Lecture : One-loop divergencies in non-Abelian quantum field theories

We would like to discuss perturbative expansion for Green's functions in non-Abelian theories. It is easy to check that the mass-dimension of all operators contained in the Lagrangian $\mathcal{L}$ for a non-abelian QFT is four or less, and so the theory should be renormalizable. However, the gauge symmetry should both restrict the number of divergent structures and impose relations between different renormalization constants. We have seen this happening in QED and we should find similar things in non-Abelian theories.

We will perform the one-loop analysis in this Lecture. Our goal will be to identify all divergent Green's functions and evaluate divergencies explicitly. The first Green's function that we consider is <u>the self-energy of a gauge boson</u>. The 1-loop diagrams are: (a, b are color degrees of freedom)

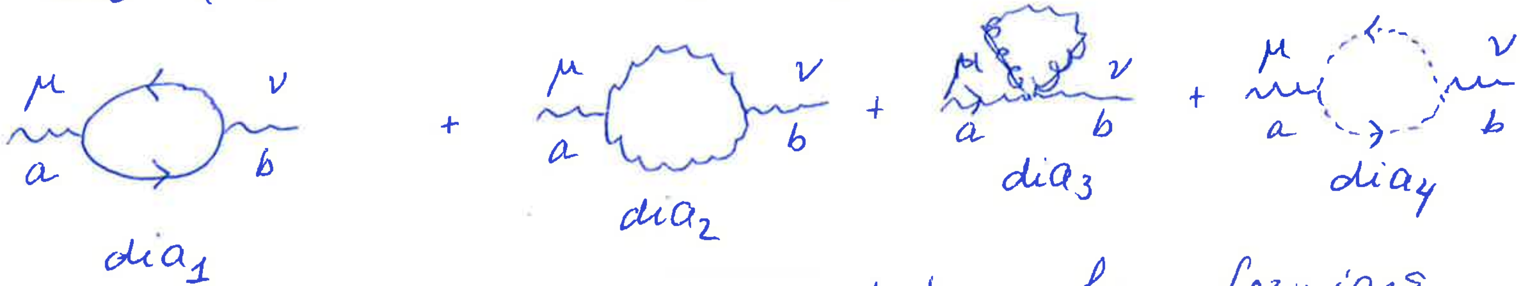


Diagram dia_1 recieves contributions from <u>fermions</u>, (dia_2, dia_3) - from gauge bosons, dia_4 - from ghosts.

The gauge - boson self-energy should be transverse: $\overset{\mu}{\sim}\!\oslash\!\overset{\nu}{\sim} \equiv i\left(g^{\mu\nu}q^2 - q^\mu q^\nu\right)\Pi(q^2)\delta^{ab}$

Similar to QED, this ensures that the gauge boson remains massless, also after the interactions are switched on, and that the longitudinal parts of A^a_μ behave as free fields.

We start with the diagaram dia_1. If we write an expression for it, we will realize that it is the same as what we already considered in QED provided that we replace the electric charge squared in the QED expression by

$$e^2 \to g^2 \sum_{i,j}\left(t^a_{ij}\, t^b_{ji}\right) \equiv g^2\, Tr\left(t^a t^b\right)$$

We write $Tr(t^a t^b) \equiv C(r)\,\delta^{ab}$ since the Lie algebra generators are orthogonal.

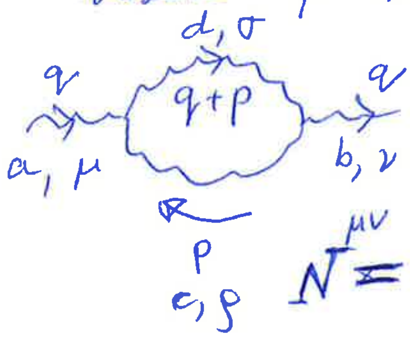
In general $C(r)$ is the representation-dependent constant; for ~~fermions~~ ~~the~~ fundamental representation, the standard choice is $C(r)=\frac{1}{2}$.

