# Lecture 10 Non-abelian gauge theories

In this Lecture, we will discuss what happens if we combine the idea of non-abelian symmetry with the idea of gauge invariance. Let's first discuss those things separately.

The gauge-invariance is known to us from QED. It arises as a consequence of the requirement that the Dirac Lagrangian $L = \bar\psi(i\hat\partial - m)\psi$ is invariant under local phase changes of the fermion field: $\psi \to \psi'(x) \equiv e^{i\alpha(x)}\psi(x)$.

First, the mass term is trivially invariant
$-m\bar\psi(x)\psi(x) \to -m\bar\psi'(x)\psi'(x) \equiv -m\bar\psi(x)\psi(x)$. Second, the term with derivative is not:

$$\bar\psi_a \partial_\mu \psi_b \to \bar\psi_a' \partial_\mu \psi_b' = \bar\psi_a e^{-i\alpha(x)} \partial_\mu e^{i\alpha(x)}\psi_b =$$
$$= \bar\psi_a \partial_\mu \psi_b + \bar\psi_a i\left(\partial_\mu\alpha(x)\right)\psi_b .$$

Therefore, the term with derivative changes.

We can try to compensate for this term by introducing an additional term in the derivative $\partial_\mu$. Consider

$\partial_\mu \to \mathcal{D}_\mu \equiv \partial_\mu + ie A_\mu(x)$. $\mathcal{D}_\mu$ is known as "covariant" derivative and $A_\mu$ is a vector field. Let's imagine that $A_\mu \to A'_\mu$ under gauge transformations. Then

$$\bar{\psi}_a(x)\mathcal{D}_\mu \psi_b(x) \to \bar{\psi}'_a(x)\mathcal{D}'_\mu \psi'_b(x) =$$

$$= \bar{\psi}_a e^{-i\alpha(x)}\left(\partial_\mu + ieA'_\mu(x)\right)e^{i\alpha(x)}\psi_b(x) =$$

$$\equiv \bar{\psi}_a\left(\partial_\mu + i\left(\partial_\mu \alpha(x)\right) + ieA'_\mu(x)\right)\psi_b(x).$$

We want this to be equal to $\bar{\psi}_a \mathcal{D}_\mu \psi_b$.

To accomplish this, define a gauge transformation for $A_\mu$ as $\boxed{A'_\mu(x) = A_\mu(x) - \frac{\partial_\mu \alpha(x)}{e}}$. With this, $\bar{\psi}_a \mathcal{D}_\mu \psi_b \equiv \bar{\psi}'_a \mathcal{D}'_\mu \psi'_b$. Hence, the Dirac Lagrangian extended to be $\mathcal{L} = \bar{\psi}(i\hat{\mathcal{D}} - m)\psi$ where $\mathcal{D}_\mu = \partial_\mu + ieA_\mu$ is invariant under gauge-transformations. We also note that gauge ~~transformatio~~ invariance fully fixes how the gauge field $A_\mu(x)$ interacts with the fermion field, provided that we restrict dimensions of relevant operators. The kinetic term of the vector field $\mathcal{L}_{kin} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$, where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is clearly gauge-invariant:

$$F_{\mu\nu} \to F'_{\mu\nu} = \partial_\mu A'_\nu - \partial_\nu A'_\mu = F_{\mu\nu} - \frac{\partial_\mu \partial_\nu \alpha(x)}{e} + \frac{\partial_\nu \partial_\mu \alpha(x)}{e}$$

$$\equiv F_{\mu\nu}.$$

The transformation $\psi(x) \to e^{i\alpha(x)}\psi(x)$ is the <u>local version</u> of the x-independent transformation $\psi(x) \mapsto e^{i\alpha}\psi(x)$ which can be thought of as a transformation by an element of the <u>abelian</u> group $U(1)$. In this case $g \in U(1)$ is represented by $e^{i\alpha}$ and $e^{i\alpha}$ is acting on a linear ~~space~~ vector space of "complex numbers" $\psi(x)$. The group is abelian since for $g_1 = e^{i\alpha_1}$ $g_2 = e^{i\alpha_2}$, $g_1 \cdot g_2 = g_2 g_1$.

