2 Lecture 1: spinors, their properties and spinor prodcuts

Consider a theory of a single massless Dirac fermion ψ. The Lagrangian is

$$\mathcal{L} = \bar{\psi}\left(i\hat{\partial}\right)\psi. \tag{2.1}$$

The Dirac equation is

$$i\hat{\partial}\psi = 0, \tag{2.2}$$

which, in momentum space becomes

$$\hat{p}U(p) = 0, \quad \hat{p}V(p) = 0, \tag{2.3}$$

depending on whether we take positive-energy(particle) or negative-energy (anti-particle) solutions of the Dirac equation. Therefore, in the massless case no difference appears in equations for paprticles and anti-particles. Finding one solution is therefore sufficient.

The algebra is simplified if we take γ matrices in Weyl repreentation where

$$\gamma^\mu = \begin{bmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{bmatrix}. \tag{2.4}$$

and $\sigma^\mu = (1, \vec{\sigma})$ and $\bar{\sigma}^\mu = (1, -\vec{\sigma})$. The Pauli matrices are

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{2.5}$$

The matrix γ_5 is taken to be

$$\gamma_5 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{2.6}$$

We can use the matrix γ_5 to construct projection operators on to upper and lower parts of the four-component spinors U and V. The projection operators are

$$\hat{P}_L = \frac{1-\gamma_5}{2}, \quad \hat{P}_R = \frac{1+\gamma_5}{2}. \tag{2.7}$$

Let us write

$$U(p) = \begin{pmatrix} u_L(p) \\ u_R(p) \end{pmatrix}, \tag{2.8}$$

where $u_L(p)$ and $u_R(p)$ are two-component spinors. Since

$$\hat{p} = \begin{bmatrix} 0 & p_\mu\sigma^\mu \\ p_\mu\bar{\sigma}^\mu & 0(p) \end{bmatrix}, \tag{2.9}$$

and $\hat{p}U(p) = 0$, the two-component spinors satisfy the following (Weyl) equations

$$p_\mu\sigma^\mu u_R(p) = 0, \quad p_\mu\bar{\sigma}^\mu u_L(p) = 0. \tag{2.10}$$

Suppose that we have a left handed spinor $u_L(p)$ that satisfies the Weyl equation. We can use it to construct a spinor that satisfies the Weyl equation for the right-handed spinor. Indeed, let us take

$$\tilde{u}_R(p) = i\sigma_2 u_L(p)^*. \tag{2.11}$$

Then,

$$\begin{aligned} 0 &= i\sigma_2 \left[p_\mu \bar{\sigma}^\mu u_L(p)\right]^* = i\sigma_2 p_\mu \bar{\sigma}^{\mu *} u_L^*(p) \\ &= i p_\mu \sigma_2 \bar{\sigma}^\mu (-1)^{\delta_{\mu 2}} u_L(p)^* = p_\mu \sigma^\mu i\sigma_2 u_L(p)^* = p_\mu \sigma^\mu \tilde{u}_R(p), \end{aligned} \tag{2.12}$$

and we conclude that $\tilde{u}_R(p)$ is a right-handed spinor.

To get some physics insight into what left- and right-handiness means, we write Weyl equations in component form

$$u_R(p) = \frac{\vec{\sigma}\vec{p}}{p_0} u_R(p), \quad u_L(p) = -\frac{\vec{\sigma}\vec{p}}{p_0} u_L(p). \tag{2.13}$$

For a massless particle, $|p_0| = |\vec{p}|$. Hence, *for positive* p_0, $u_R(p)$ describes an incoming particle with spin along the direction of its momentum and $u_L(p)$ describes an incoming particle with spin in the direction that is opposite to its momentum. Incoming particles can also be viewed as outgoing anti-particles. We choose $u_L(p)$ to describe outgoing right-handed anti-particles and $u_R(p)$ to describe lef-handed outgoing anti-particles. Outgoing particles are described by Dirac-conjugate spinors, as usual.

