Path integral formalism

To discuss quantization of non-abelian gauge fields, it is useful to introduce the so-called path integral formalism to describe quantum fields. We will first discuss it in ordinary quantum mechanics.

Consider a quantum mechanical system described by a ~~one~~ Hamiltonian

$$H = \frac{\hat{P}^2}{2m} + V(Q), \text{ where } \hat{P} \ \& \ \hat{Q}$$

are momentum and position operators.

Let's imagine that at a time $t = t_i$ our system is in a state with definite coordinate $x = x_i$; we would like to find the probability amplitude that at a time $t = t_f$ our system is in a state with ~~defi~~ the coordinate $x = x_f$. These states are defined as $\hat{Q}|x_i\rangle = x_i|x_i\rangle$, $\hat{Q}|x_f\rangle = x_f|x_f\rangle$.

The amplitude is computed ~~f~~ as: $(\hbar \to 1)$

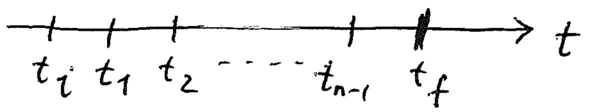
$$i\frac{\partial|\psi\rangle}{\partial t} = \hat{H}|\psi\rangle \Rightarrow |\psi(t)\rangle = e^{-iH(t-t_i)}|\psi_i\rangle, \text{ where}$$

$|\psi_i\rangle$ is the state at $t = t_i$. The amplitude is

$$\mathcal{U}(x_f, x_i; t_f, t_i) = \langle x_f| e^{-iH(t_f - t_i)}|x_i\rangle.$$

Our goal is to write this matrix element in a particular way.

To this end, let us split the time interval into n "small" intervals:



The length of each interval is $\delta t = (t_f - t_i)/n \to 0$, if $n \to \infty$.

$$\mathcal{U}(x_f, x_i; t_f, t_i) = \langle x_f | \underbrace{e^{-iH\delta t} \cdots e^{-iH\delta t} \cdots e^{-iH\delta t}}_{n \text{ times}} | x_i \rangle$$

A trick is to insert complete sets of states into "strategic places": We use

$$\hat{\mathbb{1}} = \int dx_k \; |x_k\rangle\langle x_k| \quad , \quad \langle x|y\rangle = \delta(x-y)$$

and write

$$\boxed{\begin{aligned}\mathcal{U}(x_f, x_i; t_f, t_i) = \int \prod_{k=1}^{n-1} dx_k \; & \langle x_f | e^{-iH\delta t} | x_{n-1}\rangle \\ & \otimes \langle x_{n-1} | e^{-iH\delta t} | x_{n-2}\rangle \cdots \\ & \cdots \otimes \langle x_1 | e^{-iH\delta t} | x_i \rangle .\end{aligned}}$$

We see that the primary object to investigate is $\langle q_a | e^{-iH\delta t} | q_b\rangle$, in the limit $\delta t \to 0$. To this end, we expand the exponential to first order in δt:

$$e^{-iH\delta t} \simeq 1 - iH\delta t = 1 - i\left(\frac{\hat{P}^2}{2m} + V(Q)\right)$$

Next
$$\langle q_a | q_b \rangle = \delta(q_a - q_b)$$
$$\langle q_a | V(Q) | q_b \rangle = \delta(q_a - q_b)\, V\left(\frac{q_a + q_b}{2}\right)$$

and

$$\langle q_a | \frac{\hat{P}^2}{2m} | q_b \rangle = \int \frac{dp_a}{2\pi} \frac{dp_b}{2\pi} \langle q_a p_a\rangle \langle p_a | \frac{\hat{P}^2}{2m} | p_b \rangle \langle p_b q_b \rangle$$

