is switched off at $t = \pm T$, $T \to \infty$. Then, for $\{t > T\}$ and $\{t < T\}$, the theory is free. We then write

$$\varphi(t > T) = \int \frac{d^3\vec{k}}{(2\pi)^3\sqrt{2\omega_{\vec{k}}}} \left\{ a^T_{\vec{k}} e^{-ik_\mu x^\mu} + a^{+T}_{\vec{k}} e^{ik_\mu x^\mu} \right\}$$

and
$$\varphi(t < -T) = \int \frac{d^3\vec{k}}{(2\pi)^3\sqrt{2\omega_{\vec{k}}}} \left\{ a_{\vec{k}}(-T) e^{-ik_\mu x^\mu} + a^+_{\vec{k}}(-T) e^{ik_\mu x^\mu} \right\}$$

The operators $a_{\vec{k}}(\pm T)$ and $a^+_{\vec{k}}(\pm T)$ are the operators that appear in the formula for S_{fi}. We will now derive a formula that allows us to express $a_{\vec{k}}(\pm T)$, $a^+_{\vec{k}}(\pm T)$ through ~~an~~ integral of the field $\varphi(t,x)$

As the first step, consider

$$i\int d^4x\, e^{ip_\mu x^\mu}\left(\Box + m^2\right)\varphi(x) \equiv i\int d^4x\, e^{ip\cdot x}\left(\partial_t^2 - \vec{\partial}_x^2 + m^2\right)\varphi(x)$$

We assume that $\varphi(x) \xrightarrow[|\vec{x}| \to \infty]{} 0$ and integrate the above equation by parts:

$$\int d^4x\, e^{ipx}\, \vec{\partial}\left(\vec{\partial}\varphi(x)\right) = -\int d^4x\, e^{ipx}(-i\vec{p})\left(\vec{\partial}\varphi(x)\right)$$
$$= \int d^4x\, e^{ipx}(-i\vec{p})^2\varphi(x) = \int d^4x\left(-\vec{p}^2\right)\varphi(x) \Rightarrow$$

$$\Rightarrow i\int d^4x\, e^{ip_\mu x^\mu}\left(\Box + m^2\right)\varphi(x) = i\int d^4x\, e^{ipx}\left(\partial_t^2 + \omega_{\vec{p}}^2\right)\varphi(x)$$

, where $\underline{\omega_{\vec{p}}^2 = \vec{p}^2 + m^2}$

Next, we note a useful identity

$$e^{ipx}\left(\partial_t^{\,2}+\omega_{\vec p}^{\,2}\right)\varphi(x) = -i\partial_t\left[e^{ipx}\left(i\partial_t+\omega_{\vec p}\right)\varphi(x)\right],$$

provided that $p_0=\omega_{\vec p}$. To prove this, we just execute derivative on the r.h.s. explicitely.

Using the above equation we write:

$$i\int d^4x\, e^{ip_\mu x^\mu}\left(\Box+m^2\right)\varphi(x) = i\int d^4x\, e^{ipx}\left(\partial_t^{\,2}+\omega_{\vec p}^2\right)\varphi(x)$$

$$= i\int d^4x\,(-i\partial_t)\left[e^{ipx}\left(i\partial_t+\omega_p\right)\varphi(x)\right] =$$

$$= \int d^3\vec x\; e^{ipx}\left(i\partial_t+\omega_p\right)\varphi(x)\Big|_{t=-\infty}^{t=+\infty}$$

