Lecture 11 Spontaneous symmetry breaking, the linear σ-model, Goldstone theorem

As we have discussed several times, symmetries play important role in quantum field theory. Symmetries can be either exact or approximate. In the latter case, the symmetries are broken by a small amount and one can develop some type of perturbation theory around symmetric solution. There are also cases when symmetries are broken by a "large" amount but there are manifestations of the original symmetries in the spectrum of the theory.

Symmetries can be broken in two distinct ways: explicitely or spontaneously. Explicit symmetry breaking occurs if the Lagrangian of the theory contains symmetric and not symmetric pieces at the same time.

Spontaneous symmetry breaking occurs if the Lagrangian of the theory is symmetric but the ground state of the theory — the vacuum — is not.

To understand spontaneous symmetry breaking consider familiar scalar field theory, invariant under $\varphi \to -\varphi$:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)^2 - \frac{1}{2} m^2 \varphi^2 - \frac{\lambda}{4!}\varphi^4 \quad (*)$$

Usually, $m^2 > 0$, m is associated with the mass of the "φ"-particle; the potential is $V(\varphi) = \frac{\lambda}{4!}\varphi^4$ and the minimal energy corresponds to $\varphi(x) = 0$ point (i.e. no field anywhere in space). This is the standard situation we dealt with throughout this class.

We know that we can, in this case, quantize the theory and develop perturbative expansion in λ.

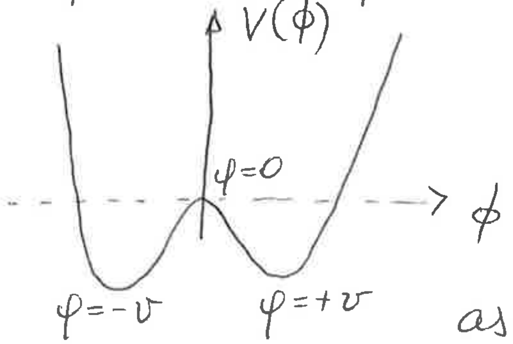
What happens if we change $m^2 \to -\mu^2$, with $\mu^2 > 0$? Then, $\boxed{\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2}\mu^2\phi^2 - \frac{\lambda}{4!}\phi^4}$

and there is now a problem – we can not treat $\mu^2\phi^2/2$ as part of the free Lagrangian because if we do, we will get tachyonic ϕ-particles and other non-sense.

So let's pretend that $\frac{1}{2}\mu^2\phi^2$ must be treated as part of $V(\phi)$, so that

$$\boxed{V(\phi) = -\frac{1}{2}\mu^2\phi^2 + \frac{\lambda}{4!}\phi^4}$$

It is instructive to plot $V(\phi)$ as a function of the field ϕ: We get the picture to the left which suggests that $\phi = 0$ is <u>not</u> the minimum of $V(\phi)$ and that, in fact, there are two minima that we denote as $\varphi = \pm v$. We find explicit expression for v in terms of parameters of the potential



$$\frac{\partial V}{\partial \phi} = -\mu^2 \phi + \frac{\lambda}{3!}\phi^3 = 0 \Rightarrow$$

$$\phi_0^2 = \frac{6\mu^2}{\lambda} \Rightarrow \quad \underline{v = \pm\sqrt{\frac{6\mu^2}{\lambda}}}$$

We would like to construct a theory and define an interaction potential by working close to a minimum of $V(\varphi)$. <u>We choose one such minimum</u>, say v, and write

$\phi(x) \equiv v + \sigma(x)$, where $\sigma(x)$ is the new field.

