5 Lecture 4: quark-gluon scattering and color ordering

Consider the amplitude for quark-gluon scattering process $0 \to q(p_1) + \bar{q}(p_4) + g(p_2) + g(p_3)$. There are three diagrams that contribute; two "abelian" and one "non-abelian", that involves triple gluon couplings. We will take the left-handed spinor $\langle 1|$ for the outgoing quark with momentum p_1 and the left-handed spinor $|4]$ for the outgoing (right-handed) anti-quark with momentum p_4. The expression for the matrix element is

$$\begin{aligned} iM = & -ig^2 \langle 1| \left\{ \frac{\hat{\epsilon}_2(\hat{p}_1 + \hat{p}_2)\hat{\epsilon}_3}{s_{12}} \left(t^a t^b\right)_{ij} + \frac{\hat{\epsilon}_3(\hat{p}_1 + \hat{p}_3)\hat{\epsilon}_2}{s_{13}} \left(t^b t^a\right)_{ij} \right\} |4] \\ & - g^2 f^{abc} t^c_{ij} \frac{\langle 1\gamma^\lambda 4]}{s_{14}} \left(\epsilon_2 \cdot \epsilon_3 (p_2 - p_3)_\lambda + \epsilon_{3\lambda}(2p_3 + p_2)\cdot \epsilon_2 + \epsilon_{2\lambda}(-2p_2 - p_3)\epsilon_3\right). \end{aligned} \tag{5.1}$$

Here, $t^{a,b}$ are the $SU(3)$ Lie algebra generators in the fundamental representation and i, j, a, b refer to color indices of quarks and gluons. The $SU(3)$ generators are normalized $\mathrm{Tr}\left[t^a t^b\right] = \delta^{ab}/2$ and, as generators of a Lie algebra, they satisfy the commutation relation

$$t^a t^b - t^b t^a = i f^{abc} t^c. \tag{5.2}$$

We can use this relation to remove the $SU(3)$ structure constants from the expression for the amplitude. Also, we rescale $t^a = T^a/\sqrt{2}$, to have $\mathrm{Tr}\left[T^a T^b\right] = \delta^{ab}$. As the result of this, the amplitude is written as the sum of two terms

$$M = \left(\frac{g}{\sqrt{2}}\right)^2 \left(M_1 (T^q T^b)_{ij} + M_2 (T^b T^a)_{ij}\right), \tag{5.3}$$

where

$$\begin{aligned} M_1 = & -\Bigg[\frac{\langle 1|\epsilon_2(\hat{1} + \hat{2})\epsilon_3|4]}{s_{12}} - \frac{\langle 1\gamma^\mu 4]}{s_{14}} \left(\epsilon_2 \cdot \epsilon_3 (p_2 - p_3)_\mu \right. \\ & \left. + \epsilon_{3\mu}(2p_3 + p_2)\cdot \epsilon_2 + \epsilon_{2\mu}(-2p_2 - p_3)\epsilon_3\right) \Bigg]. \\ M_2 = & -\Bigg[\frac{\langle 1|\epsilon_3(\hat{1} + \hat{3})\epsilon_2|4]}{s_{13}} - \frac{\langle 1\gamma^\mu 4]}{s_{14}} \left(\epsilon_2 \cdot \epsilon_3 (p_3 - p_2)_\mu \right. \\ & \left. + \epsilon_{3\mu}(-2p_3 - p_2)\cdot \epsilon_2 + \epsilon_{2\mu}(2p_2 + p_3)\epsilon_3\right) \Bigg]. \end{aligned} \tag{5.4}$$

If we write $M_1 = M(1,2,3,4)$, then $M_2 = M(1,3,2,4)$, so it is sufficient to compute one function of external momenta to get the full result. We note that out of three diagrams that contribute to the amplitude M only two contribute to the function M_1. The diagram that does not contribute has its external particles arranged in such a way that they can not be ordered (clockwise) as p_1, p_2, p_3, p_4.

The amplitude $M(1,2,3,4)$ is called "color-ordered". It is transversal (gauge-invariant) and independent of color indices of colliding particles. We now calculate the color-ordered amplitude $M(1_L, 2, 3, 4_L)$. As the first step, we consider equal photon helicities, starting from right-handed photons. The relevant formula reads

$$\gamma_\mu \epsilon^\mu_R = \frac{\sqrt{2}}{\langle rk \rangle} \left(|k]\langle r| + |r\rangle [k| \right), \tag{5.5}$$

so that

$$\begin{aligned} \langle 1|\hat{\epsilon}_{3R} &= \frac{\sqrt{2}}{\langle r_3 3\rangle} \langle 1 r_3 \rangle [3| \\ \langle 1|\hat{\epsilon}_{2R} &= \frac{\sqrt{2}}{\langle r_2 2\rangle} \langle 1 r_2 \rangle [2|. \end{aligned} \tag{5.6}$$

Also, scalar products of polarization vectors with same helicities vanishes if the two vectors have identical reference momenta

$$\epsilon_{3R} \cdot \epsilon_{2R} \sim \langle r_2 r_3 \rangle. \tag{5.7}$$

It is then easy to see that if we choose $r_2 = r_3 = p_1$, the amplitude vanishes

$$M_1(q_{1_L}, g_{2_R}, g_{3_R}, \bar{q}_{4_L}) = 0. \tag{5.8}$$

Similar argument can be used to prove that amplitude $M_1(q_{1_L}, g_{2_L}, g_{3_L}, \bar{q}_{4_L})$ vanishes as well. Indeed,

$$\hat{\epsilon}_L = -\frac{\sqrt{2}}{[rk]} \left(|r]\langle k| + |k\rangle [r| \right), \tag{5.9}$$

so that

$$\begin{aligned} \hat{\epsilon}_{3L}|4] &= -\frac{\sqrt{2}}{[r_3 3]} |3\rangle [r_3 4], \\ \hat{\epsilon}_{2L}|4] &= -\frac{\sqrt{2}}{[r_2 2]} |2\rangle [r_2 4]. \end{aligned} \tag{5.10}$$

So, we choose $r_2 = r_3 = p_4$ and find $M_1(q_{1_L}, g_{2_L}, g_{3_L}, \bar{q}_{4_L}) = 0$.