At $t=\pm\infty$, $\varphi(x)$ is written as

$$\varphi(x) = \int\frac{d^3\vec k}{(2\pi)^3\sqrt{2\omega_k}}\left(a_{\vec k}(\pm\infty)\,e^{-ik\cdot x} + a^+_{\vec k}(\pm\infty)\,e^{ikx}\right)$$

$$\Rightarrow\; e^{ipx}\left(i\partial_t+\omega_p\right)\varphi(x) = e^{ipx}\int\frac{d^3\vec k}{(2\pi)^3\sqrt{2\omega_k}}\Big\{a_{\vec k}(\pm\infty)\left(k_0+\omega_p\right)e^{-ikx} + a^+_{\vec k}(\pm\infty)\left(\omega_p-k_0\right)e^{ikx}\Big\}$$

$$\int d^3\vec x\; e^{ipx}\left(i\partial_t+\omega_p\right)\varphi(x) =$$

$$\equiv \int\frac{d^3\vec k}{(2\pi)^3\sqrt{2\omega_{\vec k}}}\Big\{a_{\vec k}(\pm\infty)\left(k_0+\omega_p\right)e^{i(p_0-k_0)t}(2\pi)^3\delta^{(3)}(\vec p-\vec k) + a^+_{\vec k}(\pm\infty)\left(\omega_p-k_0\right)e^{i(p_0+k_0)t}(2\pi)^3\delta^{(3)}(\vec p+\vec k)\Big\}$$

$$\equiv\; a_{\vec p}(\pm\infty)\sqrt{2\omega_{\vec p}}\,,\quad\text{since}\quad k_0=\omega_{\vec k}\;\;\&\;\;\omega_{-\vec p}\equiv\omega_{\vec p}.$$

Hence

$$\boxed{i\int d^4x\, e^{ip_\mu x^\mu}\left(\Box+m^2\right)\varphi(x) \equiv \sqrt{2\omega_{\vec p}}\left(a_{\vec p}(+\infty)-a_{\vec p}(-\infty)\right)}$$

what does this equation mean? In the free theory $a_{\vec p}(\infty) = a_{\vec p}(-\infty)$ and $\varphi(x)$ satisfies the Klein-Gordon equation $(\Box + m^2)\varphi \equiv 0$. So that equation doesn't seem to be very useful there. However, in an interacting theory

$$(\Box + m^2)\varphi(x) \neq 0 \quad \text{and} \quad a_{\vec p}(+\infty) \not\equiv a_{\vec p}(-\infty).$$

The above equation then tells us that the mismatch of $a_{\vec p}(\pm\infty)$ is due to interactions.

Now, we would like this result to be used in the calculation of the S-matrix element

$$S_{fi} = \sqrt{2\omega_1 2\omega_2 2\omega_3 \dots 2\omega_n}\, \langle \Omega | a_{p_3}(\infty) \dots a_{p_n}(\infty) a^\dagger_{p_1}(-\infty) a^\dagger_{p_2}(-\infty)|\Omega\rangle$$

The first thing we do is to put T-ordering step "inside" the matrix element. This is straightforward as all $+\infty$ operators are supposed to be to the left of $-\infty$ operators in S_{fi} anyhow.

So,

$$S_{fi} = \sqrt{\prod_{i=1}^{n} 2\omega_i}\, \langle \Omega | T\left(a_{p_3}(\infty) \dots a_{p_n}(\infty)\, a^\dagger_{p_1}(-\infty)\, a^\dagger_{p_2}(-\infty)\right)|\Omega\rangle$$

Now, let's write $\Delta_{p_3} = a_{p_3}(\infty) - a_{p_3}(-\infty)$, i.e. a combination of operators that appeared in formulas for $\int (\Box + m^2)\varphi(x)$.

Then $\langle \Omega | T a_{p_3}(\infty) \dots |\Omega\rangle = \langle \Omega | T(\Delta_{p_3} + a_{p_3}(-\infty)) \dots |\Omega\rangle$

$$= \langle \Omega | T\, \Delta_{p_3} \cdot a^{(+\infty)}_{p_n}\, a_{p_3}(-\infty)\, a^\dagger_{p_1}(-\infty)\, a^\dagger_{p_2}(-\infty)|\Omega\rangle$$