Then $\mathcal{L} = \frac{1}{2}(\partial_\mu \sigma)^2 + \frac{1}{2}\mu^2(v^2 + 2v\sigma + \sigma^2)$

$$- \frac{\lambda}{4!}\left(v^4 + \sigma^4 + 4v^3\sigma + 4v\sigma^3 + 6v^2\sigma^2\right) \Rightarrow$$

$$\boxed{\begin{aligned}\mathcal{L} = \frac{1}{2}(\partial_\mu\sigma)^2 &+ \left(\frac{1}{2}\mu^2 v^2 - \frac{\lambda}{4!}v^4\right) + \left(\mu^2 v - \frac{\lambda}{4!}4v^3\right)\sigma \\ &+ \left(\frac{1}{2}\mu^2 - \frac{\lambda}{4!}6v^2\right)\sigma^2 - \frac{\lambda}{4!}4v\sigma^3 - \frac{\lambda}{4!}\sigma^4\end{aligned}}$$

As you see there are many terms produced after field redefinition and there are some of them that need to be watched closely.

The σ- independent term defines the vacuum energy density:

$$-E_{vac} \equiv \frac{1}{2}\mu^2 v^2 - \frac{\lambda}{4!} v^4 = \frac{1}{2}\mu^2 \frac{6}{\lambda}\mu^2 - \frac{\lambda}{24}\cdot\frac{36\mu^4}{\lambda^2} = \frac{3}{2}\frac{\mu^4}{\lambda} \Rightarrow$$

$$\boxed{E_{vac} = -\frac{3}{2}\frac{\mu^4}{\lambda}}$$

The linear term in σ must vanish, since $\sigma = 0$ should correspond to an extremum: Indeed

$$\mu^2 v - \frac{\lambda}{4!} 4v^3 = \left(\mu^2 - \frac{\lambda}{6}v^2\right) v \equiv \emptyset.$$

The quadratic term in σ should tell us what the true mass of the correspondig φ- or σ- particle is:

$$\left(\frac{1}{2}\mu^2 - \frac{\lambda}{4!} 6v^2\right)\sigma^2 = \left(\frac{1}{2}\mu^2 - \frac{\lambda}{4}\frac{6}{\lambda}\mu^2\right)\sigma^2 = \frac{-2\mu^2\sigma^2}{2} \Rightarrow$$

$\boxed{m_\sigma^2 = 2\mu^2}$ is the σ- mass squared.

The $O(\sigma^3)$ and $O(\sigma^4)$ terms give the interaction

Hence, we write the Lagrangian in a simplified form:

$$\boxed{\mathcal{L}_\sigma = \frac{1}{2}(\partial_\mu \sigma)^2 - \frac{1}{2}(2\mu^2)\sigma^2 - \sqrt{\frac{\lambda}{6}}\,\mu\sigma^3 - \frac{\lambda}{4!}\sigma^4}$$

It clearly describes particle of mass^2 $2\mu^2$ ~~sub~~ subject to non-linear interactions.

The $\varphi \to -\varphi$ symmetry (Z_2) of the original Lagrangian is gone and the only reflection of this symmetry is the relation between mass of σ, vacuum expectation value v and couplings of σ^3 and σ^4 terms in $\mathcal{L}_\sigma$.

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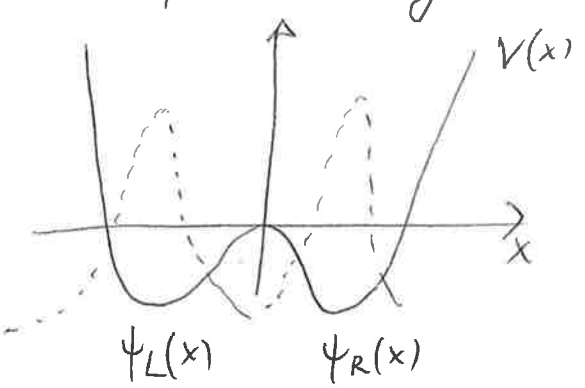
Let us comment about our choice of a particular minimum $\varphi = +v$ (or vacuum state).

As we saw, there are two states with minimal energy $\varphi = \pm v$ but we choose one of them breaking the symmetry spontaneously.