Finally, allowing for n_f massless fermion spieces, we obtain for the divergent part:

$$\boxed{\sum_{\text{ferm}} \;\; \equiv i\left(q^2 g_{\mu\nu} - q_\mu q_\nu\right)\delta^{ab}\left[\frac{-g^2}{(4\pi)^2}\,\frac{4}{3}\,n_f\,C(r) \times \Gamma(2-d/2)\right]}.$$

As usual, $d = 4-2\varepsilon$.

As the next step, we evaluate diagram dia_2. We will use Feynman gauge for gauge boson propagators:



$$= \frac{g^2}{2}\int \frac{d^dp}{(2\pi)^d}\,\frac{-i}{p^2}\,\frac{-i}{(p+q)^2} \times N^{\mu\nu} \cdot f^{acd} f^{bcd}$$

$$N^{\mu\nu} = \left[g^{\mu\rho}(q-p)^\sigma + g^{\rho\sigma}(2p+q)^\mu + g^{\sigma\mu}(-p-2q)^\rho\right]$$
$$\times \left[g^{\nu\sigma}(+2q+p)^\rho + g^{\sigma\rho}(-q-2p)^\nu + g^{\nu\rho}(+p-q)^\sigma\right]$$

The artificial degree of divergence here is <u>two</u> and so we have to be careful since all terms in the integrand lead to divergent results. ~~We~~ The first step is to combine denominators:

$$\frac{1}{p^2(p+q)^2} \equiv \int_0^1 dx\,\frac{1}{\left[(1-x)p^2 + x(p+q)^2\right]^2} =$$

$$= \int_0^1 \frac{dx}{(P^2 - \Delta)^2}$$, where $P = p + qx$ and $\Delta = -x(1-x)q^2$.

We shift the integration momentum $p \to \ell - qx$ and find a new expression for $N^{\mu\nu}$.
Since the denominator will only depend on ℓ^2, we can drop all the terms in $N^{\mu\nu}$ that are linear in ℓ. We find

$$N^{\mu\nu} \longrightarrow \left[g^{\mu\rho} q^\sigma (1+x) + g^{\rho\sigma} q^\mu (1-2x) - g^{\sigma\mu}(2-x) q^\rho\right]$$
$$\times \left[g^{\nu\sigma} q^\rho (2-x) - g^{\sigma\rho}(1-2x) - g^{\nu\rho} q^\sigma (1+x)\right]$$
$$+ \left[-g^{\mu\rho} \ell^\sigma + 2 g^{\rho\sigma} \ell^\mu - g^{\sigma\mu} \ell^\rho\right]$$
$$\times \left[g^{\nu\sigma} \ell^\sigma - 2 g^{\sigma\rho} \ell^\nu + g^{\nu\rho} \ell^\sigma\right] \equiv$$
$$\equiv \ell^2(-2g^{\mu\nu}) + \ell^\mu \ell^\nu (6-4d) - q^2\left[(2-x)^2 + (1+x)^2\right] g^{\mu\nu}$$
$$+ q^\mu q^\nu \left[(2-d)(1-2x)^2 + 2(1+x)(2-x)\right].$$

Since the integrand depends on ℓ^2 only, we average over directions of ℓ, so that we obtain $\ell^\mu \ell^\nu \to \frac{\ell^2 g^{\mu\nu}}{d}$. Hence, we obtain

$$N^{\mu\nu} \to -6 g_{\mu\nu} \ell^2 \left(1 - \frac{1}{d}\right) - g_{\mu\nu} q^2 \left[(2-x)^2 + (1+x)^2\right]$$
$$+ q_\mu q_\nu \left[(2-d)(1-2x)^2 + 2(1+x)(2-x)\right].$$

The rest of the calculation is straightforward; we perform the Wick rotation and use the two expressions

$$\int \frac{d^d p_E}{(2\pi)^d} \frac{1}{(p_E^2+\Delta)^n} = \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(n-d/2)}{\Gamma(n)} \frac{1}{\Delta^{n-d/2}}$$

$$\int \frac{d^d p_E}{(2\pi)^d} \frac{p_E^2}{(p_E^2+\Delta)^n} = \frac{1}{(4\pi)^{d/2}} \frac{d}{2} \frac{\Gamma(n-d/2-1)}{\Gamma(n)} \frac{1}{\Delta^{n-d/2-1}}$$