It is easy to construct examples of theories with <u>non-abelian</u> <u>global symmetries</u>. To give an example, consider $N$ complex scalar fields whose physics is described by the Lagrangian

$$\boxed{L = \frac{1}{2}\,\partial_\mu \varphi^{*i}\,\partial^\mu \varphi_i - \frac{1}{2} m^2 \varphi^{*i}\varphi_i - \lambda(\varphi_i^* \varphi_i)^2},$$

where we assume that there is a summation w.r.t. repeated indices $\varphi_i^*\varphi_i \to \sum_{i=1}^{N} \varphi_i^*\varphi_i$

Consider now a transformation

$\varphi_i \to \varphi_i' \equiv \Omega_{ij}\varphi_j$ where $\Omega_{ij}$ is the $SU(N)$ matrix (i.e. $\Omega^+\Omega = \mathbb{1}$ and $\det(\Omega) = 1$). Then, it is easy to see that the <u>Lagrangian</u>

$L$ is invariant; indeed $\varphi_i^{*\prime} = (\Omega_{ij}^*)\varphi_j^*$,

so that $\varphi_i^{*\prime}\varphi_i^{\prime} = (\Omega_{ij})^* \varphi_j^* \, \Omega_{ik}\varphi_k =$

$= (\Omega^\dagger)_{ji}\Omega_{ik}\,\varphi_j^*\varphi_k = \delta_{jk} = \varphi_j^*\varphi_j$.

Clearly, the kinetic term ~~remains~~ is invariant as well, $\partial_\mu\varphi\,\partial^\mu\varphi \to \partial_\mu\varphi'\partial^\mu\varphi'$, and this ensures that $L$ is invariant.

The reason these transformations are non-abelian are clear, since $\Omega \in SU(N)$ $\Omega_1\Omega_2 \neq \Omega_2\Omega_1$, in general & so the order of the transformation matters.

To streamline the notation, note that we can consider $N$ fields $\{\varphi_i\}$ as a column $\varphi = \begin{pmatrix} \varphi_1 \\ \vdots \\ \varphi_n \end{pmatrix}$ and write the transform. rule as $\underline{\varphi \to \varphi' = \Omega\varphi}$. We then say that $\varphi$ transforms under the fundamental representation of $SU(N)$. The Lagrangian $L$ is then written as

$$\boxed{L = \frac{1}{2}\partial_\mu\varphi^\dagger\,\partial_\mu\varphi - \frac{1}{2}m^2\varphi^\dagger\varphi - \lambda(\varphi^\dagger\varphi)^2}$$

The invariance simply follows from the unitarity of the matrix $\Omega$:

$\varphi \to \varphi' = \Omega\varphi \qquad \varphi^\dagger \to \varphi^{\dagger\prime} = \varphi^\dagger\Omega^\dagger \Rightarrow \varphi^{\dagger\prime}\varphi' = \varphi^\dagger\Omega^\dagger\Omega\varphi = \varphi^\dagger\varphi$ ∎

Suppose that we would like to find a way to make the Lagrangian $L$ invariant under $x$-dependent $SU(N)$ transformations $\varphi(x) \to \varphi'(x) \equiv \Omega(x)\varphi(x)$. Clearly, $\varphi^\dagger(x)\varphi(x) \equiv \varphi'^\dagger(x)\varphi'(x)$, so the mass term and the interaction term are invariant. The problems, as always, appear in the kinetic term where derivatives are involved. We have:

$$\partial_\mu \varphi \to \partial_\mu \varphi' = \partial_\mu \Omega(x)\varphi(x) = \Omega(x)\left[\partial_\mu\varphi + \Omega^{-1}(\partial_\mu\Omega(x))\varphi\right] = \neq \Omega\,\partial_\mu\varphi .$$

Similar to the $U(1)$ case that we discussed earlier, we change derivative to a <u>covariant</u> derivative $\partial_\mu \to \partial_\mu + ig\hat{A}_\mu$, where $\hat{A}_\mu(x)$ for now, is a $N\times N$ matrix <u>and</u> a vector field.

Upon transforming $\varphi$, and assuming that $A$ transforms as well, we find

$$D_\mu\varphi \to (D'_\mu\varphi') = \Omega(x)\left[\partial_\mu\varphi + \Omega^{-1}(\partial_\mu\Omega)\varphi + ig\Omega^{-1}\hat{A}'_\mu\Omega\,\varphi\right]$$

To make the r.h.s. be $\Omega D_\mu\varphi$, we require that

$$ig\,\Omega^{-1}\hat{A}'_\mu(x)\Omega + \Omega^{-1}(x)(\partial_\mu\Omega) \equiv ig\hat{A}_\mu \Rightarrow$$

$$\boxed{A_\mu(x) \to \hat{A}'_\mu(x) \equiv \Omega\hat{A}_\mu\Omega^{-1} - \frac{(\partial_\mu\Omega)\Omega^{-1}}{ig}}$$

It follows from this expression that $A_\mu(x)$ transforms under the adjoint representation of the $SU(N)$ group and, therefore, can be considered as an element of Lie algebra of the group $SU(N)$.