This is in direct contradiction with what we would have done in quantum mechanics. In that case $\varphi \to x$, $\partial_\mu \varphi \to \frac{dx}{dt}$ and

$V(x) = -\frac{\mu^2 x^2}{2} + \frac{\lambda}{4!} x^4$ is a well-known example of a two-well potential. The true ground state in this case is a linear combination of wave functions $\psi(x) = \frac{1}{\sqrt{2}} \left(\psi_L(x) + \psi_R(x) \right)$ where $\psi_L(x)$ and $\psi_R(x)$ are shown pictorially:



So the ground state in quantum mechanics is obviously symmetric. In quantum field theory we choose either $\psi_L(x)$ or $\psi_R(x)$ as our ground state, breaking the $x \to -x$ symmetry.

The reason we must take $\psi(x) = \frac{1}{\sqrt{2}}\left(\psi_L(x) + \psi_R(x)\right)$ as the ground state in quantum mechanics is tunneling, so if we take ~~$\varphi_0(x) = v$ or $\psi_R(x) = v$ as the~~ $\varphi(x) = v$ or $\varphi(x) = -v$ as the ground state in QFT, we claim, in essense that the tunneling rate is zero. The reason we do this in QFT and not in quantum mechanics, is the fact that we consider the theory in the infinite volume. The tunneling rate is proportional to the exponential of the height of the barrier and the volume $\exp(-V_{eff} \cdot L^3)$ and vanishes as $L \to \infty$. In other words, for tunneling $\varphi(x) = v$ must become $\varphi(x) = -v$ for $\forall$ points x and there are infinitely many of them, so to speak. We conclude that in QFT spontaneous symmetry breaking happens, in contrast to Quantum Mechanics.

We will now try to consider a theory that is a little bit more complex than what we just discussed.

We consider a set of ~~scalar~~ fields $\vec{\Phi} = (\phi_1, \dots \phi_N)$ and write Lagrangian that is $O(N)$-symmetric

$$\boxed{\mathcal{L} = \frac{1}{2}(\partial_\mu \vec{\Phi})^2 + \frac{1}{2}\mu^2 \vec{\Phi}^2 - \frac{\lambda}{4}\vec{\Phi}^2 \cdot \vec{\Phi}^2} \qquad (*)$$

The Lagrangian is invariant under the transformation $\phi_i \to R_{ij}\phi_j$ with $R \in O(N)$ since $R^T \cdot R \equiv 1$.

We have to repeat the calculation that we did, since we can not treat μ^2 as the mass of the fields $\phi_{i \in 1..N}$ anymore. The potential

$V(\vec{\Phi}) = -\frac{1}{2}\mu^2 \vec{\Phi}^2 + \frac{\lambda}{4}(\vec{\Phi}^2)^2$ is minimal is

$|\vec{\Phi}| = \Phi_{vac}$, with $\phi_{vac} = \frac{\mu^2}{\lambda} = v^2$ but the direction of the vector $\vec{\phi}$ at that value is arbitrary. We choose $\vec{\Phi}_{vac} = (0, 0, \dots, 0, v)$, i.e. the vacuum field is along the "N-th" direction in the field space. To parametrize deviations from the field Φ_{vac}, we write

$\vec{\Phi} = (\pi_1, \dots \pi_{N-1}, v + \sigma) \equiv \vec{\pi} + (v+\sigma)\vec{e}_N$, with the constraint $\vec{\pi} \cdot \vec{e}_N \equiv 0$. So, we write the Lagrangian through fields $\vec{\pi}$ and σ and obtain

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \vec{\pi})^2 + \frac{1}{2}(\partial_\mu \sigma)^2 - \frac{\lambda}{4}\left[\left(\vec{\pi}^2 + (v+\sigma)^2\right)\left(\vec{\pi}^2 + (v+\sigma)^2\right)\right]$$

$$+ \frac{\mu^2}{2}\left(\vec{\pi}^2 + (v+\sigma)^2\right).$$

Let us again trace a few important terms

The vacuum energy term we just ignore, as before.