We now construct the four-component spinors from the two-component ones

$$U_L(p) = N_p \begin{pmatrix} u_L(p) \\ 0 \end{pmatrix}, \qquad U_R(p) = N_p \begin{pmatrix} 0 \\ u_R(p) \end{pmatrix}, \tag{2.14}$$

where N_p is the normalization constant that we will determine later. We now introduce the following notations for the four-component spinors

$$\begin{aligned} U_L(p) &= p\,], \quad U_R(p) = p\,\rangle, \\ \overline{U_L(p)} &= \langle\, p, \quad U_R(p) = [\,p. \end{aligned} \tag{2.15}$$

Very often, for simplicity of notation, we will replace the momentum label in the spinor by its label , e.g. $p_i] \to i]$, etc. The conjugate spinors are obtained in the standard way. We find

$$\overline{U_L}(p) = N_p \left(0, u_L(p)^\dagger\right), \quad \overline{U_R}(p) = N_p \left(-u_R(p)^\dagger, 0\right). \tag{2.16}$$

It is now easy to derive first results for the spinor products.

$$\overline{U_L}(p) U_L(q) = \langle\, p\; q\,] = 0, \quad \overline{U_R}(p) U_R(q) = [\,p\; q\,\rangle = 0, \tag{2.17}$$

However, $\langle pq \rangle$ and $[pq]$ spinor products do not need to vanish and we next compute them. For this, we will need explicit expressions for left- and right-handed spinors.

Let us choose the left-handed spinor to be

$$U_L(p) = N_p \begin{pmatrix} u_L(p) \\ 0 \end{pmatrix}, \qquad u_L(p) = \begin{pmatrix} a \\ b \end{pmatrix}, \quad \frac{\vec{p}\vec{\sigma}}{p_0} \begin{pmatrix} a \\ b \end{pmatrix} = -\begin{pmatrix} a \\ b \end{pmatrix}. \tag{2.18}$$

where a, b are two complex numbers. We assume that they are normalized in the standard way

$$u_L^+(p) u_L(p) = 1 \quad \rightarrow \quad |a|^2 + |b|^2 = 1. \tag{2.19}$$

To construct the right-handed spinor, we write

$$U_R(p) = N_p \begin{pmatrix} 0 \\ i\sigma_2 u_L(p)^* \end{pmatrix}, \tag{2.20}$$

and since

$$i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{we find} \quad i\sigma_2 u_L(p)^* = \begin{pmatrix} b^* \\ -a^* \end{pmatrix}. \tag{2.21}$$

We now have all the spinors fixed and can compute the spinor products. We will need Dirac-conjugate spinors as well. They are

$$\overline{U}_L(p) = N_p \left(0, 0, a^*, b^*\right), \qquad \overline{U}_R(p) = N_p \left(b, -a, 0, 0\right), \tag{2.22}$$

We now compute the normalization condition using the completeness relation

$$\begin{aligned} \sum_{\lambda \in (L,R)} U_\lambda(p) \otimes \overline{U}_\lambda(p) &= N_p^2 \left[\begin{pmatrix} u_L(p) \\ 0 \end{pmatrix} \otimes \left(0, u_L^+(p)\right) + \begin{pmatrix} 0 \\ u_R(p) \end{pmatrix} \otimes \left(u_R^+(p), 0\right) \right] \\ &= N_p^2 \begin{bmatrix} 0 & u_L \otimes u_L^+(p) \\ u_R \otimes u_R^+(p) & 0 \end{bmatrix}. \end{aligned} \tag{2.23}$$

To proceed further, we need the density matrix of the two-component spinors u_L and u_R. Since those spinors describe normalized quantum mechanical states with $\pm$ spin projections on the axis $\vec{n} = \vec{p}/p_0$, it follows that $u_L(p) \otimes u_L^+(p)$ and $u_R(p) \otimes u_R^+(p)$ are *projection operators* whose explicit expression is known from (spin one-half) quantum mechanis

$$\begin{aligned} u_L(p) \otimes u_L^+(p) &= \frac{1 - \vec{n}\vec{\sigma}}{2} = \frac{p_0 - \vec{p}\vec{\sigma}}{2p_0} = \frac{p_\mu \sigma^\mu}{2p_0} \\ u_R(p) \otimes u_R^+(p) &= \frac{1 + \vec{n}\vec{\sigma}}{2}. = \frac{p_0 + \vec{p}\vec{\sigma}}{2p_0} = \frac{p_\mu \bar{\sigma}^\mu}{2p_0}. \end{aligned} \tag{2.24}$$

