Lecture 14 | The Standard model of particle physics: leptons and quarks

We will discuss how to use the $SU(2)_L \otimes U(1)_Y$ theory that we talked about in the previous Lecture to describe interactions of ~~leptons~~ fermions. We will focus on leptons. We know that there are three types of them, known as electron (e), muon (μ) and tau (τ), all of which have identical electric charges and different masses. ~~and~~ There ~~types~~ are also 3 types of neutral, massless (almost massless, to be precise) fermions — neutrinos (ν_e, ν_μ, ν_τ). We know that charged leptons interact with photons — standard electromagnetic interaction — and all leptons participate in weak interactions of two types: charged & neutral. Charged interaction change e to ν_e, μ to ν_μ and τ to ν_τ. These interactions have maximal parity violation — only left-handed fields participate. Neutral interactions do not change particle type and have both parity conserving and parity-violating terms. Now, we can try to connect this with our $SU_L(2) \otimes U_R(1)$ theory.

<u>First</u>; ~~massless neutrinos do not need to~~ we should treat left-handed fermion fields and right-handed fermion fields separately - they have different interactions;

<u>Second</u>: we do not need right-handed neutrinos since neutrinos do not have masses (almost, see below).

<u>Third</u>: charged leptons should interact with both $SU(2)_L$ & $U_R(1)$ bosons. In particular, since the massless field A_μ (the photon) couples to <u>both</u> left and right ~~fields~~ leptons, it is important that ~~charged leptons~~ e, μ, τ are charged under $SU_L(2)$ & $U_R(1)$.

~~We therefore group~~

<u>Fourth</u>: there is no evidence so far for transitions between different types of charged leptons $e \to \mu$, $\mu \to \tau$, $e \to \tau$.

Therefore, we treat leptons as 3 <u>independent</u> <u>families</u>: $[e, \nu_e]$, $[\mu, \nu_\mu]$, $[\tau, \nu_\tau]$.

Within <u>each</u> family, left and right-handed fields are treated separately: $[e, \nu] \to [e_L, e_R, \nu_L, \nu_R]$

We assume that the right-handed neutrinos are absent <u>and</u> that left-handed neutrinos & left-handed electrons are in a single doublet of $SU_L(2)$ ($e_L \to \nu_L$ transitions are known to exist, this is charged vector current). So we write

$$\psi_L \equiv \begin{pmatrix} \nu \\ e \end{pmatrix}_L \quad ; \quad \psi_L = \frac{1+\gamma_5}{2}\psi \; ; \; \text{and} \quad e_R = \frac{1-\gamma_5}{2} e.$$

ψ_L is the doublet of $SU(2)_L$, e_R is a singlet of $SU(2)_L$; ψ_L has a charge Y_1 under $U(1)_R$ and e_R has a charge Y_2 under the $U_1(R)$.

The kinetic term is:

$$\boxed{\mathcal{L} = \overline{\psi}_L \not{\epsilon}\, i \hat{D}_L \psi_L \quad + \bar{e}_R\, i \hat{D}_L e_R}\,, \text{ where}$$

$$\hat{D}_L^{\mu} = \partial^{\mu} - ig\, W_{\mu}^{i} \tau^{i} - ig' B_{\mu} \cdot Y_1/2$$

$$D_R^{\mu} = \partial^{\mu} - ig' B_{\mu}\, Y_2/2$$

We can now calculate ~~the~~ how electroweak bosons couple to fermions. Of particular importance to us are electric charges of fermions - we know that neutrino is neutral and electron charge is e.

Now: photon is a linear comb. of $W_\mu^{(3)}$ & B_μ:

$$W_\mu^{(3)} = \cos\theta Z_\mu + \sin\theta A_\mu ; \qquad \cos\theta = \frac{g}{\sqrt{g^2+g'^2Y^2}} ;$$

$$B_\mu = -\sin\theta Z_\mu + \cos\theta A_\mu ; \qquad \sin\theta = \frac{g'Y}{\sqrt{g^2+g'^2Y^2}})$$

where Y is the hypercharge of the Higgs doublet.

