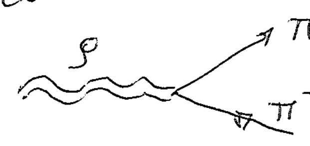
Addition to "Optical theorem, Cutkosky rules, unstable particles (Lect. 4)

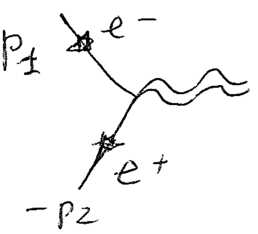
Let us consider a simple calculation of a cross-section of a process where unstable particle is produced. We will look at $e^+e^- \to \rho \to \pi^+\pi^-$. ρ-meson is a hadron, with the mass of about 770 MeV and the width of about 150 MeV. It has spin 1, and so behaves like a "massive photon".

ρ's couple to pions by rules similar to "scalar QED" but with a different charge



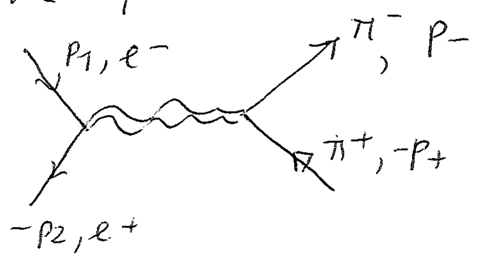
$= i g_{\rho\pi\pi} \left(p_-^\mu - p_+^\mu \right).$

ρ's couple to electrons in the same way as photons, but with different couplings

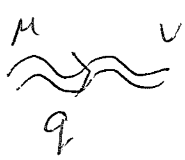


$= -i g_{ee\rho} \, \bar{v}(p_2) \gamma^\mu u(p_1).$

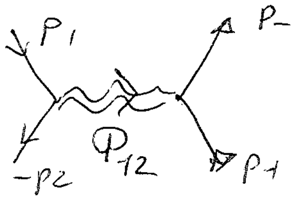
The process we want to consider is



Summation of self-energy corrections to ρ-propagator introduces a Breit-Wigner modification



$$= \frac{-i g^{\mu\nu}}{q^2 - m_\rho^2} \to \frac{-i g^{\mu\nu}}{q^2 - m_\rho^2 + i m_\rho \Gamma_\rho}$$

$$M = $$  $$\simeq \frac{-i\, g_{ee\rho}\, g_{\rho\pi\pi}}{\mathcal{P}_{12}^2 - m_\rho^2 + i m_\rho \Gamma_\rho}\, \bar{v}(p_2)\gamma^\mu u(p_1) \times (p_-^\mu - p_+^\mu)$$

$$\sum_{pol} |M|^2 \equiv \frac{g_{ee\rho}^2\, g_{\rho\pi\pi}^2}{(\mathcal{P}_{12}^2 - m_\rho^2)^2 + m_\rho^2 \Gamma_\rho^2}\, \mathrm{Tr}(\hat{p}_2 \gamma^\mu \hat{p}_1 \gamma^\nu) \times (p_-^\mu - p_+^\mu)(p_-^\nu - p_+^\nu)$$

$$= \frac{4 g_{ee\rho}^2\, g_{\rho\pi\pi}^2}{(\mathcal{P}_{12}^2 - m_\rho^2)^2 + m_\rho^2 \Gamma_\rho^2} \left(p_2^\mu p_1^\nu + p_2^\nu p_1^\mu - g^{\mu\nu} p_1 \cdot p_2 \right) \times (p_-^\mu - p_+^\mu)(p_-^\nu - p_+^\nu)$$

We will perform the calculation of the total cross-section ~~assuming~~ now, assuming that collision occurs in the center of mass fra

$$\sigma_{tot} = \underbrace{\frac{1}{4}}_{\text{spin average}} \times \underbrace{\frac{1}{4 p_1 \cdot p_2}}_{\text{flux}} \times \int \frac{d^3\vec{p}_-}{(2\pi)^3 2E_-} \frac{d^3\vec{p}_+}{(2\pi)^3 2E_+} (2\pi)^4 \delta^{(4)}(p_1 + p_2 - p_- - p_+) \times \sum_{pol} |M|^2 =$$

$$= \frac{1}{8s} \frac{4 g_{ee\rho}^2\, g_{\rho\pi\pi}^2}{(s - m_\rho^2)^2 + m_\rho^2 \Gamma_\rho^2} \left(p_2^\mu p_1^\nu + p_2^\nu p_1^\mu - g^{\mu\nu} p_1 \cdot p_2 \right) \times H_{\mu\nu},$$

where

$$\boxed{H_{\mu\nu} = \int \frac{d^3\vec{p}_-}{(2\pi)^3 2E_-} \frac{d^3\vec{p}_+}{(2\pi)^3 2E_+} (2\pi)^4 \delta^{(4)}(p_{12} - p_- - p_+) \times (p_-^\mu - p_+^\mu)(p_-^\nu - p_+^\nu)} \quad (*)$$