Hence, we find

$$\sum_{\lambda \in (L,R)} U_\lambda(p) \otimes \overline{U}_\lambda(p) = \frac{N_p^2}{2p_0} \begin{bmatrix} 0 & p_\mu \sigma^\mu \\ p_\mu \bar{\sigma}^\mu & 0 \end{bmatrix} = \frac{N_p^2}{2p_0}\, p_\mu \gamma^\mu. \tag{2.25}$$

. Since the density matrix for a massless Dirac fermion with momentum p should be equal to $\hat{p}$, we conclude that the normalization constant should be choosen as $N_p = \sqrt{2p_0}$.

To construct spinors explicitly, we need to solve the equation

$$\vec{n}\vec{\sigma}\begin{pmatrix} a \\ b \end{pmatrix} = -\begin{pmatrix} a \\ b \end{pmatrix}, \qquad \vec{n} = \frac{\vec{p}}{|\vec{p}|}. \tag{2.26}$$

which is equivalent to finding the wave function of the spin 1/2 state polarized along $-\vec{n}$ axis. The solutions of this problem are given in any book on quantum mechanics and we just borrow them from there. So, writing the vector $\vec{n}$ as

$$\vec{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \tag{2.27}$$

we find

$$u_L(p) = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -\sin\frac{\theta}{2}e^{-i\phi/2} \\ \cos\frac{\theta}{2}e^{i\phi/2} \end{pmatrix}. \tag{2.28}$$

This solution is, of course, not unique since the phase of $u_L(p)$ is arbitrary. But, once we choose $u_L(p)$ and with the rules for constructing $u_R(p)$ from the complex-conjugate u_L, all phases for spinor products are determined.

We are now in position to compute spinor products and discuss some relations between them. We will start with

$$\begin{aligned} \langle pq\rangle &= \overline{U}_L(p)U_R(q) = N_pN_q\left(0, u_L^+(p)\right)\begin{pmatrix} 0 \\ u_R(q) \end{pmatrix} = N_pN_q u_L^+(p)u_R(q) \\ &= N_pN_q(u_L^T(p))^* u_R(q) = N_pN_q(u_R^T(q)u_L^*(p)) \\ &= N_pN_q(u_R^+(q)u_L(p))^* = \left[\overline{U}_R(q)U_L(p)\right]^* = [qp]^*. \end{aligned} \tag{2.29}$$

Next, let us consider $\langle pq\rangle[qp]$. It reads

$$\langle pq\rangle[qp] = \overline{U}_L(p)U_R(q)\overline{U}_R(q)U_L(p) = \mathrm{Tr}\left[U_L(p)\otimes\overline{U}_L(p)U_R(q)\otimes\overline{U}_R(p)\right]. \tag{2.30}$$

The two matrices that appear in that formula are

$$U_L(p)\otimes\overline{U}_L(p) = \begin{bmatrix} 0 & p_\mu\sigma^\mu \\ 0 & 0 \end{bmatrix}, \quad U_R(q)\otimes\overline{U}_R(Q) = \begin{bmatrix} 0 & 0 \\ q_\mu\bar{\sigma}^\mu & 0 \end{bmatrix}. \tag{2.31}$$

As the result

$$\mathrm{Tr}\left[U_L(p)\otimes\overline{U}_L(p)U_R(q)\otimes\overline{U}_R(p)\right] = 2p_\mu q^\mu, \tag{2.32}$$

where we have used

$$\mathrm{Tr}\left[\sigma_\mu\bar{\sigma}_\nu\right] = 2g_{\mu\nu}. \tag{2.33}$$

Therefore, we find

$$\langle pq\rangle[qp] = 2pq. \tag{2.34}$$

Since $[qp] = \langle pq\rangle^*$, we find

$$|\langle pq\rangle|^2 = |[qp]|^2 = 2pq \quad \text{or} \quad \langle pq\rangle = \sqrt{|2pq|}e^{i\phi_{pq}}, \quad [qp] = \sqrt{|2pq|}e^{-i\phi_{pq}}. \tag{2.35}$$

This formula is usually refered to as the statement that spinor products are square roots of scalar products.