Using $\langle q_a p_a \rangle = e^{i p_a \cdot q_a}$ and $\langle p_b q_b \rangle = \langle q_b p_b \rangle^*$,

and $\langle p_a | \frac{\hat{P}^2}{2m} | p_b \rangle = \underbrace{2\pi\,\delta(p_a - p_b)}_{\text{normalization}} \frac{p_a^2}{2m}$, we find

normalization $\longrightarrow \langle p_a p_b \rangle = 2\pi\,\delta(p_a - p_b)$

$$\langle q_a | \frac{\hat{P}^2}{2m} | q_b \rangle = \int \frac{dp_a}{2\pi} \frac{dp_b}{2\pi}\, e^{i p_a q_a - i p_b \cdot q_b} (2\pi)\,\delta(p_a - p_b) \otimes$$

$$\otimes \frac{p_a^2}{2m} =$$

$$= \int \frac{dp_a}{2\pi}\, e^{i p_a (q_a - q_b)} \frac{p_a^2}{2m}.$$

Now, using $\delta(q_a - q_b) = \int \frac{dp_a}{2\pi} e^{i p_a (q_a - q_b)}$,

we write

$$\langle q_a | e^{-iH\delta t} | q_b \rangle = \int \frac{dp_a}{2\pi}\, e^{i p_a (q_a - q_b) - i \delta_t \left(\frac{p_a^2}{2m} + V\left(\frac{q_a + q_b}{2}\right)\right)}$$

~~Now~~ We put this back into the formula for the transition amplitude and find

$$U(x_f, x_i; t_f, t_i) = \int \prod_{k=1}^{n-1} dx_k \prod_{k=1}^{n} \frac{dp_k}{2\pi} \otimes e^{i \sum_{k=1}^{n} p_k (x_k - x_{k-1})}$$

$$\otimes e^{-i\delta_t \sum_{k=1}^{n} \left[\frac{p_k^2}{2m} + V\left(\frac{x_k + x_{k-1}}{2}\right)\right]},$$

where $x_n = x_f$ and $x_0 = x_i$

We notice that the integral over p is ~~quadratic~~ Gaussian, so it is straightforward to perform it.

We have ($x_k - x_{k-1} = \xi_k$)

$$\int \frac{dp_k}{2\pi} e^{i p_k \cdot \xi_k - i\delta_t p_k^2/2m} = \int \frac{dp_k}{2\pi} e^{-\frac{i\delta_t}{2m}\left(p_k - \frac{m\xi_k}{\delta t}\right)^2} \times e^{i\xi_k^2/(2m\delta t)} =$$

$$= \sqrt{\frac{mi}{2\pi\delta t}}\; e^{\frac{i(x_k - x_{k-1})^2}{2m\delta t}} .$$

Next, we use this result in the expression for $\mathcal{U}(x_f, x_i; t_f, t_i)$ and obtain

$$\mathcal{U}(x_f, x_i; t_f, t_i) = \int \prod_{k=1}^{n-1} dx_k \left(\frac{mi}{2\pi\delta t}\right)^{n/2} \otimes$$

$$\otimes \exp\left[\sum_{k=1}^{n} \frac{i(x_k - x_{k-1})^2}{2m\delta t} - iV\left(\frac{x_k + x_{k-1}}{2}\right)\delta t\right].$$

We can now realize that this integral can be written as

$$\left\| \; \mathcal{U}(x_f, x_i; t_f, t_i) = \int [\mathcal{D}x(t)]\, e^{i\int_{t_i}^{t_f} \mathcal{L}(x,\dot{x})\,d\tau} \right|_{\substack{x_f = x(t_f) \\ x_i = x(t_i)}}$$

with $\mathcal{L}(x,\dot{x}) = \frac{\dot{x}^2}{2m} - V(x)$, being the Lagrangian of the system and $[\mathcal{D}x(t)]$ a symbolic notation for the measure of the integral over trajectories that take a particle from x_i at $t = t_i$ to x_f at $t = t_f$. Note that we have to integrate over all trajectories and not only those that minimize the action $S = i\int_{t_i}^{t_f} \mathcal{L}(x,\dot{x})\,d\tau$.