Next, we will study the color-ordered amplitude where the two photon polarizations are different. Specifically, we consider $M(q_{1_L}, g_{2_R}, g_{3_L}, \bar{q}_{4_L})$. The explicit expression for the amplitude reads

$$\begin{aligned} M = -\Bigg[&\frac{\langle 1|\epsilon_{2R}(\hat{1}+\hat{2})\epsilon_{3L}|4]}{s_{12}} - \frac{\langle 1\gamma^\mu 4]}{s_{14}} \left(\epsilon_{2R} \cdot \epsilon_{3L} (p_2 - p_3)_\mu \right. \\ &\left. + \epsilon_{3L\mu}(2p_3 + p_2) \cdot \epsilon_{2R} + \epsilon_{2R\mu}(-2p_2 - p_3)\epsilon_{3L} \right) \Bigg]. \end{aligned} \tag{5.11}$$

To understand how to simplify computations, we will study contributing terms in Eq.(5.11) separately. The first term reads

$$\langle 1|\hat{\epsilon}_{2R}(\hat{1}+\hat{2})\hat{\epsilon}_{3L}|4] = -\frac{2\langle 1 r_2\rangle [2|(\hat{1}+\hat{2})|3\rangle [r_3 4]}{\langle r_2 2\rangle [r_3 3]} = -\frac{2\langle 1 r_2\rangle [21]\langle 13\rangle [r_3 4]}{\langle r_2 2\rangle [r_3 3]}. \tag{5.12}$$

The third and the fourth terms in Eq.(5.11) contain the following spinor products

$$\begin{aligned} \langle 1\gamma^\mu 4]\epsilon_{3L\mu} &= \langle 1|\hat{\epsilon}_{3L}4] = -\frac{\sqrt{2}\langle 13\rangle [r_3 4]}{[r_3 3]}, \\ \langle 1\gamma^\mu 4]\epsilon_{2R\mu} &= \langle 1|\hat{\epsilon}_{2R}4] = \frac{\sqrt{2}\langle 1r_2\rangle [24]}{\langle r_2 2\rangle}, \end{aligned} \tag{5.13}$$

Hence, we conclude that if we choose $r_3 = p_4$ and $r_2 = p_1$ all contributions in Eq.(5.12) and Eq.(5.13) vanish; therfore, only the second term in Eq.(5.11) contributes. We find

$$M_1(q_{1_L}, g_{2_L}, g_{3_L}, \bar{q}_{4_L}) = \frac{\langle 1|(\hat{2}-\hat{3})|4]}{s_{14}}\,\epsilon_{2R}\cdot\epsilon_{3L}. \tag{5.14}$$

To simplify it further, we use momentum conseration

$$\langle 1|(\hat{2}-\hat{3})|4] = -2\langle 13\rangle [34], \tag{5.15}$$

and compute the product of two polarization vectors

$$\epsilon_{2R}\cdot\epsilon_{3L} = \frac{\langle r_2\gamma^\mu 2]}{\sqrt{2}\langle r_2 2\rangle}\frac{(-1)[r_3\gamma_\mu 3\rangle}{\sqrt{2}[r_3 3]} = -\frac{\langle 1\gamma^\mu 2][4\gamma_\mu 3\rangle}{2\langle 12\rangle [43]} = -\frac{\langle 13\rangle [42]}{\langle 12\rangle [43]}. \tag{5.16}$$

We therefore find (use $s_{14} = s_{23} = \langle 23\rangle [23]$)

$$M(q_{1_L}, g_{2_R}, g_{3_L}, \bar{q}_{4_L}) = \frac{2\langle 13\rangle [34]}{\langle 23\rangle [23]}\,\frac{\langle 13\rangle [42]}{\langle 12\rangle [43]} = -\frac{2\langle 13\rangle^2 [42]}{\langle 12\rangle\langle 23\rangle [23]}. \tag{5.17}$$

We can simplify this expression by multiplying it by $1 = \langle 13\rangle/\langle 13\rangle$. It follows from momentum conservation that

$$\langle 13\rangle [32] = \langle 1|\hat{3}|2] = \langle 1\left(-\hat{1}-\hat{2}-\hat{4}\right)2] = -\langle 14\rangle [42]. \tag{5.18}$$

Then,

$$M_1(q_{1_L}, g_{2_R}, g_{3_L}, \bar{q}_{4_L}) = \frac{2\langle 13\rangle^3\langle 43\rangle}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}. \tag{5.19}$$

Next, we will compute the second color-ordered amplitude $M(q_{1_L}, g_{2_L}, g_{3_R}, \bar{q}_{4_L})$. We will again go through the same exercise of trying to force as many terms as possible to vanish. We will do it slightly differently this time. The amplitude reads

$$\begin{aligned} M = -\Bigg[&\frac{\langle 1|\hat{\epsilon}_{2L}(\hat{1}+\hat{2})\hat{\epsilon}_{3R}|4]}{s_{12}} - \frac{\langle 1\gamma^\mu 4]}{s_{14}}\,(\epsilon_{2L}\cdot\epsilon_{3R}(p_2-p_3)_\mu \\ &+\epsilon_{3R\mu}(2p_3+p_2)\cdot\epsilon_{2L} + \epsilon_{2L\mu}(-2p_2-p_3)\epsilon_{3R})\Bigg]. \end{aligned} \tag{5.20}$$

Lets focus on the "non-abelian" contribution to this amplitude. There are three terms that involve

$$p_3\cdot\epsilon_{2L}, \quad p_2\cdot\epsilon_{3R}, \quad \epsilon_{2L}\cdot\epsilon_{3R}. \tag{5.21}$$

Since

$$\epsilon_{2L} \cdot \epsilon_{3R} \sim [r_2 3]\langle r_3 2\rangle, \tag{5.22}$$

we can ensure that all terms in Eq.(5.21) vanish if $r_2 \sim p_3$ and $r_3 \sim p_2$. With these choices of reference vectors, we find

$$\begin{aligned} \langle 1|\hat{\epsilon}_{2L} &= -\frac{\sqrt{2}\langle 12\rangle[3|}{[32]}, \\ \hat{\epsilon}_{3R}|4] &= \frac{\sqrt{2}|2\rangle[34]}{\langle 23\rangle}. \end{aligned} \tag{5.23}$$

Therefore, we find

$$M(q_{1_L}, g_{2_L}, g_{3_R}, \bar{q}_{4_L}) = \frac{2\langle 12\rangle[34][31]\langle 12\rangle}{s_{12}[32]\langle 23\rangle} = \frac{2[34][31]\langle 12\rangle}{[21][32]\langle 23\rangle}. \tag{5.24}$$

For further simplifications, multiply both numerator and denominator with $\langle 12\rangle\langle 42\rangle$. Then

$$M(q_{1_L}, g_{2_L}, g_{3_R}, \bar{q}_{4_L}) = \frac{2[34][31]\langle 12\rangle^2\langle 42\rangle}{[21][32]\langle 23\rangle\langle 12\rangle\langle 42\rangle}. \tag{5.25}$$