Next property of spinor products that we want to discuss is the anti-symmetry. To see it, consider

$$\langle pq\rangle = \overline{U}_L(p)U_R(q) = N_pN_q\left(0, u_L(p)^+\right)\begin{pmatrix} 0 \\ u_R(q)\end{pmatrix} = N_PN_q\left(0,0,a_p^*,b_p^*\right)\begin{pmatrix} 0 \\ 0 \\ b_q^* \\ -a_q^*\end{pmatrix} \tag{2.36}$$
$$= N_pN_q\left(a_p^*b_q^* - b_p^*a_q^*\right) = (-1)N_pN_q\left(a_q^*b_p^* - b_q^*a_p^*\right) = -\langle qp\rangle.$$

Therefore, we conclude that spinor products satisfy the following equations

$$\langle pq\rangle = -\langle qp\rangle, \quad [pq] = -[qp], \tag{2.37}$$

where the latter relation can be proved in a similar manner.

In practical computations, we often need to compute matrix elements of (products) of Dirac matrices between different spinors. For those cases there are a few identities that can be used. For example, there is a relation

$$\overline{U}_R(p)\gamma^\mu U_R(q) = \overline{U}_L(q)\gamma^\mu U_L(p) \tag{2.38}$$

that we will now prove. Using explicit representation for left- and right-handed spinors, we obtain

$$\begin{aligned}\overline{U}_L(q)\gamma^\mu U_L(p) &= N_pN_q u_L(q)^+\bar{\sigma}^\mu u_L(p),\\ \overline{U}_R(p)\gamma^\mu U_R(q) &= N_pN_q u_R(p)^+\sigma^\mu u_R(q).\end{aligned} \tag{2.39}$$

We will now use a relation between two-component left- and right-handed spinors $u_R(q) = i\sigma_2 u_L^*(q)$, to rewrite $\overline{U}_R(p)\gamma^\mu U_R(q)$ as

$$\overline{U}_R(p)\gamma^\mu U_R(q) = N_pN_q u_R(p)^+\sigma^\mu u_R(q) = N_PN_q u_L^T(p)\ \sigma_2\sigma^\mu\sigma_2\ u_L^*(q). \tag{2.40}$$

Since

$$\sigma_2\sigma^\mu\sigma_2 = \begin{cases} \sigma^\mu & \mu = 0 \\ -\sigma^\mu & \mu = 1,3 \\ \sigma^\mu & \mu = 2\end{cases}, \quad \left(\sigma^\mu\right)^T = (-1)^{\delta_{\mu 2}}\sigma^\mu, \tag{2.41}$$

we can write

$$\sigma_2\sigma^\mu\sigma_2 = \left(\bar{\sigma}^\mu\right)^T. \tag{2.42}$$

Then,

$$N_P N_q u_L^T(p)\ \sigma_2\sigma^\mu\sigma_2\ u_L^*(q) = N_p N_q u_L^T(p)\ (\bar\sigma^\mu)^T u_L^*(q) = N_p N_q u_L(q)^+\bar\sigma^\mu u(p). \qquad (2.43)$$

We conclude

$$[p\gamma^\mu q\rangle = \langle q\gamma^\mu p]. \qquad (2.44)$$

Further relations between different spinor products are obtained using Fiertz identities for σ^μ matrices. We start by writing

$$\sigma^\mu_{ab}\left(\sigma_\mu\right)_{cd} = \sigma^0_{ab}\sigma^0_{cd} - \vec\sigma_{ab}\vec\sigma_{cd} = \delta_{ab}\delta_{cd} - \vec\sigma_{ab}\vec\sigma_{cd}. \qquad (2.45)$$

To simplify the second term, we write

$$\vec\sigma_{ab}\vec\sigma_{cd} = A\delta_{ad}\delta_{cb} - B\delta_{ab}\delta_{cd}. \qquad (2.46)$$

