One-loop renormalization of the Yang-Mills theory, the running coupling constant and the β-function.

In this lecture we will discuss the one-loop renormalization of the YM theory. The Lagrangian

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \bar{\Psi}(i\hat{D}_\mu - m)\psi + \bar{C}^a \partial_\mu \hat{D}^\mu_{ab} C^b + \text{gauge fixing}$$

is interpreted as the Lagrangian written through bare fields, couplings & masses. With all the interactions terms expanded, $\mathcal{L}$ reads

$$\mathcal{L} = -\frac{1}{4}\left(\partial_\mu A^a_{0\nu} - \partial_\nu A^a_{0\mu}\right)^2 + \bar{\Psi}_0 (i\hat{\partial} - m_0)\psi_0$$

$$-\bar{C}^a_0 \partial^2 C^a_0 + g_0 A^a_{0\mu} \bar{\Psi}_0 \gamma^\mu t^a \psi_0$$

$$-g_0 f^{abc}(\partial_\mu A^a_{0\nu}) A^{b\mu}_0 A^{c\nu}_0$$

$$-\frac{1}{4} g_0^2 \left(f^{eab} A^a_{0\mu} A^b_{0\nu}\right)\left(f^{ecd} A^{c\mu}_0 A^{d\nu}_0\right)$$

$$-g_0 \bar{C}^a_0 f^{abc} \partial^\mu A^b_{0\mu} C^c_0 .$$

Now, we dropped the gauge-fixing term (justified by taking $\xi \to \infty$) & expressed everything in the Lagrangian through bare input parameters such as fields, the coupling constants & masses.

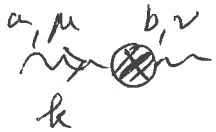
We now rescale fields as $A^a_{0\nu} = Z_3^{1/2} A^a_\nu$, $\psi_0 = Z_2^{1/2}\psi$, $c_0 = Z_c^{1/2} c$ and write the Lagrangian in terms of renormalized fields and the counter-terms

$$\mathcal{L} = -\frac{1}{4}\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right)^2 + \bar\psi(i\hat\partial - m)\psi - \bar c^a \partial^2 c^a$$
$$+ g A^a_\mu \bar\psi \gamma^\mu t^a \psi - g f^{abc}(\partial_\mu A^a_\nu) A^{b\mu} A^{c\nu}$$
$$- \frac{1}{4} g^2 f^{eab} A^a_\mu A^b_\nu \times f^{ecd} A^{c\mu} A^{d\nu} - g \bar c^a f^{abc} \partial^\mu A^b_\mu c^c$$
$$- \frac{1}{4}\delta_3\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right)^2 + \bar\psi(i\delta_2 \not\partial - \delta_m)\psi - \delta_2^c \bar c^a \partial^2 c^a$$
$$+ g\delta_1 A^a_\mu \bar\psi \gamma^\mu t^a \psi - g\delta_1^{3g} f^{abc}(\partial_\mu A^a_\nu) A^b_\mu A^c_\nu$$
$$- \frac{1}{4} g^2 \delta_1^{4g}\left(f^{eab} A^a_\mu A^b_\nu\right)\left(f^{ecd} A^c_\mu A^d_\nu\right)$$
$$- g\delta_1^c \bar c^a f^{abc} \partial^\mu A^b_\mu c^c$$

Here, the counterterms are given by

$$\boxed{\begin{array}{lll} \delta_2 = Z_2 - 1 & \delta_3 = Z_3 - 1 & \delta_2^c = Z_2^c - 1 \\ \delta_m = Z_2 m_0 - m & \delta_1 = \frac{g_0}{g} Z_2 (Z_3)^{1/2} - 1 & \\ \delta_1^{3g} = \frac{g_0}{g}(Z_3)^{3/2} - 1 & \delta_1^{4g} = \frac{g_0^2}{g^2}(Z_3)^2 - 1 & \\ \delta_1^c = \frac{g_0}{g} Z_2^c (Z_3)^{1/2} - 1. & & \end{array}}$$

We can read of the Feynman rules that are generated by the counter-term Lagrangian:

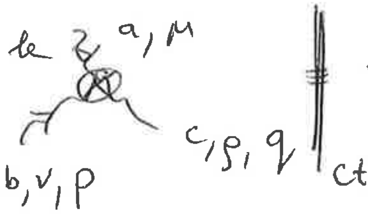


$= -i\,(k^2 g^{\mu\nu} - k^\mu k^\nu)\,\delta^{ab}\,\delta_3$

$\equiv i\,(\not p\,\delta_2 - \delta_m)\,\delta_{ij}$

$= i g\, t^a \gamma^\mu \delta_1.$

Note that ~~after~~ ~~for~~ there are also counter-terms for other vertices – for example, the 3-gluon vertex receives a counter-term which is



$\Big|_{ct} = \delta_1^{3g}\,\Big($  $\Big)_{LO}$

and the four-gluon vertex, which is

$\Big|_{ct} = \delta_1^{4g}\,\Big($  $\Big)_{LO}$

Also, as we see there are counter-terms for the vertices that involve ghosts and gauge fields and ghost self-energy.