Now, in the denominator use

$$[32]\langle 42\rangle = -[32]\langle 24\rangle = -[3|\hat{2}|4\rangle = [3|\hat{1}|4\rangle = [31]\langle 14\rangle, \tag{5.26}$$

so that

$$M(q_{1_L}, g_{2_L}, g_{3_R}, \bar{q}_{4_L}) = \frac{2[34]\langle 12\rangle^2\langle 42\rangle}{[21]\langle 14\rangle\langle 23\rangle\langle 12\rangle}. \tag{5.27}$$

To simplify it further, note that since $s_{12} = s_{34}$, we have

$$[34]\langle 43\rangle = [21]\langle 21\rangle \leftrightarrow \frac{[34]}{[21]} = \frac{\langle 12\rangle}{\langle 34\rangle}. \tag{5.28}$$

$$M(q_{1_L}, g_{2_L}, g_{3_R}, \bar{q}_{4_L}) = -\frac{2\langle 12\rangle^3\langle 42\rangle}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}. \tag{5.29}$$

Amplitudes for other helicity configurations can be obtained from the computed ones using complex conjugation.

6 Lecture 5: gluon scattering amplitudes and the idea of color ordering

As the next example, we consider gluon scattering $0 \to g_{p_1} + g_{p_2} + g_{p_3} + g_{p_4}$. The color labels will be chosen as $a_1, .., a_4$. There are four diagrams, – three with three-gluon vertices and one with the four-gluon vertex, see Fig. 1. Feynman rules for three-gluon vertices have already been shown; the Feynman rules for four-gluon vertex are (clockwise $(\mu, a), (\nu, b)(\lambda, c), (\sigma, d)$):

$$\begin{aligned} & -ig^2 \left[f^{abe} f^{cde} \left((g^{\mu\lambda} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\lambda} \right) + f^{ace} f^{bde} \left(g^{\mu\nu} g^{\lambda\sigma} - g^{\mu\sigma} g^{\mu\lambda} \right) \right. \\ & \left. + f^{ade} f^{bce} \left(g^{\mu\nu} g^{\lambda\sigma} - g^{\mu\lambda} g^{\nu\sigma} \right) \right]. \end{aligned} \tag{6.1}$$

Each of the three Feynman diagrams with three-gluon vertices and each term in Eq.(6.1) contains a color structure that reads $f^{abe} f^{cde}$, where a, b, c, d are choosen from a set $\{a_1, a_2, a_3, a_4\}$ and e is a dummy index. We will try to simplify the color structure.

First, lets write $f^{abe} f^{cde}$ is a canonical form. To do so, we will combine two equations for generators of $SU(3)$ algebra

$$[T^a, T^b] = i\sqrt{2} f^{abe} T^e, \quad \text{Tr} \left[T^a T^b \right] = \delta^{ab}. \tag{6.2}$$

Then,

$$\begin{aligned} -2 f^{abe} f^{cde} = \text{Tr} \left[[T^a, T^b][T^c, T^d] \right] = \text{Tr} \left[T^a T^b T^c T^d \right] - \text{Tr} \left[T^b T^a T^c T^d \right] \\ - \text{Tr} \left[T^a T^b T^d T^c \right] + \text{Tr} \left[T^b T^a T^d T^c \right]. \end{aligned} \tag{6.3}$$

Since we can do the same in all diagrams that contribute to four-gluon scattering ampliutde, we conclude that the full amplitude can be represented as

$$\begin{aligned} \mathcal{M} = \mathcal{M}_1 \text{Tr} \left[T^{a_1} T^{a_2} T^{a_3} T^{a_4} \right] + \mathcal{M}_2 \text{Tr} \left[T^{a_1} T^{a_2} T^{a_4} T^{a_3} \right] + \mathcal{M}_3 \text{Tr} \left[T^{a_1} T^{a_3} T^{a_2} T^{a_4} \right] \\ + \mathcal{M}_4 \text{Tr} \left[T^{a_1} T^{a_3} T^{a_4} T^{a_2} \right] + \mathcal{M}_5 \text{Tr} \left[T^{a_1} T^{a_4} T^{a_2} T^{a_3} \right] + \mathcal{M}_6 \text{Tr} \left[T^{a_1} T^{a_4} T^{a_3} T^{a_2} \right], \end{aligned} \tag{6.4}$$

where $M_{1..6}$ are functions of momenta and polarization vectors. We will now try to understand which Feynman diagrams contribute to those functions. Consider first Feynman diagram in Fig.1. The diagram has two triple-gluon vertices. The color factor is $f^{a_1 e a_4} f^{a_2 a_3 e}$ and this is

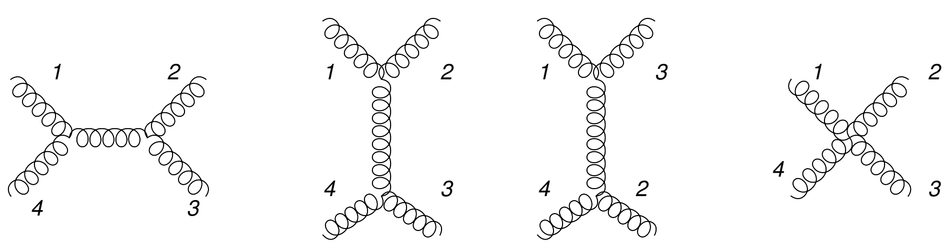

Figure 1. Four-gluons scattering diagrams

expressed through traces of $SU(3)$ generators, as

$$
\begin{aligned}
&f^{a_1 e a_4} f^{a_2 a_3 e} \sim \mathrm{Tr}\left[T^{a_1}T^{a_4}T^{a_2}T^{a_3}\right] - \mathrm{Tr}\left[T^{a_1}T^{a_2}T^{a_3}T^{a_4}\right] \\
&- \mathrm{Tr}\left[T^{a_1}T^{a_4}T^{a_3}T^{a_2}\right] + \mathrm{Tr}\left[T^{a_1}T^{a_3}T^{a_2}T^{a_4}\right].
\end{aligned}
\tag{6.5}
$$