Let us now discuss the transformation rules for the covariant derivative operator: $\mathcal{D}_\mu$

$$\mathcal{D}_\mu \equiv \partial_\mu + igA_\mu \to \partial_\mu + igA'_\mu =$$
$$= \partial_\mu + ig\Omega\hat{A}_\mu\Omega^{-1} - (\partial_\mu\Omega)\Omega^{-1} =$$
$$= \Omega\left(\Omega^{-1}\partial_\mu - \Omega^{-1}(\partial_\mu\Omega)\Omega^{-1} + ig\hat{A}_\mu\Omega^{-1}\right) =$$
$$= \Omega\left(\Omega^{-1}\partial_\mu\Omega\Omega^{-1} - \Omega^{-1}(\partial_\mu\Omega)\Omega^{-1} + ig\hat{A}_\mu\Omega^{-1}\right) =$$
$$= \Omega\left(\Omega^{-1}(\partial_\mu\Omega)\Omega^{-1} + \partial_\mu\Omega^{-1} - \Omega^{-1}(\partial_\mu\Omega)\Omega^{-1} + ig\hat{A}_\mu\Omega^{-1}\right)$$
$$= \Omega\mathcal{D}_\mu\Omega^{-1}. \Rightarrow \boxed{\mathcal{D}_\mu \to \mathcal{D}'_\mu \equiv \Omega\mathcal{D}_\mu\Omega^{-1}}$$

This is a useful equation as ~~it~~ gives immediately transformation rules for various quantities. Indeed, a re-derivation of what we already ~~discovered~~ know:

$$\mathcal{D}_\mu\varphi \to \mathcal{D}'_\mu\varphi' = \Omega\mathcal{D}_\mu\Omega^{-1}\Omega\varphi \equiv \Omega(\mathcal{D}_\mu\varphi)$$

Hence $$\underline{(\mathcal{D}_\mu\varphi)^+(\mathcal{D}_\mu\varphi) \Rightarrow (\mathcal{D}_\mu\varphi)^+\Omega^+\Omega(\mathcal{D}_\mu\varphi) = (\mathcal{D}_\mu\varphi)^+(\mathcal{D}_\mu\varphi)}$$

We can also use the transformation rules for the covariant derivative to derive kinetic term for the non-abelian gauge field.

Let's go back to the $U(1)$ case, where

$D_\mu = \partial_\mu + ie A_\mu$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

We establish the connection between the two if we notice that

$$[D_\mu, D_\nu] = [\partial_\mu + ie A_\mu, \partial_\nu + ie A_\nu] \equiv$$
$$= ie(\partial_\mu A_\nu - \partial_\nu A_\mu) = ie F_{\mu\nu}.$$

We can calculate $[D_\mu, D_\nu]$ also in the non-abelian case ($D_\mu = \partial_\mu + ig\hat{A}_\mu$)

$$[D_\mu, D_\nu] = ig(\partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu + ig[\hat{A}_\mu, \hat{A}_\nu])$$
$$\equiv ig\hat{F}_{\mu\nu}.$$ Note that $[\hat{A}_\mu, \hat{A}_\nu] \neq 0$

because $\hat{A}_\mu$ and $\hat{A}_\nu$ are the $N \times N$ matrices from the Lie algebra of $SU(N)$. We can now easily establish the transformation rule for $F_{\mu\nu}$, in case of non-abelian field.

Since $D_\mu \to D_\mu' = \Omega D_\mu \Omega^{-1}$, we find

$$[D_\mu, D_\nu] = ig\hat{F}_{\mu\nu} \to [D_\mu', D_\nu'] = ig F_{\mu\nu}' =$$
$$= \Omega [D_\mu, D_\nu] \Omega^{-1} = \hat{\Omega} ig F_{\mu\nu} \Omega^{-1} \Rightarrow$$

$$\boxed{F_{\mu\nu} \to F_{\mu\nu}' = \Omega F_{\mu\nu} \Omega^{-1}}$$

To write the kinetic term for the non-abelian gauge field, we use $\mathcal{L}_{kin} = -\frac{1}{4} \mathrm{Tr}(\hat{F}_{\mu\nu} \hat{F}^{\mu\nu})$

where trace refers to $SU(N)$ indices

As the next step, we consider explicit example of a simplest (and historically first) non-abelian gauge theory – the one with the gauge group $SU(2)$. We will also consider the case of a fermion, rather than scalar, theory.