We can write the result as

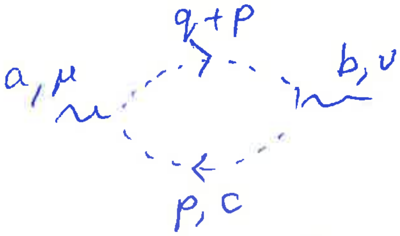
$$= \frac{ig^2}{(4\pi)^{d/2}} f^{acd} f^{bcd} \, \Gamma(2-d/2) \times \int_0^1 \frac{dx}{\Delta^{2-d/2}}$$

$$\times \Big\{ g^{\mu\nu} q^2 \cdot \Big[\frac{3(d-1)}{(2-d)} x(1-x) + \frac{1}{2}(2-x)^2 + \frac{1}{2}(1+x)^2 \Big]$$

$$- q^\mu q^\nu \Big[(1-d/2)(1-2x)^2 + (1+x)(2-x) \Big] \Big\}.$$

Note that this diagram alone is not transverse, i.e. it is not proportional to $(g^{\mu\nu} q^2 - q^\mu q^\nu)$!

Next step - diagram 3: . This diagram however vanishes in dimensional regularization, so we just drop it. The remaining diagram is #4, it involves ghosts. The expression for dia 4 is straightforward to obtain, following the Feynman rules. The result reads:



$$= (-1) \int \frac{d^d p}{(2\pi)^d} \frac{i^2}{p^2 (p+q)^2} g^2 f^{dac} f^{cbd} (p+q)^\mu p^\nu$$

The (−1) prefactor appears due to <u>anticommuting nature</u> of the ghost fields. Combining propagators using Feynman parameters, shifting the loop momentum and integrating, we obtain

$$= \frac{i g^2}{(4\pi)^{d/2}} f^{acd} f^{bcd} \, \Gamma(2 - d/2) \cdot \int_0^1 dx \times \frac{1}{\Delta^{2-d/2}}$$

$$\times \left\{ - g^{\mu\nu} q^2 \frac{x(1-x)}{2(1-d/2)} + q^\mu q^\nu x(1-x) \right\}$$

The calculation can be completed by integration over x. Since $\Delta \equiv -x(1-x) q^2$, all integrals are of the form

$$\int_0^1 dx \; x^{\alpha-1} (1-x)^{\beta-1} = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \text{ where}$$

$\Gamma(\alpha)$ are the Gamma-functions. Γ-functions can be reduced to just a single one, using $z\Gamma(z) = \Gamma(1+z)$ to shift the argument by an integer.

If we put everything together, we find


$+$


$$\equiv \frac{ig^2}{(4\pi)^{d/2}} f^{acd} f^{bcd}\, \Gamma(2-d/2) \times \left[g^{\mu\nu} q^2 - q^\mu q^\nu \right] \times \frac{6d-4}{4(d-1)} \times (-q^2)^{-\varepsilon} \frac{\Gamma^2(1-\varepsilon)}{\Gamma(2-2\varepsilon)}.$$

Inclusion of the ghost diagram is essential for achieving the transversality of the gauge-boson self-energy contribution.

To write down the final answer, note that the color factor $f^{acd} f^{bcd}$ is the Casimir operator of the gauge group in the adjoint representation; for $SU(N)$, $f^{acd} f^{bcd} = \delta^{ab} N$.

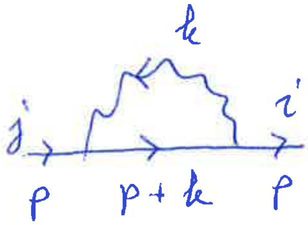
We denote it $f^{acd} f^{bcd} \equiv C_2(G)\, \delta^{ab} = \delta^{ab} N$.