We assume further that scattering occurred, so $p_3 \neq p_1 \neq p_2$. Then

$[a^{\dagger}_{p_3}(-\infty), a^{+}_{p_{1,2}}(-\infty)] = 0$, one can move $a_{p_3}(-\infty)$ all the way to the right to annihilate the vacuum ~~state~~ $|\Omega\rangle$. Repeating these steps for other operators and using

$$-i \int d^4x \, e^{-ip_\mu x^\mu} (\Box + m^2) \varphi(x) = \sqrt{2\omega_{\bar p}} \left(a^{+}_{\bar p}(\infty) - a^{+}_{\bar p}(-\infty) \right),$$

we write

$$\boxed{S_{fi} \equiv i^n \int \prod_{i=1}^{n} dx_i \, e^{i\left(\sum_{j=3}^{n} p_j x_j - p_1 x_1 - p_2 x_2\right)} \prod_{i=1}^{n} (\Box_i + m^2) \times \times \langle \Omega | T \varphi(x_1) \cdots \varphi(x_n) | \Omega \rangle}$$

This is the <u>Lehmann - Symanzik - Zimmermann formula</u> that <u>relates Green's functions with S-matrix elements.</u>

There are a few important ~~remarks~~ points that need to be ~~made~~ discussed in connection with that formula. We will approach them by first considering a particular representation of a two-point function (due to <u>Källen - Lehmann</u>).

Consider $G(x) = \langle \Omega | T \phi(x) \phi(0) | \Omega \rangle$ and take its Fourier transform

$$G(p) = \int d^4x \, e^{ipx} \langle \Omega | T \phi(x) \phi(0) | \Omega \rangle =$$

$$= \int d^4x \, e^{ipx} \{ \theta(x_0) \langle \Omega | \phi(x) \phi(0) | \Omega \rangle + \theta(-x_0) \langle \Omega | \phi(0) \phi(x) | \Omega \rangle \}$$

To simplify this equation, we use the completeness relation $\hat{1} = \sum |n\rangle\langle n|$, where $\{|n\rangle\}$ is the set of states that form a basis of the Hilbert space (we do not know these states, but we imagine that they exist). ~~The simplify this equation,~~ We write

$$\langle\Omega|\phi(x)\,\phi(0)|\Omega\rangle \equiv \langle\Omega|\phi(x)\sum_n |n\rangle\langle n|\;\phi(0)|\Omega\rangle \equiv$$

$$\equiv \sum_n \langle\Omega|\phi(x)|n\rangle\,\langle n|\phi(0)|\Omega\rangle$$

The field $\phi(x)$ is related to the field $\phi(0)$ by $\phi(x) = e^{i\hat{P}_\mu x^\mu}\phi(0)\,e^{-i\hat{P}_\mu x^\mu}$, where $\hat{P}_\mu = (\hat{H}, \vec{\hat{P}})$ is the 4-vector built of the Hamiltonian operator and the 3-momentum operator of the full theory. Then

$$\langle\Omega|\phi(x)|n\rangle = e^{-ip_n\cdot x}\langle\Omega|\phi(0)|n\rangle$$ since

$\hat{P}_\mu|\Omega\rangle = 0$ (vacuum energy is zero and vacuum momentum is zero as well)

Therefore

$$\langle\Omega|\phi(x)\,\phi(0)|\Omega\rangle \equiv \sum_n e^{-ip_n\cdot x}|\langle\Omega|\phi(0)|n\rangle|^2$$

$$\langle\Omega|\phi(0)\,\phi(x)|\Omega\rangle \equiv \sum_n e^{ip_n\cdot x}|\langle\Omega|\phi(0)|n\rangle|^2$$

Hence, the function $G(p)$ reads:

$$\boxed{G(p) = \int d^4x\; e^{ipx}\sum_n\left\{\theta(x_0)\,e^{-ip_n x} + \theta(-x_0)\,e^{ip_n x}\right\}\cdot|\langle\Omega|\phi(0)|n\rangle|^2.}$$