So, we consider a doublet of Dirac fermions $\psi \equiv \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ that transforms with matrices from the group $SU(2)$:

$\psi \to \psi' = \Omega \psi$, where $\Omega$ is a unitary $2\times 2$ matrix with a special property $\det(\Omega) = 1$.

The Lagrangian reads $L = L_A + L_\psi$, where $L_\psi = \bar{\psi}(i\hat{D} - m)\psi$, where $D_\mu = \partial_\mu + ig\hat{A}_\mu$ and the vector field $\hat{A}_\mu$ belongs to the Lie algebra of $SU(2)$.

Any element of Lie algebra of $SU(2)$ is written as a linear combination of three $SU(2)$ generators $\tau^a$ $\Rightarrow$

$$\hat{A}_\mu \equiv \sum_{a=1}^{3} A_\mu^a \tau^a; \quad \{A_\mu^{(1)}, A_\mu^{(2)}, A_\mu^{(3)}\} \text{ are \underline{three gauge fields}.}$$

The generators satisfy the commutation relation $[\tau^a, \tau^b] = i f^{abc} \tau^c$, where $f^{abc}$ are the $SU(2)$ structure constants ($f^{abc} = \varepsilon^{abc}$).

Also $\tau^a = \sigma^a/2$, where $\sigma^a$ are the Pauli matrices.

The generators are normalized as $Tr(\tau^a \tau^b) \equiv \frac{1}{2}\delta^{ab}$, which is also the standard normalization for other $SU(N)$ groups.

Hence, $L_\psi = \bar{\psi}\,(i\hat{\partial} - m)\psi - g\sum_{a=1}^{3} \bar{\psi}\gamma^\mu \tau^a \psi A^a_\mu$ and the last term defines the interaction between the fermion and the gauge field.

The field-strength tensor reads:

$$\hat{F}_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu + ig[\hat{A}_\mu, \hat{A}_\nu] =$$

$$= \tau^a(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) + ig\, A^b_\mu A^c_\nu [\tau^b, \tau^c] =$$

$$= \tau^a(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) + ig\, A^b_\mu A^c_\nu\, i\varepsilon^{bca}\tau^a \Rightarrow$$

$$\hat{F}_{\mu\nu} = \tau^a F^a_{\mu\nu}, \quad \text{where} \quad \boxed{F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g\varepsilon^{abc} A^b_\mu A^c_\nu}$$

Hence, the "kinetic" term for the gauge field is $\mathcal{L}_A = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} = -\frac{1}{2} Tr(\hat{F}_{\mu\nu}\hat{F}^{\mu\nu})$.

Note an important consequence of non-abelian gauge symmetry. <u>In contrast to QED,</u> <u>the theory with</u> ~~fermions~~ <u>gauge fields</u> <u>only</u> $\mathcal{L}_A = -\frac{1}{2} Tr(\hat{F}_{\mu\nu}\hat{F}^{\mu\nu})$ is <u>non-trivial</u>

Since this Lagrangian <u>is not quadratic</u> in the fields $A^a_\mu$. In fact, $\mathcal{L}_A$ contain <u>triple</u> and <u>quartic</u> terms which implies that there ~~are~~ interactions between non-abelian gauge bosons.

The full Lagrangian of the gauge SU(2) theory with fermions reads

$$\boxed{\mathcal{L} = -\frac{1}{2}\,\mathrm{Tr}\left(\hat{F}_{\mu\nu}\hat{F}^{\mu\nu}\right) + \bar{\psi}\left(i\hat{D} - m\right)\psi}$$, where

$$\hat{F}_{\mu\nu} \equiv \sum_{a=1}^{3} \tau^a F^a_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu + ig\left[\hat{A}_\mu, \hat{A}_\nu\right]$$

$$\hat{A}_\mu = \sum_{a=1}^{3} \tau^a A^a_\mu \,, \quad D_\mu = \partial_\mu + ig\hat{A}_\mu$$

and $\{\tau^a\}$, $a = 1, .. 3$ are generators of the SU(2) Lie Algebra that satisfy therefore the commutation relation $[\tau^a, \tau^b] = i\varepsilon^{abc}\tau^c$ and are normalized as $\mathrm{Tr}[\tau^a\tau^b] = \frac{1}{2}\delta^{ab}$.