We conclude that this diagram does not contribute to two color-structures, $\mathrm{Tr}\left[T^{a_1}T^{a_3}T^{a_4}T^{a_2}\right]$ and $\mathrm{Tr}\left[T^{a_1}T^{a_2}T^{a_4}T^{a_3}\right]$, i.e. where gluons 1 and 4 are not adjacent. Also, signs of various terms in Eq.(6.5) are different. To understand the meaning of this, consider a diagram with two three-gluon vertices where gluons appear clockwise as $((4,1),(2,3))$. This diagram is proportional to

$$
\begin{aligned}
&V_{3g}(1,-14,4)V_{3g}(2,3-23)f^{a_1 e a_4} f^{a_2,a_3,e} \\
&\sim V_{3g}(1,-14,4)V_{3g}(2,3,-23)\Big[\mathrm{Tr}\left(T^{a_1}T^{a_4}T^{a_2}T^{a_3}\right) - \mathrm{Tr}\left(T^{a_1}T^{a_2}T^{a_3}T^{a_4}\right) \\
&- \mathrm{Tr}\left(T^{a_1}T^{a_4}T^{a_3}T^{a_2}\right) + \mathrm{Tr}\left(T^{a_1}T^{a_3}T^{a_2}T^{a_4}\right)\Big].
\end{aligned}
\tag{6.6}
$$

Now, we can use the fact that the color-stripped three-gluon vertex functions $V_{3g}(i,j,k)$ is anti-symmetric w.r.t. permutations of any pair of gluons, e.g. $V_{3g}(i,j,k) = -V_{3g}(j,i,k)$. Hence, if we synchronize the order of arguments in functions V_{3g} with the order of arguments in color traces, the signs of all terms in Eq.(6.6) are the same. Indeed,

$$
\begin{aligned}
&V_{3g}(1,-14,4)V_{3g}(2,3,-23)\mathrm{Tr}\left(T^{a_1}T^{a_4}T^{a_2}T^{a_3}\right) \\
&= (-1)V_{3g}(1,4,-14)V_{3g}(2,3,-23)\mathrm{Tr}\left(T^{a_1}T^{a_4}T^{a_2}T^{a_3}\right), \\
&V_{3g}(1,-14,4)V_{3g}(2,3,-23)\mathrm{Tr}\left(T^{a_1}T^{a_2}T^{a_3}T^{a_4}\right) \\
&= V_{3g}(4,1,-14)V_{3g}(2,3,-23)\mathrm{Tr}\left(T^{a_4}T^{a_1}T^{a_2}T^{a_3}\right),
\end{aligned}
\tag{6.7}
$$

etc. Hence, we conclude that all terms in Eq.(6.5) will enter with the sign minus provided that the color-stripped diagram is drawn with the appropriate order of gluons in each of the three cases; namely, the order should be identical to the order of generators in the color trace.

One can perform a similar analysis for other diagrams that contribute to luon scattering and come to the conclusion that color-ordered amplitudes are, in fact, one and the same function that differ only by permutations of its arguments

$$
\mathcal{M} = \overline{\mathcal{M}}(1,2,3,4)\mathrm{Tr}\left(T^{a_1}T^{a_2}T^{a_3}T^{a_4}\right) + \overline{\mathcal{M}}(1,3,2,4)\mathrm{Tr}\left(T^{a_1}T^{a_2}T^{a_4}T^{a_3}\right) + \cdots \tag{6.8}
$$

These color-ordered (or "color-stripped", or "partial") amplitudes are obtained from "color-stripped" Feynman rules shown in Fig. 2.

We note that *all* graphs that are properly ordered and *only* such graphs should be included when color-ordered amplitude for a particular process is computed. As we will see on a simple example of four-gluon scattering, this implies that diagrams for which external gluons are improperly ordered can be dropped.

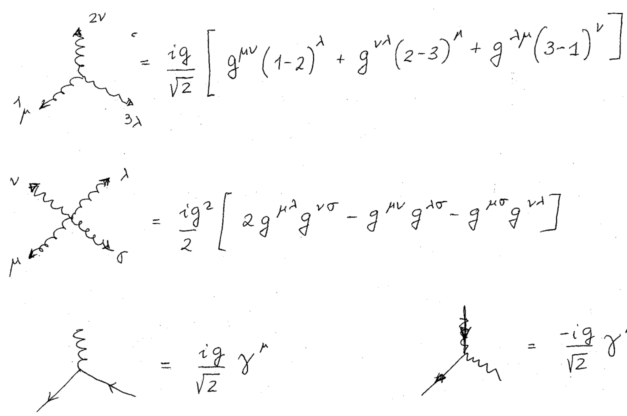

Figure 2. Color stripped Feynman rules.

We now turn to the computation of the four-gluon color-ordered amplitude $M(1,2,3,4)$. Three diagrams, shown in Fig. 4 contribute. The result reads

$$
\begin{aligned}
\overline{\mathcal{M}}(g_1,g_2,g_3,g_4) = \left(\frac{ig}{\sqrt{2}}\right)^2 \Bigg\{ & \frac{-i}{s_{14}} \left[\epsilon_1\cdot\epsilon_4(4-1)_\lambda + \epsilon_{1\lambda}(1+14)\cdot\epsilon_4 + \epsilon_{4\lambda}(-14-4)\cdot\epsilon_1\right] \times \\
& \left[\epsilon_2\epsilon_3(2-3)_\lambda + \epsilon_{3\lambda}\,(3+23)\cdot\epsilon_2 + \epsilon_{2\lambda}\,(-23-2)\cdot\epsilon_3\right] \\
& + \frac{-i}{s_{12}} \left[\epsilon_1\cdot\epsilon_2(1-2)_\lambda + \epsilon_{2\lambda}(2+12)\cdot\epsilon_1 + \epsilon_{1\lambda}\,(-12-1)\cdot\epsilon_2\right] \times \\
& \left[\epsilon_3\cdot\epsilon_4(3-4)_\lambda + \epsilon_{4\lambda}\,(4+43)\cdot\epsilon_3 + \epsilon_{3\lambda}\,(-3-34)\cdot\epsilon_2\right] \\
& - i\left[2\epsilon_1\cdot\epsilon_3\epsilon_2\cdot\epsilon_4 - \epsilon_1\cdot\epsilon_2\epsilon_3\cdot\epsilon_4 - \epsilon_1\cdot\epsilon_4\epsilon_2\cdot\epsilon_3\right] \Bigg\}.
\end{aligned}
\tag{6.9}
$$

This expression looks rather complex and we will now discuss how to simplify its computation dramatically. We will start with a very special case, taking helicities of all colliding gluons to be the same. It follows from Eq.(6.9) that every term in the amplitude contains a scalar product of polarization vectors. From Lecture 2 we know that if two polarization vectors have the same helicity and the same reference vector, their scalar product vanishes since

$$
\epsilon_L(p_i,r_i)\cdot\epsilon_L(p_j,r_j) \sim [r_i r_j], \quad \epsilon_R(p_i,r_i)\cdot\epsilon_R(p_j,r_j) \sim \langle r_i r_j\rangle. \tag{6.10}
$$