There is no linear term in $\vec{\pi}$ and the linear-σ term cancels out, as it should ($\sigma = 0$ is an extremum):

$$-\frac{\lambda}{4} 4 v^3 \sigma + \frac{\mu^2}{2} 2 v \sigma = -\mu^2 v \sigma + \mu^2 v \sigma = \phi.$$

Terms quadratic in σ are as before

$$\frac{\mu^2}{2}\sigma^2 - \frac{\lambda}{4}\left(2\sigma^2 v^2 + 4 v^2 \sigma^2\right) = \left(\frac{\mu^2}{2} - \frac{6v^2\lambda}{4}\right)\sigma^2 =$$
$$= -2\mu^2\sigma^2/2$$

Terms quadratic in $\vec{\pi}$ are

$$-\frac{\lambda}{4}\left(2\vec{\pi}^2 v^2\right) + \frac{\mu^2}{2}\vec{\pi}^2 = \frac{\vec{\pi}^2}{2}\left(\mu^2 - \lambda v^2\right) \equiv \phi,$$ so

there is no mass term for the field $\vec{\pi}$!

Continuing with other terms, we find the result to be:

$$\boxed{\begin{aligned}\mathcal{L} = {} & \frac{1}{2}\left(\partial_\mu \vec{\pi}\right)^2 + \frac{1}{2}\left(\partial_\mu \sigma\right)^2 - \frac{2\mu^2\sigma^2}{2} - \sqrt{\lambda}\,\mu\,\sigma^3 \\ & - \sqrt{\lambda}\,\mu\left(\vec{\pi}^2\right)\sigma - \frac{\lambda}{4}\left(\sigma^4 + \left(\vec{\pi}^2\right)^2\right) - \frac{\lambda}{2}\vec{\pi}^2\sigma^2\end{aligned}} \quad (**)$$

So the story is similar to φ^4 case discussed at the beginning of the lecture but now, in addition to a massive field σ we have $N-1$ massless fields $(\pi_1, \dots \pi_{N-1})$. We also clearly have the symmetry to change $\vec{\pi}$-fields into each other by ~~rotating~~

applying rotations in π-space: $\hat{R}_a \in O(N-1)$

$\pi_i \to R_{ij}\pi_j$, $\mathcal{L} \to \mathcal{L}$. As we will see, the appearance of massless particles is not accidental; it is, in fact a general feature of spontaneously broken theories.

As the first step towards understanding this phenomenon, we compare the symmetry of the original Lagrangian Eq.(*) pg.7 and the new Lagrangian Eq.(**) pg.8.

The first one has the $O(N)$ symmetry group; $O(N)$ is described by $\frac{N(N-1)}{2}$ generators that is the number of "independent symmetries".

The secon one has the $O(N-1)$ symmetry group, so the number of independent symmetries is $\frac{(N-1)(N-2)}{2}$. Hence, in transition from (*) to (**) we lost

$$\frac{N(N-1)}{2} - \frac{(N-1)(N-2)}{2} = (N-1)\left(\frac{N}{2} - N + 1\right) = \underline{\underline{N-1}}$$

symmetries of the theory were lost.

Note that this number is exactly the number of massless π-fields that we found by an explicit calculation.

We will now show that this is a general result known as Goldstone theorem.

The theorem states that - for any spontaneously broken continuous symmetry - there is a massless spin-zero boson in the spectrum.