The important point ~~that~~ is that there are <u>more</u> <u>divergent Green's functions</u> ~~diagrams~~ than the counter-terms. Indeed,

δ_1^{3g} and δ_1^{4g} are fully fixed by δ_1, δ_2 and δ_3, so that the renormalization of the 3-gluon vertex and the renormalization of the 4-gluon vertex are <u>not</u> <u>independent</u> and follow from the gauge symmetry of the theory.

Since we computed the quark self-energy, the gluon self-energy, ~~the~~ and the divergence of the quark-gluon vertex, ~~we should be~~ in the previous ~~at~~ lecture, we should be able to obtain the counter-terms and the relation between bare and renormalized coupling constant.

We will also describe a particular renormalization scheme, known as "minimal subtractions", which avoids introducing explicit subtraction point by defining renormalized Green's functions as the ones from which ~~all~~ <u>only</u> (~~also~~ this is almost true) are removed. ~~[illegible]~~ "Minimal subtractions"

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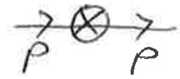
imply using dimensional regularization,
which -in turn- leads to the following subtlety: if we want the renormalized coupling constant g to be dimensionless, we can not do what we just did because the bare coupling constant must have mass dimension. Indeed, the action is dimensionless ($\hbar = 1$), so dimensionality of A and ψ is fixed from kinetic terms; then the dimensionality of the interaction term fixes $\dim[g_0] \sim M^{\epsilon}$. Hence, we must re-write the Lagrangian using $g \to g\mu^{\epsilon}$, including in the counter-terms $\delta_1 = \frac{g_0}{g\mu^{\epsilon}} Z_2 (Z_3)^{1/2} - 1$.

Now, let us go back to our calculation of the divergences in various Green's functions discussed in the previous Lecture. Take the self-energy of the quark. We computed it to be

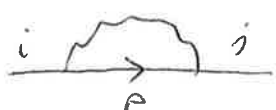
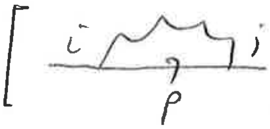
[]$_{div}$ $= \delta_{ij}\, C_F \frac{i g^2}{(4\pi)^{d/2}} \frac{\Gamma(1+\varepsilon)}{\varepsilon} \hat{p}$ + finite

Now, the expansion parameter is not g, but rather $g \cdot \mu^{\varepsilon}$, so that

[]$_{div}$ $\longrightarrow \delta_{ij} \frac{C_F\, i\, g^2 \mu^{2\varepsilon}}{(4\pi)^{d/2}} \frac{\Gamma(1+\varepsilon)}{\varepsilon} \hat{p}$ + finite.

To make this Green's function finite, we need to add the counter-term  $= i\not{p}\,\delta_2 \cdot \delta_{ij}$

The "minimal subtraction" (MS) counterterm is required to remove poles in ε from the Green's function:

[]$_{\text{Ren}}$ = []$_{\text{div}}$ $+ \text{finite} + i\not{p}\,\delta_2\,\delta_{ij}$

and
$$\boxed{\delta_2^{\,MS} \equiv -\frac{C_F\, g^2}{(4\pi)^2} \times \frac{1}{\varepsilon}} \,.$$

One unpleasant consequence of this subtraction scheme is the appearance of various unnecessary terms in the renormalized Green's function: they come from the expansion of $\dfrac{\Gamma(1+\varepsilon)}{(4\pi)^{d/2}}\dfrac{1}{\varepsilon}$ in ε:

$$\frac{\Gamma(1+\varepsilon)}{(4\pi)^{d/2}}\frac{1}{\varepsilon} \approx \frac{1}{(4\pi)^2} \times \left\{ \frac{1}{\varepsilon} - \gamma_E + \ln(4\pi) \right\}$$

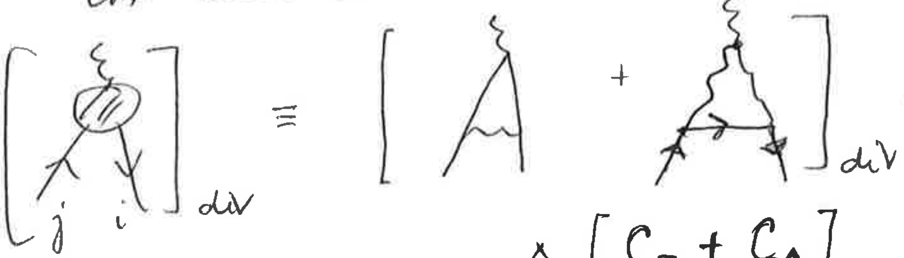
Both, the Euler γ_E and $\ln(4\pi)$ are universal; they appear in all divergent Green's functions in exactly that combination.