Hence, to compute $\mathcal{M}(g_{1L},g_{2L},g_{3L},g_{4L})$ or $\mathcal{M}(g_{1R},g_{2R},g_{3R},g_{4R})$, we just need to choose identical reference vectors for all gluons and observe that both amplitudes vanish

$$
\mathcal{M}(g_{1L},g_{2L},g_{3L},g_{4L}) = \mathcal{M}(g_{1R},g_{2R},g_{3R},g_{4R}) = 0. \tag{6.11}
$$

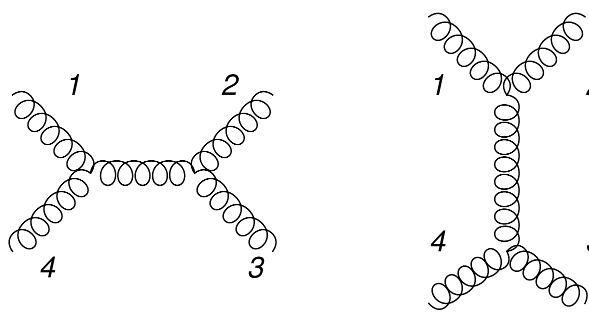

Figure 3. Four-gluons scattering diagrams

Note that the above argument generalizes to the case of n-gluon scattering in a straightforward way.

As the next step, consider the amplitude where three helicities are the same and one is different. For definitiness, we take helicities of g_1, g_2, g_3 to the R ("plus") and the helicity of g_4 to be L ("minus"). If we choose equal reference vectors for $g_{1,..,3}$, all scalar products between their polarization vectors vanish. The scalar product of a right-handed polarization vector and left-handed polarization vector reads

$$\epsilon_R(p_i, r) \cdot \epsilon_L(p_j, s) = \frac{\langle r\gamma^\mu i]}{\sqrt{2}\langle ri\rangle} \frac{(-1)[s\gamma_\mu j\rangle}{\sqrt{2}[sj]} = -\frac{\langle rj\rangle [si]}{\langle ri\rangle [sj]}. \tag{6.12}$$

In our case $p_j = p_4$ and so choose the reference vector for all the left-handed polarization vectors to be p_4, forces all the scalar product that involve ϵ_{4L} to vanish. Hence, we conclude that

$$\mathcal{M}(g_{1R}, g_{2R}, g_{3R}, g_{4L}) = 0. \tag{6.13}$$

Clearly, by a similar argument, any other amplitude for three equal helicities and one different helicity vanishes. Also, similar to all-equal helicity case, the argument generalizes to the case of n-gluon scattering in a straightforward way.

Next, we consider the case when there are two gluons with equal helicities and two gluons with different helicities. We need to consider two cases, that we take to be $\mathcal{M}(g_{1R}, g_{2R}, g_{3L}, g_{4L})$ and $\mathcal{M}(g_{1R}, g_{2L}, g_{3R}, g_{4L})$.

Let us begin by considering $\mathcal{M}(g_{1R}, g_{2R}, g_{3L}, g_{4L})$. From our previous studies, we know that scalar products of same-helicity polarization vectors vanish if reference vectors are the same and scalar product of different helciity vectors vanish if a reference vector for one gluon is the momentum of the other. Therefore, if we choose identical reference vectors for g_1 and g_2 and identical reference vectors for g_3 and g_4, we will have $\epsilon_{1R} \cdot \epsilon_{2R} = 0$ and $\epsilon_{3L} \cdot \epsilon_{4L} = 0$. Next, by choosing the reference vector for left-handed polarizations to be p_2^μ, we obtain $\epsilon_{2R} \cdot \epsilon_{3L} = \epsilon_{2R} \cdot \epsilon_{4L} = 0$. Finally, by choosing the reference vector for right-handed polarizations to be p_3^μ, we have $\epsilon_{1R} \cdot \epsilon_{3L} = 0$. Therefore, the only non-vanishing scalar product is $\epsilon_{1R} \cdot \epsilon_{4L}$.

Also, thanks to our choices of reference vectors, we have

$$p_2^\mu \epsilon_{3L\mu} = p_2^\mu \epsilon_{4L\mu} = 0, \quad p_3^\mu \epsilon_{1R\mu} = p_3^\mu \epsilon_{2R\mu} = 0. \tag{6.14}$$

It follows from Eq.(6.9) that only the second term in the sum contributes and the result reads

$$\mathcal{M}(g_{1R}, g_{2R}, g_{3L}, g_{4L}) = \frac{-2g^2}{s_{12}} \left(\epsilon_{1R} \cdot \epsilon_{4L}\right) \left(p_1 \cdot \epsilon_{2R}\right) \left(p_4 \cdot \epsilon_{3L}\right). \tag{6.15}$$

It is straighforward to compute the scalar products in Eq.(6.15). Using Eq.(6.12), we find

$$\epsilon_{1R}(p_1, p_3) \cdot \epsilon_{4L}(p_4, p_2) = -\frac{\langle 34 \rangle [21]}{\langle 31 \rangle [24]}, \tag{6.16}$$

and

$$p_{1\mu} \epsilon_{2R}^\mu = \frac{\langle 3\hat{1}2]}{\sqrt{2}\langle 32 \rangle}, \quad p_{4\mu} \epsilon_{3L}^\mu = -\frac{[2\hat{4}3\rangle}{\sqrt{2}[23]}. \tag{6.17}$$

Now, putting everything together, we obtain

$$\begin{aligned}\mathcal{M}(g_{1R}, g_{2R}, g_{3L}, g_{4L}) &= \frac{g^2}{s_{12}} \frac{\langle 34 \rangle [12]^2}{\langle 32 \rangle} \frac{\langle 43 \rangle}{[23]} = -g^2 \frac{[12]^2 \langle 34 \rangle^2}{s_{12} s_{23}} = -g^2 \frac{[12]^2 \langle 34 \rangle^2}{s_{12} s_{23}} \\ &- g^2 \frac{[12]^2 \langle 34 \rangle^2}{s_{34} s_{14}} = -g^2 \frac{[12]^2 \langle 34 \rangle^2}{\langle 34 \rangle [43] \langle 14 \rangle [41]} = -g^2 \frac{[12]^3 \langle 34 \rangle}{[12][43] \langle 14 \rangle [41]}\end{aligned} \tag{6.18}$$