Coupling of neutrino to A_μ is $\left(\tau_3 \begin{pmatrix} \nu \\ e \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\nu \\ -\frac{1}{2}e \end{pmatrix}\right)$

$$-ig\sin\theta A_\mu \cdot \frac{1}{2} - ig'\cos\theta \frac{Y_1}{2} A_\mu =$$

$$= \frac{-igg'A_\mu}{\sqrt{g^2+g'^2Y^2}} \left[\frac{Y}{2} + \frac{Y_1}{2} \right] \Rightarrow$$ we need $Y_1 \equiv -Y$, to have neutrino decouple from the electromagnetic field. At the same time, electron interaction with A_μ is given by two terms: $\frac{-igg'A_\mu}{\sqrt{g^2+g'^2Y^2}} \left[-\frac{Y}{2} + \frac{Y_1}{2} \right]$ for the left-handed field and $-ig' \cos\theta A_\mu \frac{Y_2}{2} =$

$$= \frac{-ig'g\, Y_2/2}{\sqrt{g^2+g'^2Y^2}} A_\mu$$ for the right-handed.

Since the interaction of e_R & e_L with A_μ should be the same, we find

$$\frac{Y_2}{2} = -\frac{Y}{2} + \frac{Y_1}{2} \Rightarrow \boxed{Y_2 = 2Y_1}$$

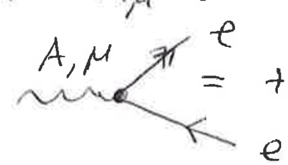
Hence, charges w.r.t. $U_R(1)$ [hypercharges]
are $Y_1 = -Y$ (of ψ_L) & $Y_2 = 2Y_1$ (of e_R)

Next, we will take $Y = 1$, $Y_1 = -1$, $Y_2 = -2$.
In "proper units" [divide above by 2], left-handed fermion have hypercharge $-1/2$, right-handed fermions: -1 & the Higgs doublet $+1/2$.

The interaction with the electrons is then

$$\frac{ig'g}{\sqrt{g^2+g'^2}} A_\mu \bar{e}\gamma^\mu e \quad , \text{ so that } \quad \frac{g'g}{\sqrt{g^2+g'^2}} \equiv |e| \neq e$$

where $|e|$ is the positron electric charge and the electron coupling to A_μ is

$-iQe\,\bar{e}\gamma_\mu e A^\mu$, $Q=-1$,  $= +ieQ\gamma^\mu$

Since e is known, we have $\begin{cases} e = g'\cos\theta \\ e = g\sin\theta \end{cases}$

and, since $\frac{m_W^2}{m_Z^2} \equiv \cos^2\theta_W$, there is a relation between e, m_W & m_Z that fixes g' and g.

Note that a relation exist $\hat{Q} = \hat{T}_3 + \frac{\hat{Y}}{2}$, where $\hat{Y}$ is the $U(1)_R$ charge operator that we derived and $\hat{T}_3$ is weak isospin.

It is easy to find interactions of electrons and neutrinos with ~~other~~ massive gauge bosons:
First, with charge bosons W^+, W^-:

$$-ig[W_\mu^1\tau^1 + W_\mu^2\tau^2]\begin{pmatrix}\nu_L\\ e_L\end{pmatrix} = -\frac{ig}{\sqrt{2}}\left(\tau^+W_\mu^- + \tau^-W_\mu^+\right)\begin{pmatrix}\nu_L\\ e_L\end{pmatrix} =$$

$$\equiv -\frac{ig}{\sqrt{2}}\left(W_\mu^-\begin{pmatrix}e_L\\ 0\end{pmatrix} + W_\mu^+\begin{pmatrix}0\\ \nu_L\end{pmatrix}\right)$$

$$\bar\Psi_L = (\bar\nu_L, \bar e_L) \Rightarrow$$

$$\boxed{\mathcal{L}_{(w^+,w^-,\nu,e)} \equiv -\frac{ig}{\sqrt{2}}\left(\bar\nu_L\gamma^\mu e_L W_\mu^- + \bar e_L\gamma^\mu\nu_L W^{+\mu}\right)}.$$