Therefore, it is useful to redefine the subtraction scheme, to remove them

We modify this by writing $g^2 \to g^2 \mu^{2\varepsilon}$ and add the counter-term: We find

$$\boxed{\delta_3^{\overline{MS}} = \frac{g^2}{(4\pi)^2}\left(\frac{5}{3}C_A - \frac{4}{3}N_F T_R\right)\left(\frac{1}{\varepsilon} - \gamma_E + \ln(4\pi)\right)}$$

Finally, we find the counter-term for quark-gluon vertex by using the result of the calculation in the last lecture:



$$= i g \gamma_\mu t^a_{ij} \frac{\mu^{2\varepsilon} g^2 \Gamma(1+\varepsilon)}{(4\pi)^{d/2}\varepsilon} \times \left[C_F + C_A\right] \Rightarrow$$

$$\boxed{\delta_1^{\overline{MS}} = -\frac{g^2}{(4\pi)^2}(C_F + C_A)\left(\frac{1}{\varepsilon} - \gamma_E + \ln 4\pi\right)}$$

With these 3 counterterms, we can ~~evaluate~~ find a relationship between ~~the~~ renormalized and bare coupling constants: $\left[\hat{\varepsilon} = 1/\varepsilon - \gamma_E + \ln 4\pi\right]$

$$g_0 \equiv g\mu^\varepsilon \frac{[1+\delta_1]}{Z_2 Z_3^{1/2}} = g\mu^\varepsilon\left[1 + \delta_1 - \delta_2 - 1/2\,\delta_3\right] =$$

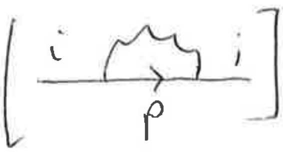
$$= g\mu^\varepsilon\left[1 + \frac{g^2}{(4\pi)^2\hat{\varepsilon}}\cdot\left(-C_F - C_A + C_F - \frac{5}{6}C_A + \frac{2}{3}N_F T_R\right)\right] =$$

The modified subtraction scheme is called $\overline{MS}$ (modified minimal subtraction scheme). It amounts to subtracting away $\left(1/\varepsilon - \gamma_E + \ln(4\pi) \right)$ from divergent Green's functions. $\left[1/\varepsilon \to 1/\varepsilon \, e^{-\gamma_E \varepsilon + \varepsilon \ln 4\pi} \right]$.

Hence, $$\boxed{\delta_2^{\overline{MS}} = - \frac{C_F g^2}{(4\pi)^2} \left[\frac{1}{\varepsilon} - \gamma_E + \ln(4\pi) \right].}$$

Next, we will trace the dependence of the renormalized Green's function on $\underline{\mu}$: since

$$\left[\; \right]_{REN} = \left[\; \right]_{div} + \text{finite} + i\hat{p}\, \delta_2\, \delta_{ij}$$



$$= \delta_{ij}\, C_F \frac{i g^2}{(4\pi)^2} \log(\mu^2)\, \hat{p} + \text{finite, } \mu\text{-independent contributions.}$$

The meaning of the scale μ is similar to the subtraction point $(p^2 = -\mu^2)$, but the relationship is not exact.

Next, we will ~~take~~ consider the gluon self-energy. We have seen that

$$\left[\; \right]_{div} = \frac{i g^2 \Gamma(1+\varepsilon)}{(4\pi)^{d/2} \varepsilon} \left(q^2 g^{\mu\nu} - q^\mu q^\nu \right) \delta^{ab} \times \left[\frac{5}{3} C_A - \frac{4}{3} N_F \cdot T_R \right]$$

$$\Rightarrow \quad g_0 = g\mu^{\varepsilon}\left[1 + \frac{g^2}{(4\pi)^2\hat{\varepsilon}}\left(-\frac{11}{6}C_A + \frac{2}{3}N_F T_R\right)\right]$$

Conveniently, this relationship is written for the non-Abelian analog of the fine-structure constants : $\alpha_s^{(0)} = \dfrac{g_0^2}{(4\pi)}$. $\Rightarrow$

$$\boxed{\alpha_s^{(0)} \equiv \alpha_s \mu^{2\varepsilon}\left[1 - \left(\frac{\alpha_s}{2\pi}\right)\frac{1}{\hat{\varepsilon}}\left(+\frac{11}{6}C_A - \frac{2}{3}N_F T_R\right) + \dots\right]}$$

The term in [..] brackets is called the strong-coupling constant renormalization constant.