We would like to get rid of "wrong" brackets in this expression $\langle 34 \rangle / \langle 14 \rangle$. We can further simplify the last expression if we use momentum conservation

$$[2\hat{3}4\rangle = -[2\hat{1}4\rangle. \tag{6.19}$$

This implies $[23]\langle 34 \rangle = -[21]\langle 14 \rangle$, so that $\langle 34 \rangle / \langle 14 \rangle = [12]/[23]$. Using this in Eq.(6.18), we obtain

$$\mathcal{M}(g_{1R}, g_{2R}, g_{3L}, g_{4L}) = g^2 \frac{[12]^4}{[12][23][34][41]}. \tag{6.20}$$

The second helicity amplitude that we need to consider is $\mathcal{M}(g_{1R}, g_{2L}, g_{3R}, g_{4L})$. The calculation is similar to what we already did: taking reference vectors for right-handed polarization vectors to be $p_{2\mu}$ and for left-handed polarizations to be $p_{3\mu}$ implies that the only scalar product of polarization vectors that survives is $\epsilon_{1R} \cdot \epsilon_{4L}$. The expression for the amplitude reads

$$\mathcal{M}(g_{1R}, g_{2L}, g_{3R}, g_{4L}) = \frac{-2g^2}{s_{12}} \left(\epsilon_{1R} \cdot \epsilon_{4L}\right) \left(p_1 \cdot \epsilon_{2L}\right) \left(p_4 \cdot \epsilon_{3R}\right). \tag{6.21}$$

To compute the scalar products, we use

$$\epsilon_{1R}(p_1, p_2) \cdot \epsilon_{4L}(p_4, p_3) = -\frac{\langle 24 \rangle [31]}{\langle 21 \rangle [34]}, \tag{6.22}$$

and

$$p_{1\mu} \epsilon_{2L}^\mu = -\frac{[3\hat{1}2\rangle}{\sqrt{2}[32]}, \quad p_{4\mu} \epsilon_{3R}^\mu = -\frac{\langle 2\hat{4}3]}{\sqrt{2}\langle 23 \rangle}. \tag{6.23}$$

We find

$$\mathcal{M}(g_{1R}, g_{2L}, g_{3R}, g_{4L}) = \frac{g^2[13]^2\langle 24\rangle^2}{s_{12}[32]\langle 23\rangle} = \frac{g^2[13]^2\langle 24\rangle^2}{s_{12}s_{23}} = \frac{g^2[13]^2\langle 24\rangle^2}{s_{34}s_{14}}. \tag{6.24}$$

To simplify, we multiply this expression with $[13]^2$, use

$$\begin{aligned} &\frac{\langle 24\rangle^2}{[13]^2\langle 34\rangle\langle 41\rangle} = \frac{\langle 24\rangle^2}{\langle 4\hat{1}3]\langle 34\rangle[13]} = -\frac{\langle 24\rangle^2}{\langle 4\hat{2}3]\langle 34\rangle[13]} \\ &= \frac{\langle 24\rangle}{[23]\langle 34\rangle[13]} = \frac{\langle 24\rangle}{[23][1\hat{3}4\rangle} = -\frac{\langle 24\rangle}{[23][12]\langle 24\rangle} = \frac{-1}{[23][12]} \end{aligned} \tag{6.25}$$

to find

$$\mathcal{M}(g_{1R}, g_{2L}, g_{3R}, g_{4L}) = \frac{g^2[13]^4}{[12][23][34][41]}. \tag{6.26}$$

As we see, use of spinor-helciity methods allows us to find very compact expressions for scattering amplitudes for four-gluon scattering. This completes the calculation of spinor-helicity ampltudes for gluon scattering. Every amplitude that we have not computed explicitly can be obtained from the complex conjugation.

7 Lecture 6: gluon scattering cross-sections

In this Lecture we will discuss how to use the color-ordered helicity amplitudes calculated in the prevoius lecture to compute scattering cross-sections. A difficult part here is the sum over colors. Recall that the scattering amplitude is written as

$$\mathcal{M}(g_1^{a_1}, g_2^{a_2}, g_3^{a_3}, g_4^{a_4}) = \sum_{\sigma \in P(2,3,4)} \overline{\mathcal{M}}(g_1, g_{\sigma_2}, g_{\sigma_3}, g_{\sigma_4}) \times \mathrm{Tr}\left[T^{a_1} T^{a_{\sigma_2}} T^{a_{\sigma_3}} T^{a_{\sigma_4}}\right], \tag{7.1}$$

where σ_i is an element of the permutation set of three numbers a_2, a_3, a_4. To compute the cross-section, we need to square the amplitude and sum over colors and helicities. The helicity sums are easy but the sum over colors seems complicated. We will discuss how it can be performed.

To sum over colors, we need to deal with products of traces, summed over color indices

$$\mathrm{Tr}\left[T^{a_1} T^{a_{\sigma_1}} T^{a_{\sigma_2}} T^{a_{\sigma_3}}\right] \times \mathrm{Tr}\left[T^{a_1} T^{a_{\xi_2}} T^{a_{\xi_3}} T^{a_{\xi_4}}\right] \Pi \delta_{a_{\sigma_i} a_{\xi_j}}. \tag{7.2}$$

In general, products of traces can be computed with the help of the following identity

$$X_{ij;km} = \sum_{a=1}^{N^2-1} \left(T^a\right)_{ij} \left(T^a\right)_{km} = \delta_{im}\delta_{kj} - \frac{1}{N}\delta_{ij}\delta_{km}. \tag{7.3}$$

To prove Eq.(7.3), we use transformation properties of the right-hand side under $SU(N)$, the fact that T^a's are traceless and that $\mathrm{Tr}\left[T^a T^b\right] = \delta_{ab}$. We can use these identities to, e.g., transform the product of traces into trace of products and simple terms. Indeed, consider

$$\begin{aligned} &\mathrm{Tr}\left[A_1 T^a A_2\right] \mathrm{Tr}\left[B_1 T^a B_2\right] = A_{1,i_1 i_2} A_{2,j2,i1} B_{1,k_1 k_2} B_{2,m_2 k_1} \left[\delta_{i_2 m_2}\delta_{k_2 j_2} - \frac{1}{N}\delta_{i_2 j_2}\delta_{k_2 m_2}\right] \\ &= \mathrm{Tr}\left[A_1 B_2 B_1 A_2\right] - \frac{1}{N}\mathrm{Tr}\left[A_1 A_2\right] \mathrm{Tr}\left[B_1 B_2\right]. \end{aligned} \tag{7.4}$$

The result will be a complicated collection of traces to compute. It can be done but it is not easy. However, it turns out that there is a simpler way to do it and we will describe it now.