To a large extent, <u>quantum field theory is quantum mechanics with ∞ degrees of freedom</u>, so the formula that we just derived should apply directly. We then have

$\hat{\Phi}(x)|\phi_a\rangle = \phi_a(\vec{x})|\phi_a\rangle$

$$\langle \phi_b(\vec{x}) | e^{-iHT} | \phi_a(\vec{x}) \rangle = \int \mathcal{D}\varphi \, e^{i \int_0^T d^4x \, \mathcal{L}},$$

where $\mathcal{L} = \frac{1}{2} (\partial_\mu \varphi)^2 - V(\varphi)$ and the fields $\varphi(x,t)$ are supposed to satisfy $\varphi(t=0, \vec{x}) = \varphi_a(\vec{x})$ and $\varphi(t=T, x) = \varphi_b(\vec{x})$.

The main quantities that we need to construct QFT, ~~are~~ ~~the~~ are Green's functions, so we need to understand how to compute them. We will focus on a 2-point correlation function; all other ones are computed by simple generalization of the arguments that we now discuss.

Let us consider an integral

$$I = \int \mathcal{D}\varphi(x) \, \varphi(x_1) \varphi(x_2) \exp\left[i \int_{-T}^{T} d^4x \, \mathcal{L}(\phi) \right]$$

where $\varphi(-T, \vec{x}) = \varphi_a(\vec{x})$ and $\varphi(T, \vec{x}) = \varphi_b(\vec{x})$.

Let us now split the integration measure $\mathcal{D}\varphi(x)$ by separating integrals over fields at $t = x_1^0$ & $t = x_2^0$:

$$\int \mathcal{D}\varphi(x) = \int \mathcal{D}\varphi_1(\vec{x}_1) \, \mathcal{D}\varphi_2(\vec{x}_2) \int\limits_{\substack{\varphi(x_1^0, \vec{x}) = \varphi_1(\vec{x}) \\ \varphi(x_2^0, \vec{x}) = \varphi_2(\vec{x})}} \mathcal{D}\varphi(x)$$

Now: consider $x_1^0 < x_2^0$. Then, the path integral splits into 3 parts:

$$I = \int \mathcal{D}\varphi_1(\vec{x}_1)\, \mathcal{D}\varphi_2(\vec{x}_2)\, \langle \varphi_b | e^{-iH(T - x_2^0)} | \varphi_2 \rangle$$

$$\otimes\ \varphi_2(x_2)\, \langle \varphi_2 | e^{-iH(x_2^0 - x_1^0)} | \varphi_1 \rangle\, \varphi_1(x_1)$$

$$\otimes\ \langle \varphi(x_1) | e^{-iH(x_1^0 + T)} | \varphi_a \rangle$$

To simplify this formula, write

$$\hat{\varphi}(x_1)\, |\varphi_1\rangle = \varphi_1(x_1)\, |\varphi_1\rangle \quad \text{and}$$

$$I = \int \mathcal{D}\varphi_1 \mathcal{D}\varphi_2\, \langle \varphi_b | e^{-iH(T - x_2^0)}\, \hat{\varphi}(x_2)\, |\varphi_2\rangle\langle\varphi_2|$$

$$e^{-iH(x_2^0 - x_1^0)}\, \hat{\varphi}_1\, |\varphi_1\rangle\langle\varphi_1|\, e^{-iH(x_1^0 + T)}\, |\varphi_a\rangle =$$

$$= (\text{completeness}) = \int \langle \varphi_b | e^{-iH(T - x_2^0)}\, \hat{\varphi}(x_2)\, e^{-iH(x_2^0 - x_1^0)}$$

$$\cdot\, \hat{\varphi}_1(x_1)\, e^{-iH(x_1^0 + T)}\, |\varphi_a\rangle \Rightarrow$$

$$I = \langle \varphi_b | e^{-iHT}\, \varphi_H(x_2)\, \varphi_H(x_1)\, e^{iHT} | \varphi_a \rangle \Big|_{x_2^0 > x_1^0},$$

where $\varphi_H(x) = e^{iHx_0}\, \hat{\varphi}(\vec{x})\, e^{-iHx_0}$ <u>is the field operator</u> in the <u>Heisenberg representation.</u>