The strength of the interactions with neutral gauge bosons is derived in a similar way
The relevant part of the covariant derivative is

$$\bar\Psi_L\gamma^\mu\left[-igW_\mu^{(3)}\tau_3 + ig'B_\mu/2\right]\begin{pmatrix}\nu_L\\ e_L\end{pmatrix} + ig'\bar e_R\gamma^\mu B_\mu e_R \Longrightarrow$$

$$\Rightarrow \bar\Psi_L\gamma^\mu\left[-ig\cos\theta\,\tau_3 - \frac{ig'}{2}\sin\theta\right]\begin{pmatrix}\nu_L\\ e_L\end{pmatrix}Z_\mu \mp ig'\bar e_R\gamma^\mu e_R\sin\theta\, Z_\mu$$

$$= \bar\nu_L\gamma^\mu\nu_L\left(\frac{-i}{2}\right)(g\cos\theta + g'\sin\theta)$$

$$+ \bar e_L\gamma^\mu e_L\left(\frac{-i}{2}\right)(-g\cos\theta + g'\sin\theta) - ig'\sin\theta\,\bar e_R\gamma^\mu e_R Z_\mu =$$

$$= \bar\nu_L\gamma^\mu\nu_L\left(\frac{-ie}{2\cos\theta\sin\theta}\right) + \bar e\gamma^\mu(a + b\gamma_5)e, \text{ where}$$

$$\boxed{a = \frac{ie\,(1-4\sin^2\theta)}{4\cos\theta\sin\theta}, \quad b = \frac{+ie}{4\cos\theta\sin\theta}}.$$

So, as expected, we have short-range neutral vector currents in the theory; coefficient "a" for ~~leptons~~ electrons is somewhat peculiar, since it is numerically small.

Next topic that we want to discuss is generation of masses. We know that the mass term in a Dirac Lagrangian mixes left- and right-fields: $m\bar{\Psi}\Psi \equiv m(\bar{\Psi}_L\Psi_R + \bar{\Psi}_R\Psi_L)$. We need some type of mass terms in the Standard model since we know that most of the leptons are massive (some neutrino types may be exceptions)

We know that we can not write the mass term $m\bar{\Psi}_L e_R$ because Ψ_L & e_R have different $SU(2)\otimes U(1)$ charges, so that the mass term $m\bar{\Psi}_L e_R$ is not gauge-invariant. To do things properly, we need to make use of the Higgs mechanism. The idea is to write the interaction between Ψ_L, e_R and the Higgs doublet ϕ which is $SU_L(2) \otimes U_R(1)$ invariant and that may become a mass term after spontaneous symmetry breaking.

Suppose we write the <u>Yukawa term</u>

$$\mathcal{L}_Y = g\left[\bar{\Psi}_L \cdot \phi\, e_R + h.c.\right]$$

~~How~~ Since ψ_L & ϕ are both $SU(2)_L$ doublets, $(\bar{\Psi}_L \phi)$ is $SU(2)_L$ invariant. To check invariance w.r.t. $U(1)_R$, we recall that the charges of ψ_L, ϕ & e_R w.r.t. $U(1)_R$ are $Y_\phi = 1$, $Y_L = -1$ and $Y_R = -2$. Hence, $\bar{\Psi}_L \phi\, e_R$ and h.c. are invariant w.r.t. $U(1)_R$. $\Rightarrow$ $\mathcal{L}_Y$ is a valid ~~$\mathcal{L}_{Y}$~~ term in gauge-invariant. Let us check what happens after the symmetry breaking:

$$\phi = \begin{pmatrix} 0 \\ \frac{v+h}{\sqrt{2}} \end{pmatrix} \Rightarrow$$

$$\mathcal{L}_Y = g\left[(\bar{\nu}_L\ \bar{e}_L)\begin{pmatrix} 0 \\ \frac{v+h}{\sqrt{2}} \end{pmatrix} e_R + h.c.\right] = g\left(\bar{e}_L \cdot e_R + h.c.\right)\frac{v+h}{\sqrt{2}}$$

$$\equiv \frac{gv}{\sqrt{2}}\left(\bar{e}_L e_R + \bar{e}_R e_L\right) + \frac{gh}{\sqrt{2}}\left(\bar{e}_L \cdot e_R + \bar{e}_R e_L\right) =$$

$$\equiv m_e\left(\bar{e}_L e_R + \bar{e}_R e_L\right) + \frac{m_e}{v} h \left(\bar{e}_L e_R + \bar{e}_R e_L\right) \Rightarrow$$

$$\boxed{\mathcal{L}_Y \equiv m_e \bar{e} \cdot e + \frac{m_e h \bar{e} e}{v}}$$

We see that the Yukawa interaction generated the mass term for leptons <u>and</u> the term for Higgs-electron interaction. The latter is proportional to the electron mass $H \langle^{e}_{e} \sim \frac{m_e}{v}$