Using $\dfrac{6d-4}{4(d-1)} \approx \dfrac{5}{3} + \dfrac{\varepsilon}{9} + O(\varepsilon^2)$; $\Gamma(2-d/2) = \Gamma(\varepsilon) \approx \dfrac{\Gamma(1+\varepsilon)}{\varepsilon}$

$\dfrac{\Gamma^2(1-\varepsilon)}{\Gamma(2-2\varepsilon)} \approx 1 + 2\varepsilon + O(\varepsilon^2)$, we find

$$\Bigg|_{\substack{\text{gluons} \\ +\text{ghosts}}} \equiv \frac{ig^2 \Gamma(1+\varepsilon)}{(4\pi)^{d/2}} \left\{ \delta^{ab} C_2(G) \left(\frac{5}{3\varepsilon} + \frac{31}{9} \right) \right\} (-q)^{-\varepsilon} \times \left(g^{\mu\nu} q^2 - q^\mu q^\nu \right).$$

Next divergent contribution is quark self-energy.
We will consider massless quarks



$$\equiv (ig)^2 (t^a t^a)_{ij} \int \frac{d^d k}{(2\pi)^d} \frac{\gamma^\mu (\hat{p}+\hat{k}) \gamma_\mu}{(p+k)^2} \times \frac{1}{k^2}$$

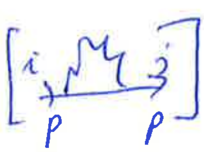
We will only compute divergent terms, so we will systematically neglect all $O(\varepsilon)$ terms in the numerator. Then $\gamma^\mu(\hat{p}+\hat{k})\gamma_\mu = -2(\hat{p}+\hat{k})$

$$\frac{1}{(p+k)^2} \frac{1}{k^2} \equiv \int\limits_0^1 \frac{dx}{(K^2-\Delta)^2} \; ; \quad K = k + px \text{ and } \Delta = -x(1-x)p^2. \;\Rightarrow$$

$$\equiv 2g^2 (t^a t^a)_{ij} \int\limits_0^1 dx \int \frac{\hat{K} + \hat{p}(1-x)}{(K^2-\Delta)^2} \frac{d^d K}{(2\pi)^d}$$

We can now drop linear term in K, perform the Wick-rotation and integrate over K; A useful formula that gives divergent part of the integral is

$$\int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2-\Delta)^2} \xrightarrow{\text{div}} \frac{i}{(4\pi)^2} \frac{\Gamma(1+\varepsilon)}{\varepsilon}.$$ Therefore,

div 

$$\equiv 2g^2 (t^a t^a)_{ij} \frac{i}{(4\pi)^2} \frac{\Gamma(1+\varepsilon)}{\varepsilon} \hat{p} \int\limits_0^1 dx\,(1-x)$$

$$\equiv \delta_{ij} C_2(2) \frac{ig^2}{(4\pi)^2} \frac{\Gamma(1+\varepsilon)}{\varepsilon} \hat{p}.$$ Here, $C_2(2)$

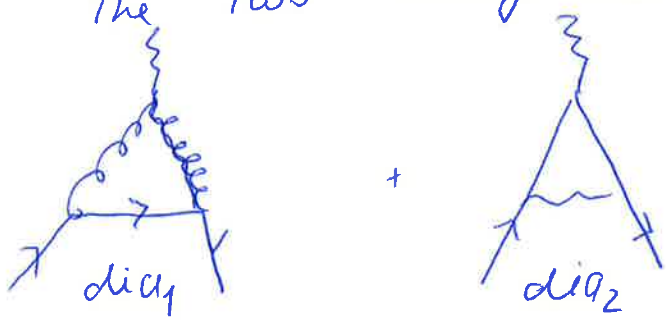
is the Casimir of the ~~adj~~ fundamental representation: $C_2(r)\,\delta_{ij} = \sum_{a=1}^{N^2-1} \left(t^a \cdot t^a\right)_{ij}$.

For the group $SU(N)$, $C_2(r) = \dfrac{N^2-1}{2N}$. Hence,

div [] $= \boxed{\delta_{ic}\, C_2(r)\, \dfrac{ig^2}{(4\pi)^2}\, \dfrac{\Gamma(1+\varepsilon)}{\varepsilon}\, \hat{p}.}$

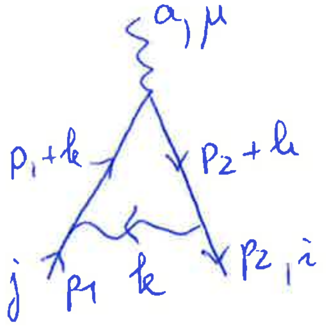
Finally, we need to compute two one-loop diagrams that lead to divergent contributions to the quark-gluon-quark 3-point function.