Of course, if we have chosen $x_2^0 < x_1^0$, we'll have $\varphi_H(x_2)$ & $\varphi_H(x_1)$ to appear in ~~different~~ the opposite order. Hence, we conclude that

$$I = \langle \varphi_b | e^{-iHT}\, \underline{\underline{T}}\{\varphi_H(x_2)\, \varphi_H(x_1)\}\, e^{-iHT} | \varphi_a \rangle$$

This is almost that we want apart from external states, φ_a & φ_b. However, we can use the trick that we used already when discussing the $T \to \infty$ limit of perturbative expansion of Green's functions: take $T \Rightarrow T(1-i\theta)$ & take the limit $T \to \infty$. As the result

$$e^{-iHT}|\varphi_a\rangle = \sum_n e^{-iE_n T}|n\rangle\langle n|\varphi_a\rangle \to$$

$$\to e^{-iE_0 \cdot T}\,|\Omega\rangle\langle\Omega|\varphi_a\rangle,$$ where

$|\Omega\rangle$ is the vacuum state. With this

$$I = e^{-2iE_0 T}\langle\varphi_b|\Omega\rangle\langle\Omega|\varphi_a\rangle\langle\Omega|T\{\varphi_H(x_2)\varphi_H(x_1)|\Omega\rangle$$

$$\Rightarrow \quad \frac{I}{e^{-2iE_0 T}\langle\varphi_b|\Omega\rangle\langle\Omega|\varphi_a\rangle} = \langle\Omega|T\{\varphi_H(x_2)\varphi_H(x_1)|\Omega\rangle$$

Now, use

$$\langle\varphi_b|\Omega\rangle\langle\Omega|\varphi_a\rangle e^{-2iE_0 T} = \langle\varphi_b|e^{-iHT}\,e^{-iHT}|\varphi_a\rangle$$

$$\equiv \int \mathcal{D}\varphi \exp\left[i\int_{-T}^{T} d^4x\, \mathcal{L}\right] \Rightarrow$$

$$\boxed{\langle\Omega|T\{\varphi_H(x_2)\varphi_H(x_1)\}|\Omega\rangle \equiv \lim_{T\to\infty(1-i\varepsilon)} \frac{\int \mathcal{D}\varphi\, \varphi(x_1)\varphi(x_2)\, e^{iS_T}}{\int \mathcal{D}\varphi\, e^{iS_T}}},$$

where $S_T = \int_{-T}^{T} d^4x\, \mathcal{L}$, with

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\varphi)^2 - V(\varphi).$$

It is clear that higher correlation functions require just taking more insertions of the φ-fields under the integration sign in the numerator of the above expr.

We can make this very clear & compact by defining the generating functional for the Green's functions $Z[J]$:

$$Z[J] = \int \mathcal{D}\varphi \, e^{iS[\varphi, J]}, \text{ where}$$

$$S[\varphi, J] = \int d^4x \left(\mathcal{L}(\varphi) + \varphi(x) \cdot J(x) \right)$$

Then

$$\boxed{\langle \Omega | T\{\varphi(x_1) \dots \varphi(x_n)\} | \Omega \rangle = \frac{1}{Z[0]} \frac{\delta}{i\delta J(x_1)} \cdots \frac{\delta}{i\delta J(x_n)} Z[J].}$$

It is possible to continue along these lines and study the case of the free field and then develop perturbation theory ~~etc.~~ for quantum fields, but we will not do that. Instead, we will focus on another aspect of the functional integral which becomes very helpful for quantization of gauge fields.

We will start with QED. The key object is an integral $\int \mathcal{D}A\, e^{iS[A]}$, with

$$\mathcal{D}A = \mathcal{D}A_0 \mathcal{D}A_1 \mathcal{D}A_2 \mathcal{D}A_3 \quad \& \quad S[A] = \int d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}\right)$$

Suppose we want to compute the integral.