Using the fact that $\delta_{ab}\sigma^i_{ab} = 0$ and that $\vec\sigma_{ab}\vec\sigma_{bc} = 3\delta_{ac}$, we find two equations for A and B

$$0 = A - 2B, \quad 3 = 2A - B, \Rightarrow B = 1, \quad A = 2. \qquad (2.47)$$

For the σ-matrices, the result reads

$$\left(\sigma^\mu\right)_{ab}\left(\sigma_\mu\right)_{cd} = \left(\bar\sigma^\mu\right)_{ab}\left(\bar\sigma_\mu\right)_{cd} = 2\left(\delta_{ab}\delta_{cd} - \delta_{ad}\delta_{bc}\right) = 2\epsilon^{ace}\epsilon_{bde} = 2(i\sigma_2)_{ac}(i\sigma_2)_{bd}. \qquad (2.48)$$

The significance of this equation is that the order of spinor indices, as they appear on the left- and the right-hand sides is different and this can be used for simplifications of spinor products. Indeed, consider

$$\begin{aligned}
&\langle p\gamma^\mu q]\langle k\gamma_\mu l] = \overline{U}_L(p)\gamma^\mu U_L(q)\overline{U}_L(k)\gamma_\mu U_L(l) \\
&= N_p N_q u_L^+(p)\bar\sigma^\mu u_L(q)\ N_k N_l u_L^+(k)\bar\sigma_\mu u_L(l) \\
&= N_p N_q N_k N_l [u_L(p)]^*_a \left(\bar\sigma_\mu\right)_{ab} [u_L(q)]_b [u_L(k)]^*_c \left(\bar\sigma_\mu\right)_{cd} [u_L(l)]_d \\
&= 2N_p N_q N_k N_l\ \ [u_L(p)]^*_a i\sigma^2_{ac}[u_L(k)]^*_c\ \ [u_L(q)]_b i\sigma^2_{bd}[u_L(l)]_d
\end{aligned} \qquad (2.49)$$

To simplify this expression, we use the relation between left- and right-handed spinors

$$i\sigma_2 u_L^* = u_R, \qquad (2.50)$$

to write

$$\begin{aligned}
&N_p N_k\ [u_L(p)]^*_a i\sigma^2_{ac}[u_L(k)]^*_c = N_p N_k [u_L(p)]^*_a [u_R(k)]_a = N_p N_k u_L(p)^+ u_R(k) = \langle pk\rangle, \\
&N_q N_l [u_L(q)]_b i\sigma^2_{bd}[u_L(l)]_d = -u_R^+(q)u_L(l) = [lq].
\end{aligned} \qquad (2.51)$$

Hence, we find an identity

$$\langle p\gamma^\mu q]\langle k\gamma_\mu l] = 2\langle pk\rangle[lq]. \qquad (2.52)$$

Similarly

$$[k\gamma^\mu l\rangle\langle p\gamma_\mu q] = 2[kq]\langle pl\rangle. \tag{2.53}$$

Finally, spinor products obey Schouten identities of the following form

$$\begin{aligned}&\langle ij\rangle\langle kl\rangle + \langle ik\rangle\langle lj\rangle + \langle il\rangle\langle jk\rangle = 0,\\ &[ij][kl] + [ik][lj] + [il][jk] = 0.\end{aligned} \tag{2.54}$$

To prove the Schouten identities note that, due to the antisymmetry of a spinor product $\langle ij\rangle = -\langle ji\rangle$, the left hand sides of the above equations are antisymmetric w.r.t. j, k and l. However, a fully anti-symmetric combination of three two-components object (well, effectively two-component) is zero. Indeed, suppose we write a right-handed spinor (so effectively two component object) as

$$\phi\rangle = j\rangle\langle kl\rangle + k\rangle\langle lj\rangle + l\rangle\langle jk\rangle. \tag{2.55}$$

This spinor can be decomposed as

$$\phi\rangle = a_\xi \xi\rangle + a_\eta \eta\rangle, \tag{2.56}$$

where ξ and η are two basis spinors. It is easy to see that $\langle j\phi\rangle = \langle k\phi\rangle = \langle l\phi\rangle = 0$ (this follows from $\langle jj\rangle = 0$ and the antisymmetry of spinor products). These equations imply

$$a_\xi\langle j\xi\rangle + a_\eta\langle j\eta\rangle = 0, \tag{2.57}$$

and similar equations for other spinors. The solution $a_\xi = a_\eta = 0$ which implies that $\phi\rangle = 0$. The Schouten identities then follows.