To calculate $H_{\mu\nu}$, it is convenient to think what is can depend upon. Since the integrand only depends on p_{12}, we can write

$$\boxed{H_{\mu\nu} = A\left(g_{\mu\nu} P_{12}^2 - P_{12\mu} P_{12\nu}\right) + B\, P_{12\mu} P_{12\nu},}$$

where A & B are functions of P_{12}^2.

Note that if we multiply $H_{\mu\nu}$ with $P_{12}^{\mu} P_{12}^{\nu}$, we find $H_{\mu\nu} P_{12}^{\mu} P_{12}^{\nu} = B (P_{12}^2)^2$

on one hand and

$$H_{\mu\nu} P_{12}^{\mu} P_{12}^{\nu} = \int \frac{d^3\vec{p}_-}{(2\pi)^3 2E_-} \frac{d^3\vec{p}_+}{(2\pi)^3 2E_+} (2\pi)^4 \delta^{(4)}(P_{12} - P_- - P_+) \times \left((P_-^{\mu} - P_+^{\mu}) \cdot P_{12\mu} \right)^2$$

on the other hand. Since $P_{12} = P_- + P_+$ and $p_-^2 = p_+^2 = m_\pi^2$, we find $(P_- - P_+)(P_- + P_+) \equiv \emptyset$.

Therefore, $B = 0$ and

$$\boxed{H_{\mu\nu} = A \left(g_{\mu\nu} P_{12}^2 - P_{12\mu} P_{12\nu} \right).}$$

To find A, we contract $g_{\mu\nu}$ with $H_{\mu\nu}$

$$g_{\mu\nu} \cdot H^{\mu\nu} = 3 P_{12}^2 \cdot A = \int \frac{d^3\vec{p}_-}{(2\pi)^3 2E_-} \frac{d^3\vec{p}_+}{(2\pi)^3 2E_+} (2\pi)^4 \delta(P_{12} - P_- - P_+) \times (2m_\pi^2 - 2p_- \cdot p_+) =$$

$$= (s - 4m_\pi^2) \times \int \frac{d^3\vec{p}_-}{(2\pi)^3 2E_-} \frac{d^3\vec{p}_+}{(2\pi)^3 2E_+} (2\pi)^4 \delta(P_{12} - P_- - P_+)$$

The phase-space calculation is straightforward; we will not repeat it. The result is

$$\int \frac{d^3\vec{p}_-}{(2\pi)^3 2E_-} \frac{d^3\vec{p}_+}{(2\pi)^3 2E_+} (2\pi)^4 \delta(P_{12} - P_- - P_+) \equiv \frac{\beta}{8\pi},$$

where $\beta = \sqrt{1 - \frac{4m_\pi^2}{s}}$.

So that. $\boxed{A = -\frac{1}{3}\,\beta^{3/2} \times \frac{1}{8\pi}}$. The result for the cross-section is easy to obtain if we ~~obsee~~ notice that $(p_2^{\mu} p_1^{\nu} + p_2^{\nu} p_1^{\mu} - g^{\mu\nu} p_{12}) P_{12\mu} P_{12\nu} \equiv \emptyset$.

Hence,

$$\sigma_{tot} = \frac{1}{8s} \frac{4 g_{ee\rho}^2 g_{\rho\pi\pi}^2}{(s-m_\rho^2)^2 + m_\rho^2 \Gamma_\rho^2} A\,(2p_1p_2 - \cancel{4}p_1p_2) \propto s$$

$$= \frac{1}{64\pi s}\left(\frac{\beta^{3/2}}{3}\right) \times \frac{4 g_{ee\rho}^2 g_{\rho\pi\pi}^2}{(s-m_\rho^2)^2 + m_\rho^2\Gamma_\rho^2} \times s \Rightarrow$$

$$\boxed{\sigma_{tot} = \frac{\beta^{3/2}}{48\pi} \frac{g_{ee\rho}^2 g_{\rho\pi\pi}^2\, s}{(s-m_\rho^2)^2 + m_\rho^2 \Gamma_\rho^2}}$$