To this end, we write

$$-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{2} \partial_\mu A_\nu F^{\mu\nu} \Rightarrow \frac{1}{2} A_\nu \partial_\mu F^{\mu\nu} =$$

$$= \frac{1}{2} A_\nu \left(\Box g^{\nu\mu} - \partial_\nu \partial_\mu \right) A_\nu .$$

If we write $A_\mu(x) = \int \frac{d^4k}{(2\pi)^4} e^{ikx} A_\mu(k)$, we find $\mathcal{D}A(x) \to \mathcal{D}A_\mu(k)$ and

$$S[A] = \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} A_\mu(k) \left[-k^2 g_{\mu\nu} + k_\mu k_\nu \right] A_\nu(k)$$

This is a typical Gaussian integral, of the type

$$\int d^N\vec{x}\; e^{-\vec{x}^T \hat{A} \vec{x}} = \frac{\text{const}}{[\det(A)]^{1/2}} ,$$

(we do not need to know what the constant is)

The matrix $-k^2 g_{\mu\nu} + k_\mu k_\nu$ has an eigenvalue 0, since $\overline{\overline{A}}_\nu(k) \equiv k_\mu f(k^2)$ gives

$$(-k^2 g_{\mu\nu} + k_\mu k_\nu)\, \overline{\overline{A}}_\nu(k) = \phi \Rightarrow [\det(A)] = 0$$

$\Rightarrow$ the integral over $A_\mu(k)$ of $e^{iS(A)}$ will diverge. (i.e. does not exist)

The problem, as was recognized by L. Fadeev & V. Popov, is that when we try do perform an integral over $A_\mu(x)$, we integrate over infinitely many physically-equivalent field configurations that differ from each other by a gauge transform:

$$A_\mu(x) \to A_\mu(x) + \frac{1}{e}\partial_\mu \alpha(x)$$

$\int_{-\infty}^{\infty} dx f(x)$ for $f(x) = f(x+a)$ for $\forall a$

They suggested that the integration over $\mathcal{D}A_\mu$ should be re-arranged in such a way that the integral over non-equivalent physical field configurations is ~~ea~~ identified & extracted.

To this end, let us introduce the following integral: $\left[1 = \int d\vec{a}\, \delta(\vec{g}(\vec{a})) \cdot \det[\partial g_i / \partial a_j] \right]$

$$1 = \int \mathcal{D}\alpha(x)\, \delta\left(G(A_\alpha)\right) \times \det\left[\frac{\delta G[A^\alpha]}{\delta \alpha}\right]$$

Here, $G(A_\alpha) = \partial_\mu A^\mu + \frac{1}{e}\partial^2 \alpha - \omega(x)$.

Since $\det\left[\frac{\delta G(A^\alpha)}{\partial \alpha}\right] = \det\left[\frac{1}{e}\partial^2\right]$ is independent of $\alpha(x)$, for the purpose of the following discussion it is irrelevant. So let's write

$$\int \mathcal{D}A_\mu\, e^{iS[A_\mu]} = \int \mathcal{D}A_\mu\, \mathcal{D}\alpha\, \delta\left(G[A_\alpha]\right) \otimes \det\left[\frac{1}{e}\partial^2\right] \otimes e^{iS[A]} =$$

$$= \det\left[\frac{1}{e}\partial^2\right] \int d\alpha \int \mathcal{D}A_\mu\, \delta\left(G[A_\alpha]\right) \times e^{iS[A]}$$

~~Here~~ We can now perform a gauge transformation $\equiv$ change integration variables $A_\mu \to \bar{A}_\mu = A_\mu \pm \frac{1}{e}\partial_\mu \alpha$.