To prove the theorem, we consider generalization -10-
of our explicit computation. Consider the
Lagrange function $\mathcal{L} = \frac{1}{2}(\partial_\mu \vec{\Phi})^2 - V(\vec{\Phi})$.
Let $\vec{\phi}_0^{(0)}$ be the field that minimizes $V(\vec{\Phi})$:

$$\left.\frac{\partial V}{\partial \vec{\Phi}}\right|_{\vec{\Phi}=\vec{\Phi}^{(0)}} \equiv \phi$$

We want to understand the mass spectrum. To this end, we expand around $\vec{\Phi} = \vec{\Phi}^{(0)}$. We write $\vec{\Phi} = \vec{\Phi}_0 + \delta\vec{\Phi}$ and Taylor expand to second order in $\delta\vec{\phi}$. We find

$$V(\vec{\Phi}) \equiv V(\vec{\Phi}^{(0)}) + \frac{\partial V}{\partial \vec{\Phi}_i^{(0)}}\delta\phi^i + \frac{1}{2}\frac{\partial^2 V}{\partial\vec{\Phi}_i^{(0)}\partial\vec{\Phi}_j^{(0)}}\delta\phi^i\delta\phi^j + \ldots$$

Since $\left.\frac{\partial V}{\partial \vec{\Phi}}\right|_{\vec{\Phi}=\vec{\Phi}_0} = \phi$, since $\vec{\Phi}^{(0)}$ is a minimum,
we have $V(\vec{\Phi}) \equiv V(\vec{\Phi}^0) + m_{ij}^2\,\delta\phi^{(i)}\delta\phi^{(j)} + \ldots$

The "mass matrix" is

$$m_{ij}^2 = \frac{1}{2}\left.\frac{\partial^2 V}{\partial\phi_i\,\partial\phi_j}\right|_{\vec{\phi}=\vec{\phi}_0}$$

Since $\vec{\phi}^{(0)}$ is a true minimum, we know that m_{ij}^2 ~~can~~ ~~have~~ ~~posit~~ must have positive or zero eigenvalues which will correspond to masses of excitations of the $\vec{\phi}$-fields.
We need to find a way to establish the number of zero eigenvalues in m_{ij}^2, for general $V(\vec{\phi})$.

To understand this, recall that $V(\bar{\Phi})$ was invariant under symmetry transformations of the full symmetry group, so that if $\phi_i \to \phi_i + \Delta_{ij}\phi_j$, $V(\bar{\Phi})$ doesn't change $\Rightarrow V(\bar{\Phi}+\Delta\bar{\Phi}) \equiv V(\bar{\Phi})$

This equation should hold ~~for~~ to all orders in $\Delta\bar{\Phi}$; expanding to first order, we obtain

$0 = \frac{\partial V}{\partial \bar{\phi}_i}\Delta_{ij}\phi_j$. We now take the derivative of this equation w.r.t. $\bar{\phi}_k$ and find

$$\boxed{0 = \frac{\partial^2 V}{\partial\bar{\phi}_i\,\partial\bar{\phi}_k}\Delta_{ij}\bar{\phi}_j + \frac{\partial V}{\partial\bar{\phi}_i}\Delta_{ik}}$$

Now, let's take $\bar{\phi}$ to be the vacuum field $\bar{\phi}_i \Rightarrow \phi_i^{(0)}$, where $\frac{\partial V}{\partial\bar{\phi}_i^{(0)}} \equiv \emptyset$. Then, the above equation, we obtain

$$\boxed{0 \equiv m^2_{ki}\,\Delta_{ij}\vec{\phi}_j^{(0)}}$$

This equation, when written as $0 = m^2_{ki}\xi_i$, ~~which~~ tells us that the mass matrix has vanishing eigenvalues. The number of this eigenvalues should ~~correspond~~ be equal to a number of transformations in the symmetry group of the original theory that do not leave the chosen vacuum state invariant. Alternatively, if $\Delta_{ij}\bar{\Phi}_j^{(0)} = 0$, i.e. the vacuum state is invariant under particular transformation,

the correspondiy "direction" in m^2_{ki} space doesn't
need to vanish. Hence, we have proved the Goldstone theorem classically.
The appearance of very large number of massless particles seems like a generic problem for these kind of theories - we do not see many of such particles in Nature. However, as we will see in the next lecture something interesting happens to these particles if we combine the ~~Gold~~ spontaneous symmetry breaking idea with gauge ~~th~~ symmetry.