We have so far considered the group $SU(N)$ where S tells us that group elements should have determinant 1. This implies that $SU(N)$ generators are traceless; indeed, an element of $SU(N)$ is

$$g \approx e^{iT^a\theta^a} \approx 1 + iT^a\theta^a \;\Rightarrow\; 1 = \det(g) \approx 1 + i\theta^a \mathrm{Tr}[T^a] \;\Rightarrow\; \mathrm{Tr}[T^a] = 0. \tag{7.5}$$

Let us imagine now that we extend the $SU(N)$ group to $U(N)$. This amounts to the introduction of a "phase" generator T_{N^2} which commutes with all generators of $SU(N)$ and that is normalized as $\mathrm{Tr}[T_{N^2} T_{N^2}] = 1$. Hence, we take T_{N^2} to be a diagonal matrix with elements $1/\sqrt{N}$. With this extension, equation for $X_{ij;km}$ simplifies

$$X_{ij,km}^{U(N)} = \sum_{a=1}^{N^2} T_{ij}^a T_{km}^a = \sum_{a=1}^{N^2-1} T_{ij}^a T_{km}^a + T_{ij}^{N^2} T_{km}^{N^2} = \delta_{im}\delta_{kj}. \tag{7.6}$$

We would like to use this result to compute color factors to describe scattering of gluons, effectively changing $SU(N) \to U(N)$. Are we allowed to do that? The answer is yes, because $U(1)$ gluons do not couple to other, $SU(N)$ ones, sine $f^{abc} = 0$ if a, b or c is N^2. Therefore, we can use a simple $U(N)$ formula to sum over colors.

Hence, we need to compute products of traces summed over colors. First, we calculate

$$(T^q T^a)_{ij} = N\delta_{ij}. \tag{7.7}$$

Then

$$\begin{aligned} &\mathrm{Tr}\left[T^a T^b T^c T^d\right] \mathrm{Tr}\left[T^a T^b T^c T^d\right]^+ = \mathrm{Tr}\left[T^a T^b T^c T^d\right] \mathrm{Tr}\left[T^d T^c T^b T^a\right] \\ &= T^a_{ik}\left(T^b T^c T^d\right)_{ki}\left(T^d T^c T^b\right)_{jm} T^a_{mj} = \delta_{ij}\delta_{mk}\left(T^b T^c T^d\right)_{ki}\left(T^d T^c T^b\right)_{jm} \\ &= \mathrm{Tr}\left[T^b T^c T^d T^d T^c T^b\right] = N\mathrm{Tr}\left[T^b T^c T^c T^b\right] = N^3 \mathrm{Tr}\left[1\right] = N^4. \end{aligned} \tag{7.8}$$

The second product of traces that we need to compute is more complicated

$$\begin{aligned} &\mathrm{Tr}\left[T^a T^b T^c T^d\right] \mathrm{Tr}\left[T^a T^b T^d T^c\right]^+ = \mathrm{Tr}\left[T^a T^b T^c T^d\right] \mathrm{Tr}\left[T^c T^d T^b T^a\right] \\ &= \mathrm{Tr}\left[T^b T^c T^d T^c T^d T^b\right] = N\mathrm{Tr}\left[T^c T^d T^c T^d\right]. \end{aligned} \tag{7.9}$$

To compute the last trace note that, for any matrix A, we have

$$(T^c A T^c)_{ij} = T^c_{ik} T^c_{mj} A_{km} = \delta_{ij}\delta_{km} A_{km} = \mathrm{Tr}\left[A\right]\delta_{ij}. \tag{7.10}$$

This implies that

$$T^c T^d T^c|_{d=N^2} = \sqrt{N}\hat{1}, \quad \text{and} \quad T^c T^d T^c|_{d\in N^2-1} = 0. \tag{7.11}$$

Therefore, the last term in Eq.(7.9) becomes

$$N\mathrm{Tr}\left[T^c T^d T^c T^d\right] = N^{3/2}\mathrm{Tr}\left[T^{N^2}\right] = N^2. \tag{7.12}$$

Hence,

$$\mathrm{Tr}\left[T^a T^b T^c T^d\right] \mathrm{Tr}\left[T^a T^b T^d T^c\right]^+ = N^2. \tag{7.13}$$

The remaining contirbutions are

$$\begin{aligned} &\mathrm{Tr}\left[T^a T^b T^c T^d\right] \mathrm{Tr}\left[T^a T^c T^b T^d\right]^+ = N^2, \\ &\mathrm{Tr}\left[T^a T^b T^c T^d\right] \mathrm{Tr}\left[T^a T^c T^d T^b\right]^+ = N^2, \\ &\mathrm{Tr}\left[T^a T^b T^c T^d\right] \mathrm{Tr}\left[T^a T^d T^c T^b\right]^+ = N^2, \\ &\mathrm{Tr}\left[T^a T^b T^c T^d\right] \mathrm{Tr}\left[T^a T^d T^b T^c\right]^+ = N^2. \end{aligned} \tag{7.14}$$

Suppose now we square Eq.(7.1) and sum it over gluon color indices. Then, in the resulting sum over colors, there will be terms proportional to squared of color-ordered amplitudes and terms that are interferences. According to our calculation above, color-factors that multiply squares of amplitudes are N^4, while color factors that multiply *all* interference terms are N^2. Hence, we find

$$\sum_{\text{colors}} |\mathcal{M}|^2 = N^4 \sum_{I=1}^{6} |\overline{\mathcal{M}}_I|^2 + N^2 \sum_{I \neq J} \overline{\mathcal{M}}_I \overline{\mathcal{M}}_J^*. \tag{7.15}$$

We will simplify this formula using the following identity for color-stripped amplitudes

$$\sum_{I=1}^{6} \overline{\mathcal{M}}_I = 0. \tag{7.16}$$

This equation is *not obvious* and we will explain shortly why it is valid. But let us see first how it helps. We find

$$\sum_{I \neq J} \overline{\mathcal{M}}_I \overline{\mathcal{M}}_J^* = -\sum_{I} |\overline{\mathcal{M}}_I|^2, \tag{7.17}$$

so that Eq.(7.15) becomes

$$\sum_{\text{colors}} |\mathcal{M}|^2 = N^2(N^2-1) \sum_{I=1}^{6} |\overline{\mathcal{M}}_I|^2. \tag{7.18}$$

We conclude that for four-gluon scattering, the full amplitude squared is given by the sum of squares of color-ordered amplitudes.