Then $\mathcal{D}A_\mu = \mathcal{D}\bar{A}_\mu$ $\quad S[A] = S[\bar{A}]$

and $G[A_\alpha] = G_{\alpha=0}[\bar{A}]$. $\Rightarrow$

$$\int \mathcal{D}A_\mu \, e^{iS[A_\mu]} \equiv \det\left[\frac{1}{e}\partial^2\right] \int d\alpha \int \mathcal{D}\bar{A} \, e^{iS[\bar{A}]} \otimes \delta(\partial_\mu \bar{A}^\mu - \omega(x)).$$

Now, we achieved what we wanted; the integral over $\bar{A}$ restricted to configurations that satisfy the gauge-fixing condition & the infinite integral $\int d\alpha$ that describes gauge redundances. We can make the final result somewhat nicer by integrating over all functions ω, centered around $\omega(x)=0$ since the result of functional integration can not depend on $\omega(x)$ [one more gauge transformation will ~~sell~~ remove $\omega(x)$ explicitely]

We write $\quad 1 = N(\xi) \int \mathcal{D}\omega \exp\left[-i\int d^4x \frac{\omega^2(x)}{2\xi}\right]$

&

$$\int \mathcal{D}A_\mu \, e^{iS[A_\mu]} \equiv \det\left[\frac{1}{e}\partial^2\right] N[\xi] \int d\alpha \int \mathcal{D}\bar{A} \, e^{iS[\bar{A}]} \otimes \int \mathcal{D}\omega \exp\left[-i\int d^4x \frac{\omega^2(x)}{2\xi}\right] \otimes \delta(\partial_\mu \bar{A}^\mu - \omega(x)) =$$

$$= \det\left[\frac{1}{e}\partial^2\right] N[\xi] \int d\alpha \int \mathcal{D}\bar{A} \, e^{iS_\xi[\bar{A}]},$$

where $\boxed{S_\xi[\bar{A}] = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu \bar{A}^\mu)^2.}$

Now, if we imagine computation of Green's functions of gauge-invariant operators, all the computations that we just discussed go through and we find

$$\langle \Omega | T\, \mathcal{O}(A) | \Omega \rangle = \frac{\int \mathcal{D}A\, \mathcal{O}(A)\, e^{iS_\xi[A]}}{\int \mathcal{D}A\, e^{iS_\xi[A]}} .$$

For the action $S_\xi[A]$, we can easily find the photon propagator

$$\mathcal{D}^{\mu\nu}(k) = \frac{-i}{k^2 + i\varepsilon}\left(g^{\mu\nu} - (1-\xi)\frac{k^\mu k^\nu}{k^2} \right) \quad \&$$

more general Feynman rules if we couple the theory to e.g. Dirac fermion field. We'll not go into this direction.

Our next ~~discussion~~ step concerns non-abelian gauge fields.

In that case, we'll repeat the same procedure as in QED, except that the integration over gauge configurations becomes tricky. We write ($\Omega = e^{i\alpha^a t^a}$)

$$\boxed{G[A] = \partial_\mu \left[\Omega \hat{A} \Omega^{-1} + \frac{i}{g} \Omega \partial_\mu \Omega^{-1} \right]}.$$

To compute $\frac{\delta G[A]}{\delta \alpha}$, we consider infinitesimal transformation:

$$(A^a)^\alpha_\mu = A^a_\mu + \frac{1}{g}\partial_\mu \alpha^a + f^{abc} A^b_\mu \alpha^c \equiv$$

$$\equiv A^a_\mu + \frac{1}{g}(D_\mu \alpha)^a , \text{ where}$$

D_μ is the covariant derivative in the adjoint representation.

Then $\frac{\delta G[A^a]}{\delta \alpha^b} \equiv \frac{1}{g}(\partial^\mu D_\mu)^{ab} = \frac{1}{g}\partial^\mu [\partial_\mu \delta^{ab} + f^{acb} A^c_\mu]$

The difference between abelian & non-abelian cases is that $\det\left[\frac{\delta G[A^a]}{\delta \alpha^b}\right]$ depends on the field A^a_μ in the non-abelian case and, therefore, remains "inside" the functional integral, so that we have

$$\int \mathcal{D}A^a e^{iS[A]} = \left[\int \mathcal{D}\alpha\right] \int \mathcal{D}A^a e^{iS[A]} \delta(G[A^a]) \times \det\left[\frac{\delta G[A^a]}{\delta \alpha^b}\right]$$

Since $\delta(G[A^a]) \equiv \delta(\partial_\mu A^{a,\mu})$, it can be treated in the same way as in the QED case.