We would like to rewrite this cross-section in a useful way:

Consider decay of the ρ-meson to pions.

$$M = -i g_{\rho\pi\pi}\, \epsilon^{\mu}\cdot(p_-^{\mu} - p_+^{\mu})$$

$$\frac{1}{3}\sum_{pol\,\rho} |M|^2 = \frac{1}{3} g_{\rho\pi\pi}^2 \left(-g_{\mu\nu} + \frac{P_\mu P_\nu}{M_\rho^2}\right)(p_-^{\mu} - p_+^{\mu})(p_-^{\mu} - p_+^{\nu}),$$

where $P = (M_\rho, 0)$ is the ρ-momentum

$$\Gamma_{\rho\to\pi\pi} \equiv \frac{1}{2m_\rho} \times \frac{1}{3} g_{\rho\pi\pi}^2 \left(-g_{\mu\nu} + \frac{P_\mu P_\nu}{M_\rho^2}\right) \times$$
$$\times \int \frac{d^3\vec{p}_-}{(2\pi)^3 2E_-} \frac{d^3\vec{p}_+}{(2\pi)^3 2E_+} (2\pi)^4 \delta^{(4)}(P - p_- - p_+)$$
$$\times (p_-^{\mu} - p_+^{\mu})(p_-^{\nu} - p_+^{\nu})$$

$$\Gamma_{\rho\to\pi\pi} = \frac{1}{2m_\rho}\ \frac{1}{3}\, g_{\rho\pi\pi}^2 \left(-g_{\mu\nu} + \frac{P_\mu P_\nu}{M_\rho^2}\right) A(M_\rho^2)\left(+g_{\mu\nu} M_\rho^2 \ldots\right)$$

$$= \frac{1}{6m_\rho}\, g_{\rho\pi\pi}^2 (-3)\left(-\frac{1}{3}\right) \beta_\pi^{3/2} \times \frac{1}{8\pi}\, m_\rho^2 \Rightarrow$$

$$\boxed{\Gamma_{\rho\to\pi\pi} \equiv \frac{g_{\rho\pi\pi}^2\, m_\rho}{48\pi}\, \beta_\pi^{3/2}}$$

As the next step, we can repeat the same procedure, but considering decay $\rho \to e^+e^-$:

$$M = \epsilon_\mu (-i g_{\rho ee})\, \bar{u}(p_2)\gamma^\mu u(p_1)$$

$$|M|^2 = g_{\rho ee}^2 \left(-g_{\mu\nu} + \frac{P_\mu P_\nu}{M_\rho^2}\right) \mathrm{Tr}\{\hat{p}_2 \gamma^\mu \hat{p}_1 \gamma^\nu\}$$

$$= 4 g_{\rho ee}^2 (-g_{\mu\nu})\{p_2^\mu p_1^\nu + p_2^\nu p_1^\mu - g^{\mu\nu} p_1\cdot p_2\}$$

$$\int \frac{d^3\vec{p}_2}{(2\pi)^3 2E_2}\, \frac{d^3\vec{p}_1}{(2\pi)^3 2E_1}\, (2\pi)^4 \delta^{(4)}(p_2 + p_1 - P_\mu)\, \cancel{(2\pi)^4 \delta^{(4)}} (p_2^\mu p_1^\nu + p_2^\nu p_1^\mu - g^{\mu\nu} p_1\cdot p_2)$$

$$\equiv \tilde{A}\left(-g^{\mu\nu} + \frac{P_\mu P_\nu}{M_\rho^2}\right); \text{ Contracting with } -g^{\mu\nu};$$

$$3\tilde{A} = \int \frac{d^3\vec{p}_2}{(2\pi)^3 2E_2}\, \frac{d^3\vec{p}_1}{(2\pi)^3 2E_1}\, (2\pi)^4 \delta^{(4)}(P_{12} - p_1 - p_2)\cdot 2p_1 p_2$$

$$= M_\rho^2/8\pi \quad \Rightarrow \tilde{A} = \frac{1}{3}\,\frac{M_\rho^2}{8\pi}$$

$$\Gamma_{\rho\to e^+e^-} \equiv \frac{1}{2m_\rho}\ \frac{1}{3}\ 4 g_{\rho ee}^2 (-g_{\mu\nu})\, \tilde{A}\left(-g^{\mu\nu} + \frac{P^\mu P^\nu}{M_\rho^2}\right)$$

$$\Gamma_{\rho\to e^+e^-} = \frac{1}{2m_\rho}\,\frac{1}{3}\,4g_{\rho ee}^2\cdot\cancel{3}\,\frac{\cancel{1}}{\cancel{3}}\,\frac{M_\rho^2}{8\pi} \Rightarrow$$

$$\boxed{\Gamma_{\rho\to e^+e^-} \equiv \frac{g_{\rho ee}^2}{12\pi}\, m_\rho}$$

Let's go back to the formula for cross-section and rewrite it using expressions for partial width to get rid of couplings.