To simplify this, insert $\int \frac{d^4q}{(2\pi)^4}(2\pi)^4\delta^{(4)}(q-p_n) = 1$ into the integrand for $G(p)$. We find

$$G(p) = \sum_n \int \frac{d^4q}{(2\pi)^4} |\langle\Omega|\phi(0)|n\rangle|^2 (2\pi)^4 \delta(q-p_n) \times$$
$$\times \int d^4x \left\{ \theta(x_0)\, e^{i(p-q)_\mu x^\mu} + \theta(-x_0)\, e^{i(p+q)_\mu x^\mu} \right\}.$$

First, consider $\sum_n |\langle\Omega|\phi(0)|n\rangle|^2 (2\pi)^4 \delta^{(4)}(q-p_n)$.

This is a function $\underline{\text{of}}$ q^2 (Lorentz-invariance) that is non-vanishing for $q_0 > 0$ (p_n^0 is the energy of the state $|n\rangle$; must be positive-definite). Hence, we write

$$\boxed{\sum_n |\langle\Omega|\phi(0)|n\rangle|^2 (2\pi)^4 \delta^{(4)}(q-p_n) \equiv \rho(q^2)\,\theta(q_0).}$$

$\Rightarrow$
$$G(p) = \int \frac{d^4q}{(2\pi)^4} \rho(q^2)\,\theta(q^0) \times \int d^4x \left\{ \theta(x_0)\, e^{i(p-q)_\mu x^\mu} + \theta(-x_0)\, e^{i(p+q)_\mu x^\mu} \right\}.$$

To integrate over x, we use

$$\int d^3\vec{x}\, e^{i\vec{k}\vec{x}} = (2\pi)^3 \delta^{(3)}(\vec{k}) \quad \text{and}$$
$$\int_0^\infty dx_0\, e^{i\omega x_0} = \frac{i}{\omega + i0}\,; \qquad \int_{-\infty}^0 dx_0\, e^{i\omega x_0} = \frac{-i}{\omega - i0}$$

Hence
$$G(p) = \int \frac{d^4q}{(2\pi)^4} \rho(q^2)\,\theta(q^0) \left\{ (2\pi)^3 \delta^{(3)}(\vec{p}-\vec{q}) \frac{i}{p_0 - q_0 + i0} + (2\pi)^3 \delta^{(3)}(\vec{p}+\vec{q}) \frac{(-i)}{p_0 + q_0 - i0} \right\} \qquad (*)$$

To simplify this expression, note that $\rho(q^2)$ does not depend on the sign of $\vec{q}$.

Therefore, we can change $\vec{q} \to -\vec{q}$ in the second term of Eq.(*, pg.7).

Then

$$G(p) = i\int \frac{d^4q}{(2\pi)^4}\, \rho(q^2)\,\theta(q^0)\,(2\pi)^3\,\delta^{(3)}(\vec{p}-\vec{q})\,\frac{2q_0}{p_0^2-q_0^2+i0}$$

$$G(p) \equiv i\int \frac{d^4q}{(2\pi)^4}\, \rho(q^2)\,\theta(q^0)(2\pi)^3\,\delta^{(3)}(\vec{p}-\vec{q})\,\frac{2q_0}{p^2-q^2+i0}.$$

Now, we can write $d^4q\,\theta(q^0)$ as

$$\frac{d^4q\,\theta(q^0)}{(2\pi)^4} = \frac{(dq^2)}{2\pi}\,\frac{d^3\vec{q}}{(2\pi)^3\,2q_0}.$$

Using this representation in the integrand of $G(p)$ and integrating over $\vec{q}$ (i.e. removing the δ-function), we obtain:

$$G(p) = i\int_0^\infty \frac{dq^2}{2\pi}\,\frac{\rho(q^2)}{p^2-q^2+i0} \equiv i\int_0^\infty \frac{dM^2}{(2\pi)}\,\frac{\rho(M^2)}{p^2-M^2+i0}.$$

This is the Källen-Lehmann representation. $\rho(M^2)$ is called spectral density. You should also note the similarity of this representation with dispersion relations discussed in the previous lecture.