The two diagrams are



; The diagram dia2 is very much QED-like. The first diagram involves 3-gluon vertex and, therefore, is determined by the non-abelian nature of the theory.

We will start the calculation with dia2. As before, we only care about divergent contributions only.



$$\equiv \int \frac{d^d k}{(2\pi)^d} (ig)^3 \times \left(t^b t^a t^b\right)_{ij} \times$$

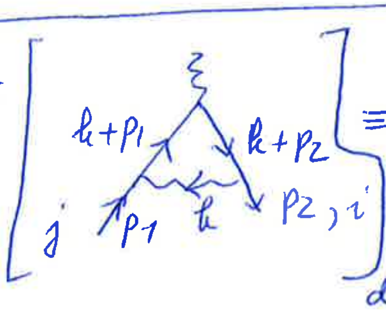
$$\times\ \hat{\gamma}_\rho \frac{i}{\hat{k}+\hat{p}_2}\, \hat{\gamma}_\mu \frac{1}{\hat{k}+\hat{p}_1}\, \hat{\gamma}_\rho \frac{-i}{k^2}$$

We calculate the "color factor":

$$t^b t^a t^b = t^b\left(t^a t^b - t^b t^a + t^b t^a\right) =$$
$$= t^b[t^a, t^b] + (t^b \cdot t^b)\, t^a =$$
$$= t^b\, i f^{abc} t^c + C_2(r)\cdot t^a = \frac{1}{2} i f^{abc}[t^b, t^c] + C_2(r)\, t^a$$
$$= \frac{1}{2} i f^{abc}\, i f^{bcd} t^d + C_2(r)\, t^a = \left[-\frac{1}{2}C_2(G) + C_2(r)\right] t^a \Rightarrow$$

$$\boxed{(t^b t^a t^b)_{ij} = -\left(\frac{C_2(G)}{2} - C_2(r)\right) t^a_{ij}}$$

As the result



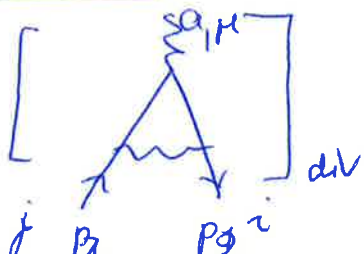
$$= -g^3\left(\frac{C_2(G)}{2} - C_2(r)\right) t^a_{ij} \times \int \frac{d^d k}{(2\pi)^d}\, \frac{\gamma_\rho \hat{k} \gamma_\mu \hat{k} \gamma^\rho}{(k^2)^3}$$

Since we can write $k_\mu k_\nu \to g_{\mu\nu} k^2/4$,

and $\gamma_\rho \gamma_\alpha \gamma_\mu \gamma^\alpha \gamma^\rho = 4\gamma_\mu$, we obtain

$$\int \frac{d^d k}{(2\pi)^d}\, \frac{\gamma_\rho \hat{k} \gamma_\mu \hat{k} \gamma^\rho}{(k^2)^3} = \gamma_\mu \times \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2)^2} \Rightarrow \frac{i\gamma_\mu \Gamma(1+\varepsilon)}{(4\pi)^{d/2}\varepsilon}$$

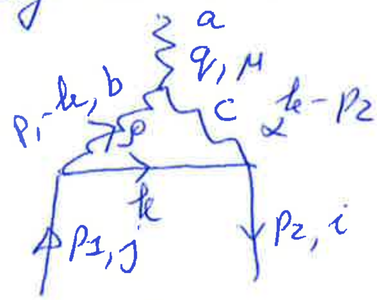
Therefore, we conclude that



$$\boxed{= i g (t^a)_{ij} \gamma^\mu \times \left[\frac{g^2\left(C_2(r) - \frac{C_2(G)}{2}\right)}{(4\pi)^2 \times \varepsilon}\right]}$$

The next diagram is dia_1. It is a bit more complex because it has non-abelian coupling.