It gives $e^{iS[A]}\delta(G[A^a])$

$$\to e^{iS_\xi[A]}, \text{ where}$$

$$S_\xi[A] = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} - \frac{1}{2\xi}(\partial^\mu A^a_\mu)^2.$$

However, $\det\left[\frac{\delta G[A^a]}{\delta \alpha^b}\right]$ remains.

Faddeev & Popov suggested to represent this determinant as the functional integral over new set of <u>anticommuting spin-zero fields</u> — the Faddeev-Popov ghosts:

$$\det\left[\frac{1}{g}\partial^\mu D_\mu\right] = \int \mathcal{D}c\, \mathcal{D}\bar{c}\, \exp\left[i\int d^4x\, \bar{c}(-\partial^\mu D_\mu)c\right]$$

where

$$\bar{c}(-\partial^\mu D_\mu)c = \bar{c}^a(-\partial^2\delta^{ac} - g\partial^\mu f^{abc} A^b_\mu)c^c$$

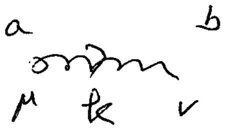
With this trick, the path integral for the non-abelian gauge fields becomes:

$$\int \mathcal{D}A^a\, \mathcal{D}c\, \mathcal{D}\bar{c}\; e^{\,i S_\xi[A] + i S_{ghost}}\,, \quad \text{where}$$

$$i\left[S_\xi + S_{ghosts}\right] = i\int d^4x \left[-\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} - \frac{1}{2\xi}\left(\partial^\mu A^a_\mu\right)^2 + \bar{c}^a\left(-\partial^2 \delta^{ac} - g\,\partial^\mu f^{abc} A^b_\mu\right) c^c \right].$$

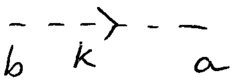
With this, quantization procedure of the Yang-Mills theory becomes straightforward.

Feynman rules (including fermions)



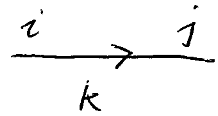
$$: \frac{-i\delta^{ab}}{k^2+i0}\left(g_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right)$$

gluon propagator



$$: \frac{i\delta^{ab}}{k^2}$$

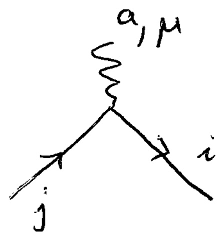
ghost propagator



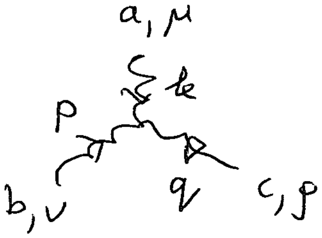
$$= \frac{i\delta_{ij}}{\hat{k} - m}$$

quark prop.

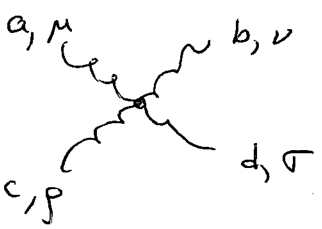
Vertices:



$$= i g \gamma^\mu (t^a)_{ij}$$



$$= g f^{abc}\left[g^{\mu\nu}(k-p)^\rho + g^{\nu\rho}(p-q)^\mu + g^{\mu\rho}(q-k)^\nu \right];$$



$$= -ig^2\left[f^{abe} f^{cde}\left(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}\right) + f^{ace} f^{bde}\left(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\sigma}g^{\nu\rho}\right) + f^{ade} f^{bce}\left(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma}\right)\right];$$

$$\equiv -g f^{abc} p^\mu .$$