We will now calculate $\rho(M^2)$. Recall:

$$\rho(M^2)\,\theta(q_0) = \sum_n |\langle\Omega|\phi(0)|n\rangle|^2\,(2\pi)^4\,\delta^{(4)}(q-p_n)$$

The sign $\sum_n$ means an integral over the phase-space of n-particles and the sum over all possible states proper.

Let's imagine that there is a single-particle state in the spectrum, and its mass is "m".

Then
$$\sum_n = \int \frac{d^3\vec{p}_1}{(2\pi)^3 2E_1} + \sum_{n>1}$$

$$\rho_1(q^2) = \int \frac{d^3\vec{p}_1}{(2\pi)^3 2E_1} \left|\langle \Omega | \phi(0) | 1_{\vec{p}_1}\rangle\right|^2 (2\pi)^4 \delta^{(4)}(q-p_1) =$$

$$= \int \frac{d^4 p_1}{(2\pi)^4} 2\pi\, \delta(p_1^2 - m^2) \left|\langle \Omega | \phi(0) | 1\rangle\right|^2 (2\pi)^4 \delta^{(4)}(q-p_1)$$

$$\equiv 2\pi\, \delta(q^2 - m^2) \left|\langle \Omega | \phi(0) | 1\rangle\right|^2.$$

Hence:

$$\boxed{\rho(M^2)\,\Theta(q_0) \equiv 2\pi\, \delta(M^2 - m^2) \left|\langle \Omega | \phi(0) | 1\rangle\right|^2 + \rho_{n>1}(M^2) \times \Theta(q_0)}$$

If we now substitute this formula into Källen-Lehmann representation, we will find $\left(Z = |\langle \Omega | \phi(0) | 1\rangle|^2\right)$

$$G(p) \equiv \underbrace{\frac{iZ}{p^2 - m^2 + i0}}_{\text{single-particle contribution}} + \underbrace{\int \frac{dM^2}{2\pi} \frac{i\rho_{n>1}(M^2)}{p^2 - M^2 + i0}}_{\text{multi-particle states}}$$

It is possible to get an interesting constraint on Z. by repeating the same calculation for the commutator $[\phi(x), \phi(0)]$ rather than the time-ordered product. We will find

$$\langle \Omega | [\phi(x), \phi(0)] | \Omega \rangle = \int_0^\infty \frac{dM^2}{2\pi} \rho(M^2) \Big[\langle 0 | [\phi(x), \phi(0)] | 0 \rangle \Big]_M$$

where $\big[\langle 0 | [\phi(x), \phi(0)] | 0 \rangle\big]_M$ refers to Fourier transform of a free field theory Green's function with $p^2 = M^2$. Taking derivative of the above equation w.r.t. x_0 and using canonical commutation relations, we find

$$\boxed{1 = \int_0^\infty \frac{dM^2}{2\pi} \rho(M^2) \equiv Z + \int_0^\infty \frac{dM^2}{2\pi} \rho_{in}(M^2)}$$

Hence, we conclude that $0 \leq Z \leq 1$, in a fully-interacting QFT.

Note that $Z = |\langle p | \phi(0) | \Omega \rangle|^2$; i.e. it is given by the matrix element between a single particle state and the vacuum. In a free field theory, $Z = 1$ and we were assuming that this is correct also in the interacting theory earlier in this lecture, when we wrote down asymptotics of $\varphi(x)$ at $t \to \pm\infty$. If this assumption doesn't hold, the LSZ formula will have to be modified.