From experiments on neutrino oscillations, we know that neutrinos have masses; therefore our construction is not complete. Let's discuss how neutrino masses can be introduced into the theory. First, note that ~~the~~, neutrino masses seem to require right-handed neutrinos, ~~and we~~ second, right-handed neutrinos seem not to participate in gauge interactions. Hence, let's declare the right-handed neutrino ν_R - a "sterile" particle, that is singlet w.r.t. both $SU(2)_L$ & $U(1)_R$. We need to write a mass term for neutrinos. But we have 2 problems: First $\bar{\Psi}_L \phi$ projects on e_L, not ν_L after electroweak symmetry breaking and, second, it is not $U(1)$ invariant.

A useful object for us is ϕ^*, since, under $U(1)_R$ its ~~transform~~ charge is -1, so that $\bar{\Psi}_L \phi^* \nu_R$ is invariant under $U(1)_R$ if ν_R is a singlet. However $\bar{\Psi}_L \phi^*$ is not invariant under $SU(2)$. It is easy to fix this, **however**. Let's define ~~a spinor~~ as $SU(2)$ spinor $\phi_c \equiv i\tau_2 \phi^*$. It transforms as a doublet under $SU(2)_L$ and ~~[illegible]~~ it has the -1 $U_R(1)$ charge.

Therefore, we can write the gauge-invariant term

$\mathcal{L}_\nu = g_\nu [\bar{\Psi}_L \phi_c \nu_R + h.c.]$ for neutrino masses.

The masses are given by $m_\nu = g_\nu v/\sqrt{2}$ and the Higgs boson interaction with neutrinos are proportional to neutrino masses. This is the only interaction -except for gravity- that involves right-handed neutrinos. The mechanism we described to generate masses for neutrinos is also used for quarks; it is simple and does not involve -directly- any physics beyond the Standard Model

Let us now describe a different way of thinking about neutrino masses which is based on the idea that neutrino is a <u>Majorana fermion.</u>

Let us recall what a Majorana fermion is

Let's start with a 4-comp. spinor, assuming that Dirac matrices are in Weyl representation

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \qquad \left[\gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \right]$$

The two two-component spinors transform differently under Lorentz transformations

At the same time $i\sigma_2 \psi_L^*$ transforms as ψ_R

and $i\sigma_2 \psi_R^*$ transforms as ψ_L.

So, in principle, if we have a 4-component left-handed spinor $\nu_L \equiv \begin{pmatrix} \chi_L \\ 0 \end{pmatrix}$, we can make a 4 component spinor with both left & right components out of it by writing $\nu_L \to \begin{pmatrix} \chi_L \\ i\sigma_2 \chi_L^* \end{pmatrix}$. This is a valid spinor w.r.t. Lorentz transformations

To "make" the right-handed spinor out of ν_L we can make use of charge conjugation:

$$\psi \mapsto \psi^c \equiv -i(\bar{\psi}\gamma_0\gamma_2)^T \quad \text{For } \psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix},$$

$\psi^c = \begin{bmatrix} -i\sigma_2 \psi_R^* \\ i\sigma_2 \psi_L^* \end{bmatrix}$. Hence, taking $\psi_R = 0$, we find $\nu_L^c = \begin{bmatrix} 0 \\ i\sigma_2 \psi_L^* \end{bmatrix} \Rightarrow \nu_M \equiv \nu_L + \nu_L^c \equiv \begin{bmatrix} \psi_L \\ i\sigma_2 \psi_L^* \end{bmatrix}$