$$\sigma_{tot} = \frac{\beta(s)^{3/2}}{48\pi}\,\frac{g_{\rho ee}^2\, g_{\rho\pi\pi}^2\, s}{(s-m_\rho^2)^2+m_\rho^2\Gamma_\rho^2} \equiv$$

$$\equiv 12\pi\,\frac{\Gamma_{\rho\to e^+e^-}}{m_\rho}\,\Gamma_{\rho\to\pi^+\pi^-}\,\frac{\beta(s)^{3/2}}{\beta(m_\rho^2)^{3/2}}\,\frac{s}{m_\rho}\,\frac{1}{(s-m_\rho^2)^2+m_\rho^2\Gamma_\rho^2}$$

$$\Rightarrow \boxed{\sigma_{tot}(s) = \frac{12\pi\,\Gamma_{\rho\to e^+e^-}\,\Gamma_{\rho\to\pi^+\pi^-}}{(s-m_\rho^2)^2+m_\rho^2\Gamma_\rho^2}\,\frac{s}{m_\rho^2}\times\left[\frac{\beta(s)}{\beta(m_\rho)}\right]^{3/2}}.$$

What happens to this formula in the limit $\Gamma_\rho \to 0$ (called narrow width approximation)?

In this limit the cross-section has a large support for $s\sim m_\rho^2$ and not much beyond that; we can approximate: $\dfrac{1}{(s-m_\rho^2)^2+m_\rho^2\Gamma_\rho^2} \xrightarrow[\Gamma_\rho \to 0]{} N\,\delta(s-m_\rho^2)$

To find N, we integrate left and right sides:

$$\int_{-\infty}^{\infty} \frac{ds}{(s-m_\rho^2)^2+m_\rho^2\Gamma_\rho^2} \equiv N \int_{-\infty}^{\infty} ds\,\delta(s-m_\rho^2) \equiv N$$

At the same time

$$\int_{-\infty}^{\infty} \frac{ds}{(s-m_\rho^2)^2+m_\rho^2\Gamma_\rho^2} = \int_{-\infty}^{\infty} \frac{dx}{x^2+m_\rho^2\Gamma_\rho^2} = \frac{1}{m_\rho\Gamma_\rho}\int_{-\infty}^{\infty}\frac{dy}{y^2+1} =$$

$$= \frac{\pi}{m_\rho\Gamma_\rho} \Rightarrow N. \quad \text{Hence}$$

$$\frac{1}{(s-m_\rho^2)^2+m_\rho^2\Gamma_\rho^2} \longrightarrow \frac{\pi}{m_\rho\Gamma_\rho}\,\delta(s-m_\rho^2) \Rightarrow$$

$$\sigma_{tot}^{\text{narrow width}}(s) = \frac{12\pi^2\,\Gamma_{\rho\to e^+e^-}\,\Gamma_{\rho\to\pi^+\pi^-}}{m_\rho\Gamma_\rho}\,\delta(s-m_\rho^2) \Rightarrow$$

$$\sigma_{tot}^{\text{narrow width}}(s) = \underbrace{\frac{12\pi^2\,\Gamma_{\rho\to e^+e^-}}{m_\rho}}_{\substack{\text{Probability}\\ \text{to produce}\\ \text{a resonance}\\ \text{(spin 1).}}} \times \underbrace{\frac{\Gamma_{\rho\to\pi^+\pi^-}}{\Gamma_\rho}}_{\substack{\text{Branching ratio to produce}\\ \text{the final state}}}\,\delta(s-m_\rho^2).$$

As an example: $e^+e^-\to\rho\to\mu^+\mu^-$, narrow width approximation

$$\sigma_{tot}^{\text{narrow width}}(s) \equiv \underbrace{\frac{12\pi^2\,\Gamma_{\rho\to e^+e^-}}{m_\rho}}_{\text{same as before}} \times \underbrace{\frac{\Gamma_{\rho\to\mu^+\mu^-}}{\Gamma_\rho}}_{\substack{\text{different}\\ \text{branching}}}\,\delta(s-m_\rho^2)$$