To see how this happens, note that $Z = |\langle p | \phi(0) | \Omega \rangle|^2$ is consistent with

$$\varphi(t \to \pm\infty) \to \sqrt{Z}\left(a_{\vec{k}}(\pm\infty) + a^{+}_{\vec{k}}(\pm\infty)\right),$$

so that

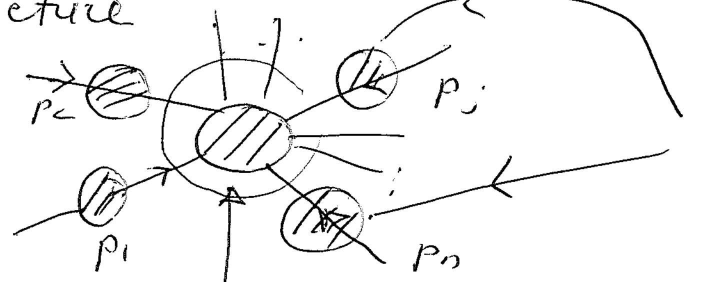
$$i\int d^4x\, e^{-ip_\mu x^\mu}(\Box + m^2)\varphi(x) = \sqrt{2\omega_{\vec{p}}}\,\sqrt{Z}\left(a_{\vec{p}}(\infty) - a_{\vec{p}}(-\infty)\right)$$

$$\Rightarrow \boxed{S_{fi} = i^n \int \prod_{i=1}^{n} dx_i\, e^{i\left(\sum_{j=3}^{n} p_j x_j - p_1 x_1 - p_2 x_2\right)} \times \prod_{i=1}^{n} (\Box_i + m^2)\left(Z_i^{-1/2}\right) \times \langle \Omega | T\varphi(x_1) \cdots \varphi(x_n) | \Omega \rangle \qquad (*)}$$

Now, suppose we take the Fourier transform of the Green's function $\langle \Omega | T\phi(x_1) \cdots \phi(x_n) | \Omega \rangle$ and write

$$\langle \Omega | T\phi(x_1) \cdots \phi(x_n) | \Omega \rangle = \int \frac{d^4q_1}{(2\pi)^4} \cdots \frac{d^4q_n}{(2\pi)^4}\, e^{i\sum q_i x_i} \times G(q_1, \ldots q_n). \qquad (**)$$

$G(p_1, \ldots p_n)$ can be represented by the following picture

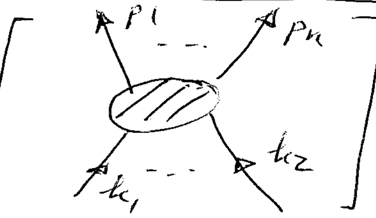
$G(p_1, \ldots p_n) \equiv$ 

Now, what happens if we substitute Eq. (**) into Eq (*) on the previous page? For each $(\Box_i + m^2)$, we will get $(-q_i^2 + m_i^2)$. For each integral over x_i, we will get $(2\pi)^4 \delta(q_i - p_i)$. Since $p_i^2 = m_i^2$, $-q_i^2 + m_i^2 = \phi$. The only reason Eq. (*) is <u>not zero</u> is ~~because~~ the pole in the "exact 2-point function" that is attached to each external leg.

Hence $\left[(-q_i^2 + m_i^2) \text{—}\!\oslash\!\text{—} \right]_{q_i^2 \to m_i^2} \equiv -i Z_i$,

as follows from Källen-Lehmann representation. Once "2-point functions attached to each leg disappear, we are left with the "amputated" Green's function. Hence, Eq. (*) on page 11 turns into ~~the~~ ~~matrix~~ ~~element~~ a relation between amputated Green's functions in momentum space and the S-matrix elements:

$$S_{fi} = \prod_{i=1}^{n} Z_i^{1/2} \times \Big[\quad \Big]_{\text{amputated}}$$



The reduction (LSZ) formulas can be generalized to cases with external fermions and bosons. Although generalization is simple ($\Box + m^2 \to i\hat{\partial} + m$, $\varphi(x) \to \psi(x)$ or $\overline{\psi}(x)$), one has to pay attention to details. The consequences of LSZ reduction formulas are <u>the Feynman rules</u> that allow you to calculate S-matrix elements from Feynman diagrams by assigning proper polarization vectors, spinors, etc. to external states.