Since $\hat{C}^2 = 1$, it follows that $\nu_M^c \equiv \nu_M$, i.e the Majorana fermion is invariant under charge conjugation. Since we now have a fermion with both left- & "right-" components, we can write a mass term

$$\Delta \mathcal{L}_m = -\frac{1}{2} m_L \left(\overline{\nu_L}\, \nu_L^c + h.c. \right).$$

This is the so-called Majorana mass term. Note that since $\nu_L^c \sim \psi_L^*$, the transformation properties of $\overline{\nu_L}\, \nu_L^c$ under gauge transformation are non-trivial. This is the reason why Majorana mass-terms are not used for "charged" fermions.

There are several ways - how this Majorana mass term - can be introduced

~~toe~~ to help with neutrino masses.
We will talk about one. ~~Suppose the~~ ~~again~~ we would like to treat ν_L as a Majorana fermion, but its mass term $m_L \overline{\nu_L} \nu_L^c$ is forbidden by gauge invariance since ν_L is part of the $SU_L(2)$ doublet. However, a right-handed sterile Majorana neutrino is possible.
At the same time, we can write a cross-term $m_D \overline{\nu_L} \nu_R$ that couples left-handed and right-handed neutrinos.
We imagine that this term ~~again~~ is generated dynamically due to Higgs mechanism.

The most general mass term ~~that is~~ consistent with $SU_L(2) \otimes U_R(1)$ gauge invariance is

$$\mathcal{L}_m = -\frac{1}{2} m_R \overline{\nu_R^c} \nu_R - m_D \overline{\nu_L} \nu_R + h.c.$$

~~with $SU_L(2) \otimes U(1)_R$ gauge invariance is the~~

We can write ~~this~~ it in a matrix form by introducing a doublet $\begin{pmatrix} \nu_L^c \\ \nu_R \end{pmatrix} = N_R$

$$N_R^c = \begin{pmatrix} \nu_L \\ \nu_R^c \end{pmatrix} \qquad \overline{N_R^c} = \left(\overline{\nu_L}, \overline{\nu_R^c} \right),$$

so that

$$\mathcal{L}_m = -\frac{1}{2} \overline{N_R^c} \begin{pmatrix} 0 & m_D \\ m_D & m_R \end{pmatrix} N_R + h.c.$$

Note that the cross-terms are peculiar:

$$\overline{N_R^c} \begin{pmatrix} 0 & m_D \\ m_D & m_R \end{pmatrix} N_R \longrightarrow m_D \left[\overline{\nu_R^c} \nu_L^c + \overline{\nu_L} \nu_R \right] \Rightarrow$$

We can prove that $m_D \overline{\nu_R{}^c} \nu_L{}^c \equiv m_D \overline{\nu}_L \nu_R$
(this is straightforward, just need to remember that fermion fields anticommute)

Now, let's write $\hat{O} \equiv \begin{pmatrix} \cos\theta & +\sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

$$\begin{pmatrix} 0 & m_D \\ m_D & m_R \end{pmatrix} \equiv \hat{O} \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \hat{O}^T .$$

It is easy to ~~fing~~ find $\tan(2\theta) \equiv \dfrac{2m_D}{m_R}$

and $m_{1,2} = \dfrac{m_R \pm \sqrt{m_R^2 + 4m_D^2}}{2}$.

Finaly, we can redefine our fields to absorb the rotation matrix O and arrive at the mass eigenstates:

We write $\hat{O}^T N_R \equiv \tilde{N}_R \Rightarrow \overline{N_R{}^c}\, O \equiv \tilde{N}_R{}^c$

$$\overline{N_R}^c \begin{pmatrix} 0 & m_D \\ m_D & m_R \end{pmatrix} N_R \equiv \tilde{N}_R \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \tilde{N}_R .$$

Note that in the limit $m_R \ggg m_D$ (heavy right-handed neutrinos), the mass eigenstates have a hierarchy $m_2 \sim m_R$, $m_1 \sim \dfrac{m_D^2}{m_R} \ll m_D$.

Hence if m_D is assumed to be a typical scale for lepton masses, the suppression factor (m_D/m_R) allows to understand why neutrino masses are so much smaller than masses of leptons.