Another useful equation is a relative of the Gordon identity, but for massless spinors

$$\begin{aligned}\langle p\gamma^\mu p] &= \overline{U}_L(p)\begin{pmatrix}0 & \sigma^\mu\\ \bar\sigma^\mu & 0\end{pmatrix}U_L(p) = \mathrm{Tr}\left[U_L(p)\otimes\bar U_L(p)\begin{pmatrix}0 & \sigma^\mu\\ \bar\sigma^\mu & 0\end{pmatrix}\right]\\ &= \mathrm{Tr}\left[\begin{pmatrix}0 & p_\nu\sigma^\nu\\ 0 & 0\end{pmatrix}\begin{pmatrix}0 & \sigma^\mu\\ \bar\sigma^\mu & 0\end{pmatrix}\right] = p_\nu \mathrm{Tr}\left(\sigma^\nu\bar\sigma^\mu\right) = 2p^\mu.\end{aligned} \tag{2.58}$$

There is a relation between matrix elements of products of γ-matrices that allows us to reverse the order of elements of the matrix prodcuts. Indeed,

$$\begin{aligned}&\langle p|\gamma_1^\mu....\gamma^{\mu_{2N+1}}|q] = \bar U_L(p)\gamma_1^\mu....\gamma^{\mu_{2Nn1}}U_L(q)\\ &= N_P N_q\left(0, u_L^+(p)\right)\begin{pmatrix}0 & \sigma^{\mu_1}....\sigma^{\mu_{2n+1}}\\ \bar\sigma^{\mu_1}....\bar\sigma^{\mu_{2n+1}} & 0\end{pmatrix}\begin{pmatrix}u_L(q)\\ 0\end{pmatrix}\\ &= N_p N_q u_L^+(p)\bar\sigma^{\mu_1}....\bar\sigma^{\mu_{2n+1}}u_L(q) = N_p N_q u_L^T(q)\bar\sigma^{\mu_{2n+1},T}...\bar\sigma^{\mu_1,T}u_L^*(p)\\ &= N_p N_q\ u_L^T(q)\sigma_2\sigma^{\mu_{2n+1}}...\sigma^{\mu_1}\sigma_2 u_L^*(p) = N_p N_q u_R(q)^+\sigma^{\mu_{2n+1}}...\sigma^{\mu_1}u_R(p),\end{aligned} \tag{2.59}$$

where we used $\sigma_2\bar\sigma^\mu\sigma_2 = (\sigma^\mu)^T$. Hence, we obtain

$$\langle p|\gamma_1^\mu....\gamma^{\mu_{2n+1}}|q] = [q|\gamma^{\mu_{2n+1}}....\gamma^{\mu_1}|p\rangle. \tag{2.60}$$

Similarly

$$[p|\gamma_1^{\mu}....\gamma^{\mu_{2n+1}}q\rangle = \langle q|\gamma^{\mu_{2n+1}}....\gamma^{\mu_1}p]. \tag{2.61}$$

A relation for even number of γ matrices reads

$$\begin{aligned} [p|\gamma^{\mu_1}....\gamma^{\mu_{2n}}|q] &= -[q|\gamma^{\mu_{2n}}...\gamma^{\mu_1}p], \\ \langle p|\gamma^{\mu_1}....\gamma^{\mu_{2n}}|q\rangle &= -\langle q|\gamma^{\mu_{2n}}...\gamma^{\mu_1}p\rangle. \end{aligned} \tag{2.62}$$

Finally, consider $\langle p\gamma^\mu q]\gamma_\mu$. This is a matrix, such that it depends on the spinors constructed out of p and q momenta and that has the property that it must contain $i\rangle[j$ and $i]\langle j$, where $i = p, q$ and $j = p, q$ since γ_μ is helicity conserving. Hence, we can write

$$\langle p\gamma^\mu q]\gamma_\mu = A_1|q]\langle p| + A_2|p\rangle[q| + A_3|p]\langle q| + A_4|q\rangle[p|. \tag{2.63}$$

The coefficients $A_{1..4}$ can be constrained by considering matrix elements of the right-hand side and the left hand side with various spinors. Taking the matrix element with respect to $\langle p$ and $q]$, we find $A_4 = 0$. Taking the matrix element with respect to $[q$ and $p\rangle$, we find $A_3 = 0$. Hence,

$$\langle p\gamma^\mu q]\gamma_\mu = A_1|q]\langle p| + A_2|p\rangle[q|. \tag{2.64}$$