The diagram reads:

$dia_1 \equiv$ 

$$\equiv \int \frac{d^d k}{(2\pi)^d}\, ig\hat{\gamma}_\alpha \frac{i}{\hat{k}}\, ig\hat{\gamma}_\rho \times$$

$$\times \frac{-i}{(p_1-k)^2}\,\frac{-i}{(k-p_2)^2} \times \left(t^c \cdot t^b\right)_{ij} \times g f^{abc} \times$$

$$\times \left[g^{\mu\rho}(q-p_1+k)_\alpha + g^{\rho\alpha}(p_1+p_2-2k)^\mu + \right.$$

$$\left. + g^{\alpha\mu}(k-p_2+q)^\rho \right].$$

Next, we compute the color factor: $(C_A = C_2(G))$

$$\boxed{\begin{aligned} (t^c t^b)_{ij} \times f^{abc} &= -\frac{1}{2} f^{acb}\left[t^c, t^b\right]_{ij} = \\ &= -\frac{i}{2} f^{acb} f^{cbd} \left(t^d\right)_{ij} \equiv -\frac{iC_A}{2}\left(t^a\right)_{ij} \end{aligned}}$$

The logarithmic divergence of the loop integral comes from $k \to \infty$ limit. Neglecting all external momenta, ~~we~~ ~~find~~ (which is allowed ~~too~~ because divergence is logarithmic), we find

$$\hat{\gamma}_\alpha \frac{1}{\hat{k}} \hat{\gamma}_\rho \left[g^{\mu\rho}(q-p_1+k)_\alpha + \dots \right]$$

$$\to k^2\gamma^\mu + 4\hat{k}k^\mu + \gamma^\mu k^2 \to 2k^2\gamma^\mu + k^2\gamma^\mu = 3k^2\gamma^\mu. \Rightarrow$$

Hence, we find the following result for dia_1

$$\text{dia}_1\Big|_{\text{div}} = i g^3 \frac{(-i C_A t^a_{ij})}{2} \times \frac{3}{2} \gamma^\mu \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2} \equiv$$

$$\equiv (i g \gamma^\mu)\, t^a_{ij} \left(\frac{C_A g^2}{\varepsilon (4\pi)^2} \times \frac{3}{2} \right) \equiv \Big($$  $$\Big)_{\text{div}}$$

The sum of the two divergent contributions gives

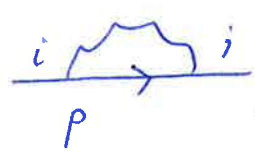
$$\Big[$$  $$\Big]_{\text{div}} \equiv \Big[$$  $+$  $$\Big]_{\text{div}} = i g \gamma_\mu t^a_{ij} \frac{g^2}{(4\pi)^2 \varepsilon} \left[C_2(r) + C_2(G) \right]$$

The divergent part of the gluon self-energy is

$$\Big[$$  $$\Big]_{\text{div}} = \frac{i g^2 \Gamma(1+\varepsilon)}{(4\pi)^{d/2} \varepsilon} \left(q^2 g^{\mu\nu} - q^\mu q^\nu \right) \delta^{ab} \times$$

$$\times \left[\frac{5}{3} C_2(G) - \frac{4}{3} N_f C(r) \right]$$

The divergent part of the massless electron self-energy is

$$\Big[$$  $$\Big]_{\text{div}} = \delta_{ij}\, C_2(r) \frac{i g^2}{(4\pi)^{d/2}} \frac{\Gamma(1+\varepsilon)}{\varepsilon} \hat{p}$$

We should note that there are other divergent Green's functions in non-abelian theory, including

3g coupling,

4g coupling,

ghost-gluon coupling,

ghost self energy

However, as we will see in the next lecture, their divergences are not independent of the divergences of Green's functions that we already computed (except for the ghost self-energy). This issue is crucial for the renormalizability of the ~~Ya~~ non-Abelian QFT, that we discuss in the next Lecture.