We can use the above equation to find an interesting result. <u>The left-hand side</u> of that equation is the bare coupling - it is independent of μ. The right-hand side <u>depends on μ</u>, both explicitly <u>and</u> implicitly since $\alpha_s \equiv \alpha_s(\mu)$. We write

$$\alpha_s^{(0)} \equiv \alpha_s \mu^{2\varepsilon}\left[1 - \frac{\alpha_s}{2\pi}\frac{1}{\hat{\varepsilon}}\beta_0 + O(\alpha_s^2)\right], \quad \boxed{\beta_0 = \frac{11}{6}C_A - \frac{2}{3}N_F T_R}$$

$$\mu\frac{d\alpha_s^{(0)}}{d\mu} = 0 \Rightarrow 0 = \mu\frac{d\alpha_s}{d\mu}\mu^{2\varepsilon}\left[\dots\right] + 2\varepsilon\mu^{2\varepsilon}\alpha_s\left[\dots\right]$$

$$\frac{\alpha_s \mu^{2\varepsilon}}{2\pi\hat{\varepsilon}}\beta_0\,\mu\frac{d\alpha_s}{d\mu} + \dots$$

We will work to first non-trivial order in <u>α_s</u>.

$$\mu \frac{d\alpha_s}{d\mu} = \frac{-2\varepsilon\alpha_s\left[1-\frac{\alpha_s}{2\pi}\frac{1}{\hat{\varepsilon}}\beta_0+\cdots\right]}{\left[1-\frac{\alpha_s}{2\pi}\frac{1}{\hat{\varepsilon}}\beta_0-\frac{\alpha_s}{2\pi}\frac{\beta_0}{\hat{\varepsilon}}+\cdots\right]} =$$

$$= -2\varepsilon\alpha_s\left[1-\frac{\alpha_s}{2\pi}\frac{1}{\hat{\varepsilon}}\beta_0\right]\left(1+\frac{\alpha_s}{\pi\hat{\varepsilon}}\beta_0+O(\alpha_s^2)\right)$$

$$= -2\varepsilon\alpha_s\left[1+\frac{\alpha_s}{2\pi\hat{\varepsilon}}\beta_0+O(\alpha_s^2)\right] = -2\varepsilon\alpha_s-\frac{\alpha_s^2\varepsilon}{\pi\hat{\varepsilon}}\beta_0+\cdots$$

Taking the limit $\varepsilon \to 0$, we find

$$\boxed{\mu\frac{d\alpha_s}{d\mu} = -\beta_0\cdot\frac{\alpha_s^2}{\pi}}\,, \text{ where } \beta_0 = \frac{11}{6}C_A-\frac{2}{3}N_F T_R.$$

Hence, we find an interesting equation for the dependence of the coupling constant on the renormalization scale.

What are the solutions? First, $\beta_0 = \frac{11}{6}C_A - \frac{2}{3}N_F T_R$

$$= \frac{11}{6}\cdot 3 - \frac{2}{3}N_f\cdot\tfrac{1}{2} = \frac{11}{3}-\frac{N_f}{3},$$ so as long as $N_f < 11$, $\beta_0 > 0$. For "real world", $N_F \sim 2$ or 3 (up, down and may-be strange quarks), so $\beta_0 \sim 3$.

Now, $\mu\frac{d\alpha_s}{d\mu} = -\beta_0\frac{\alpha_s^2}{\pi} \Rightarrow \frac{d\alpha_s}{\alpha_s^2} = -\frac{\beta_0}{\pi}\frac{d\mu}{\mu} \Rightarrow$

$$\frac{1}{\alpha_s(\mu_i)} - \frac{1}{\alpha_s(\mu_f)} = -\frac{\beta_0}{\pi}\ln\frac{\mu_f}{\mu_i} \Rightarrow$$

$$\frac{1}{\alpha_s(\mu_i)} + \frac{\beta_0}{\pi}\ln\frac{\mu_f}{\mu_i} = \frac{1}{\alpha_s(\mu_f)} \Rightarrow$$

$$\boxed{\alpha_s(\mu_f) = \frac{\alpha_s(\mu_i)}{1+\frac{\alpha_s(\mu_i)}{2\pi}\beta_0\ln\left(\mu_f^2/\mu_i^2\right)}}$$

Hence, this equation implies that if we fix the coupling constant at the scale $\mu = \mu_i$, the coupling constant at a larger scale $\mu_f > \mu_i$ will be smaller. The scale-depend. coupling is called the "running" coupling constant; β_0 - is known as the beta-function and the phenomenonon of the coupling decrease with scale is ~~called~~ known as asymptotic freedom. Right now, the scale looks as somewhat artificial; ~~so~~ it is not related to quantities like energies of the collisions, etc. We will make this connection in the next lectures; then it will become clear why $\alpha_s(\mu)$ is an important quantity.