To find A_1 and A_2, we consider matrix elements with repspect $\langle k$ and $|l]$, where k and l are independent momenta. We find

$$A_2\langle kp\rangle[ql] = \langle p\gamma^\mu q]\langle k\gamma_\mu l] = 2\langle pk\rangle[lq], \tag{2.65}$$

which means that $A_2 = 1$. We find A_1 in a similar way. We obtain

$$\langle p\gamma^\mu q]\gamma_\mu = 2\left(|q]\langle p| + |p\rangle[q|\right). \tag{2.66}$$

This concludes our discussion of the spinor algebra.

3 Lecture 2: massless spin-one particles

A polarization vector ϵ of a massless particle with momentum k is a four-vector that satisfies the following conditions

$$0 = \epsilon_\mu k^\mu, \quad \epsilon_\mu r^\mu = 0, \quad \epsilon_\mu \epsilon^{\mu,*} = -1. \tag{3.1}$$

The first condition is "transversality", the second condition is the gauge choice ($r^2 = 0, r_\mu A^\mu = 0$ and the last condition is normalization. The sum over two polarizations reads

$$\sum_{\lambda \in (1,2)} \epsilon_\lambda^\mu \epsilon_\lambda^{*\nu} = -g^{\mu\nu} + \frac{k^\mu r^\nu + k^\nu r^\mu}{k \cdot r}. \tag{3.2}$$

Given the two massless vectors k and r, it is easy to construct a four-vector that satisfies the transversality and the gauge choice conditions. In fact, we can write down two independent vectors

$$\eta_1^\mu = [r\gamma^\mu k\rangle, \quad \eta_2^\mu = \langle r\gamma^\mu k]. \tag{3.3}$$

It is obvious that, thanks to Dirac equation for massless spinors, the transversality and the gauge conditions are fullfilled. To claim that the two η vectors can be choosen as polarization vectors for massless gauge bosons, we will have to normalize them, check their orthogonality and make sure that the sum over polarizations works out correctly.

To do this, we need to find complex conjugate vectors. To this end, consider

$$\begin{aligned} \eta_1^{*\mu} &= ([r\gamma^\mu k\rangle)^* = \left(\overline{U}_R(p)\gamma^\mu U_R(k)\right)^* = \left(u_R^+(r)\sigma^\mu u_R(k) N_r N_k\right)^* \\ &= u_R^T(r)\sigma^{\mu*} u_R^*(k) N_r N_k = u_L(r)^{T*} i\sigma_2^T \sigma^{*\mu}(-i)\sigma_2^* u_L(k) N_r N_k \end{aligned} \tag{3.4}$$

Using $\sigma_2^* = -\sigma_2$, $\sigma_2^T = -\sigma_2$ and $\sigma_2 \sigma^{\mu*}\sigma_2 = \bar{\sigma}^\mu$, we find

$$u_L(r)^{T*} i\sigma_2^T \sigma^{*\mu}(-i)\sigma_2^* u_L(k) N_r N_k = N_r N_k u_L(r)^+ \bar{\sigma}^\mu u_L(k) = \langle r\gamma^\mu k], \tag{3.5}$$

which means

$$\eta_1^{*\mu} = \eta_2^\mu \quad \text{and} \quad \eta_2^{\mu*} = \eta_1^\mu. \tag{3.6}$$

Therefore,

$$\eta_1^* \cdot \eta_2 = \eta_2 \cdot \eta_2 = \langle r\gamma^\mu k]\langle r\gamma^\mu k] \sim \langle rr\rangle[kk] = 0, \tag{3.7}$$

which implies that the two η vectors are indeed orthogonal.