Let me now explain why the sum of color-ordered amplitudes vanishes. For our purposes, it can be viewed as a consequence of the fact that the $U(1)$ gluon can not interact with $SU(N)$ gluons. This statement is obvious as long as the color-information is kept. However, once we use color-ordered amplitudes, the color information disappears and the "non-interaction" of certain types of gluons with the rest manifests itself in a complex way. This is the meaning of Eq.(7.16). To see how it works in detail, consider a four-gluon scattering amplitude and take one of those gluons to be the $U(1)$ gluon and the other three $SU(N)$ gluons. Then

$$0 = \mathcal{M}\left(g_1^{N^2} g^{a_2} g^{a_3} g^{a_4}\right) = \sum_{\sigma \in P} \overline{\mathcal{M}}(g_1, g_{\sigma_2}, g_{\sigma_3}, g_{\sigma_4}) \, \mathrm{Tr}\left[T^{a^2_N} T^{a_{\sigma_2}} T^{a_{\sigma_3}} T^{a_{\sigma_4}}\right]. \tag{7.19}$$

There are six different traces on the right hand side of the above equation, but they can grouped into two groups of equal traces since T^{N^2} generator is proportional to an identity matrix. Thefore

$$\begin{aligned} \mathrm{Tr}\left[T^{a^2_N} T^{a_2} T^{a_3} T^{a_4}\right] &= \mathrm{Tr}\left[T^{a^2_N} T^{a_4} T^{a_2} T^{a_3}\right] = \mathrm{Tr}\left[T^{a^2_N} T^{a_3} T^{a_4} T^{a_2}\right], \\ \mathrm{Tr}\left[T^{a^2_N} T^{a_2} T^{a_4} T^{a_3}\right] &= \mathrm{Tr}\left[T^{a^2_N} T^{a_3} T^{a_2} T^{a_4}\right] = \mathrm{Tr}\left[T^{a^2_N} T^{a_4} T^{a_3} T^{a_2}\right]. \end{aligned} \tag{7.20}$$

Using these equalities and the fact that T^{N^2} is proportional to the identity matrix, we can re-write the right-hand side of Eq.(7.19) as

$$\begin{aligned} 0 &= \mathrm{Tr}\left[T^{a_2}T^{a_3}T^{a_4}\right]\left(\overline{\mathcal{M}}(g_1,g_2,g_3,g_4)+\overline{\mathcal{M}}(g_1,g_4,g_2,g_3)+\overline{\mathcal{M}}(g_1,g_3,g_4,g_2)\right) \\ &+ \mathrm{Tr}\left[T^{a_2}T^{a_4}T^{a_3}\right]\left(\overline{\mathcal{M}}(g_1,g_2,g_4,g_3)+\overline{\mathcal{M}}(g_1,g_3,g_2,g_4)+\overline{\mathcal{M}}(g_1,g_4,g_3,g_2)\right). \end{aligned} \tag{7.21}$$

Now, we can choose any values for remaining color indices. For example, take $a_2 = N^2$. Then, since $\mathrm{Tr}\left[T_2^a T_4^a\right] = \mathrm{Tr}\left[T_4^a T_2^a\right]$, we obtain

$$\begin{aligned} 0 &= \overline{\mathcal{M}}(g_1,g_2,g_3,g_4)+\overline{\mathcal{M}}(g_1,g_4,g_2,g_3)+\overline{\mathcal{M}}(g_1,g_3,g_4,g_2) \\ &+ \overline{\mathcal{M}}(g_1,g_2,g_4,g_3)+\overline{\mathcal{M}}(g_1,g_3,g_2,g_4)+\overline{\mathcal{M}}(g_1,g_4,g_3,g_2), \end{aligned} \tag{7.22}$$

which is Eq.(7.16).

One the other hand, Eq.(7.21) contains more information than what we have in Eq.(7.16). This is because, for a general group $SU(N)$, $\mathrm{Tr}\left[T^{a_2}T^{a_4}T^{a_3}\right]$ and $\mathrm{Tr}\left[T^{a_2}T^{a_3}T^{a_4}\right]$ are linear-independent. Therefore, the right hand side in Eq.(7.21) can vanish if an only if the coefficients of two color traces there vanish independently of each other

$$\begin{aligned} 0 &= \overline{\mathcal{M}}(g_1,g_2,g_3,g_4)+\overline{\mathcal{M}}(g_1,g_4,g_2,g_3)+\overline{\mathcal{M}}(g_1,g_3,g_4,g_2), \\ 0 &= \overline{\mathcal{M}}(g_1,g_2,g_4,g_3)+\overline{\mathcal{M}}(g_1,g_3,g_2,g_4)+\overline{\mathcal{M}}(g_1,g_4,g_3,g_2), \end{aligned} \tag{7.23}$$

These two equations can be understood as a consequence of a simple identity. We can rewrite the first equation in (7.23) (using cyclic symmetry of the color-stripped amplitudes) as

$$0 = \overline{\mathcal{M}}(g_1,g_2,g_3,g_4)+\overline{\mathcal{M}}(g_2,g_1,g_3,g_4)+\overline{\mathcal{M}}(g_2,g_3,g_1,g_4), \tag{7.24}$$

which shows that the sum of all color-ordered amplitudes where the position of one gluon is changed and the position of all other gluons are kept fixed vanishes. This is an example of a more general set of "abelian" color identities that reduce the number of independent color-ordered amplitudes. To give you an idea about the reduction in the number of independent amplitudes, let me note that "naive" estimate of the number of independent color-ordered amplitudes for n-gluon scattering is obviously $(n-1)!$. However, the abelian identities and the so-called Bern-Johannson-Carrasco identities, bring the number of independent amplitudes can be brought down to $(n-3)!$.

For now, we will complete the calculation of the ampltiude squared for gluon scattering, focusing now on the sum over gluon heliciites. We have computed two helicity ampltiudes in the previous lecture; they are given in Eq.(6.20) and Eq.(6.26). We will use $s = s_{12} = s_{34}$, $t = s_{13} = s_{24}$ and $u = s_{23} = s_{14}$ to denote kinematic invariants. We have

$$\begin{aligned} |\mathcal{M}(g_{1R},g_{2R},g_{3L},g_{4L})|^2 &= \left|\frac{g^2[12]^4}{[12][23][34][41]}\right|^2 = \frac{g^4 s_{12}^4}{s_{12}s_{23}s_{34}s_{41}} = \frac{g^4 s^2}{u^2}, \\ |\mathcal{M}(g_{1R},g_{2L},g_{3R},g_{4L})|^2 &= \left|\frac{g^2[13]^4}{[12][23][34][41]}\right|^2 = \frac{g^4 t^4}{s^2 u^2}. \end{aligned} \tag{7.25}$$