To normalize the η-vectors, we need to compute

$$\begin{aligned} \eta_1 \cdot \eta_1^* &= \eta_1 \cdot \eta_2 = [r\gamma^\mu k\rangle\langle r\gamma_\mu k] = \langle r\left(\quad [r\gamma^\mu k\rangle\gamma_\mu \quad \right)|k] \\ &= 2\langle r\left(\quad |r]\langle k| + |k\rangle[r| \quad \right)|k] = 2\langle rk\rangle[rk] = -\left(\sqrt{2}[rk]\right)\left(\sqrt{2}[rk]\right)^*. \end{aligned} \tag{3.8}$$

Therefore, for normalized vectors we choose (the signs are choosen for convenience)

$$\epsilon_1^\mu = -\frac{[r\gamma^\mu k\rangle}{\sqrt{2}[rk]}, \quad \epsilon_2^\mu = \frac{\langle r\gamma^\mu k]}{\sqrt{2}\langle rk\rangle}, \tag{3.9}$$

To understand what these vectors correspond to, we will evaluate them in a familiar kinematic case. Consider a photon propagating in the $+z$ direction, so that its momentum is $k^\mu = E\,(1,0,0,1)$. The vector r^μ is taken to be $r^\mu = E\,(1,0,0,-1)$. Then

$$
\begin{aligned}
|k] &= U_L(k) = N_k \begin{pmatrix} \phi_- \\ 0 \end{pmatrix}, \quad \phi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \\
k\rangle &= U_R(k) = N_k \begin{pmatrix} 0 \\ i\sigma_2 \phi_-^* \end{pmatrix} = N_k \begin{pmatrix} 0 \\ \phi_+ \end{pmatrix}, \quad \phi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix},
\end{aligned}
\tag{3.10}
$$

Next, we need similar formulas for r-spinors. We find

$$
\begin{aligned}
\langle r| &= \overline{U}_L(r) = N_r \left(0, \phi_+^T\right), \\
[r| &= \overline{U}_R(r) = N_r \left(-\phi_-^T, 0\right).
\end{aligned}
\tag{3.11}
$$

Therefore,

$$
[rk] = -N_r N_k \phi_-^T \phi_- = -N_r N_k, \quad \langle rk\rangle = N_r N_k. \tag{3.12}
$$

Then,

$$
\begin{aligned}
[r|\gamma^\mu k\rangle &= -N_r N_k \phi_-^T \sigma^\mu \phi_+ = -N_r N_k \,(0,1,i,0)\,, \\
\langle r\gamma^\mu k] &= ([r|\gamma^\mu k\rangle)^* = -N_r N_k \,(0,1,-i,0)\,.
\end{aligned}
\tag{3.13}
$$

Hence, we find

$$
\epsilon^\mu_{1,2} = -\frac{1}{\sqrt{2}}\,(0,1,\pm i,0)\,. \tag{3.14}
$$

We will consider calculations of scattering amplitudes assuming that all particles are outgoing. For this, we need complex-conjugates of actual polarization vectors. Hence, we write

$$
\epsilon_2^\mu = \frac{\langle r\gamma^\mu k]}{\sqrt{2}\langle rk\rangle} = -\frac{1}{\sqrt{2}}\,[0,1,-i,0] = -\frac{1}{\sqrt{2}}\,[0,1,i,0]^* = \epsilon_R^{*\mu}, \quad \text{etc.} \tag{3.15}
$$

So, to summarize, we will use polarization vectors for outgoing massless vector bosons

$$
\epsilon_R^{*\mu} = \frac{\langle r\gamma^\mu k]}{\sqrt{2}\langle rk\rangle}, \qquad \epsilon_L^{*\mu} = -\frac{[r\gamma^\mu k\rangle}{\sqrt{2}[rk]}. \tag{3.16}
$$

In what follows, I will skip the complex-conjugate notation for the sake of simlpicity.

The polarization vectors have peculiar transformation properties with respect to changing the reference vector. Indeed, consider a difference of two polarization vectors with different reference vectors

$$
\epsilon_R^\mu(k,r) - \epsilon_R^\mu(k,s) = \frac{1}{\sqrt{2}}\left(\frac{\langle r\gamma^\mu k]}{\langle rk\rangle} - \frac{\langle s\gamma^\mu k]}{\langle sk\rangle}\right) = \frac{(\langle r\gamma^\mu k]\langle sk\rangle - \langle s\gamma^\mu k]\langle rk\rangle)}{\sqrt{2}\langle rk\rangle\langle sk\rangle}. \tag{3.17}
$$

To simplify this further, we use the following equation

$$
\hat{p} = |p]\langle p| + |p\rangle[p|. \tag{3.18}
$$