The fermion field $\psi$ transforms under fundamental representation of SU(2) $\psi \to \Omega\psi$ and $\hat{A}_\mu$ - under adjoint. As follows from the dimensionality of SU(2) Lie algebra, there are $\underline{\underline{3}}$ gauge bosons $(A^{(1)}_\mu, A^{(2)}_\mu, A^{(3)}_\mu)$.

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Generalization of this result for other gauge groups such as SU(N) is straightforward. All of the above formulas hold, except that the dimensionality of the Lie algebra increases, for SU(N), it is $N^2-1$. Therefore, $\hat{A}_\mu = \sum_{a=1}^{N^2-1} \tau^a A^a_\mu$ and there are $N^2-1$ independent gauge bosons for the SU(N) ...

As a further example, we consider a theory of scalar fields that transform under adjoint representation of the group $SU(N)$. There are $N^2-1$ such real-valued fields and $\phi = \sum_{a=1}^{N^2-1} \tau^a \cdot \phi^a$ is the matrix in Lie algebra of $SU(N)$. The global transform. rule is $\phi \to \hat{\phi}' = \Omega \hat{\phi} \Omega^{-1}$, $\Omega \in SU(N)$ and the invariant Lagrangian is

$$\mathcal{L} = \mathrm{Tr}\left[(\partial_\mu \hat{\phi})(\partial_\mu \hat{\phi})\right] - m^2 \mathrm{Tr}(\hat{\phi}\hat{\phi}) - \lambda\left[\mathrm{Tr}(\hat{\phi}\hat{\phi})\right]^2.$$

We now want to write a Lagrangian that is invariant under local transformations. We define covariant derivative as

$$\partial_\mu \hat{\phi} \to [\mathcal{D}_\mu, \hat{\phi}] = \partial_\mu \phi + ig[\hat{A}_\mu, \phi]$$

Clearly $[\mathcal{D}_\mu, \hat{\phi}] \to [\Omega \mathcal{D}_\mu \Omega^{-1}, \Omega \hat{\phi} \Omega^{-1}] \equiv$

$\equiv \Omega[\mathcal{D}_\mu, \phi]\Omega^{-1}$, so it transforms properly. We can also write $\hat{\phi} = \sum_{a=1}^{N^2-1} \phi^a \tau^a$ $\hat{A} = \sum_{a=1}^{N^2-1} A^a \cdot \tau^a$, and obtain:

$$[\mathcal{D}_\mu, \hat{\phi}] = (\partial_\mu \phi^a)\tau^a + ig A_\mu^b \phi^c i f^{bca} \tau^a$$

$$= \left[(\partial_\mu \phi^a) - g f^{abc} A_\mu^b \phi^c\right]\tau^a \Rightarrow$$

$$[\mathcal{D}_\mu, \hat{\phi}]^a \equiv \boxed{(\mathcal{D}_\mu \phi)^a \equiv \partial_\mu \phi^a - g f^{abc} A_\mu^b \phi^c}$$

$\Rightarrow$ a covariant derivative for a field in the adjoint representation.

The Lagrangian is then

$$\boxed{\mathcal{L} = \mathrm{Tr}\Big[ [\mathcal{D}_\mu,\hat{\phi}]\,[\mathcal{D}^\mu,\hat{\phi}]\Big] - m^2\,\mathrm{Tr}[\hat{\phi}\hat{\phi}) - \lambda\big(\mathrm{Tr}[\hat{\phi}\cdot\hat{\phi}]\big)^2}$$

We will now derive equations of motion for gauge fields in non-abelian theory, to illustrate our statement that even in the absence of matter fields the theory is non-trivial. We consider a theory with the Lagrangian

$$\mathcal{L}_A = -\frac{1}{4}\sum_{b=1}^{N^2-1} F^b_{\mu\nu} F^{b,\mu\nu} \quad \text{and}$$

the action $S = \int d^4x\, \mathcal{L}_A$. To find equations of motion, we compute $\frac{\delta S}{\delta A^a_\mu(x)}$ and set it to zero

$$\delta S = \int d^4x\, \delta\mathcal{L}_A = -\frac{1}{2}\int d^4x\, F^b_{\mu\nu}\,\delta F^{b,\mu\nu} =$$

$$= -\frac{1}{2}\int d^4x\, F^b_{\mu\nu}\Big\{\partial^\mu \delta A^{b,\nu} - \partial^\nu \delta A^{b,\mu} - g f^{bcd}\cdot$$

$$\cdot\left(\delta A^c_\mu A^d_\nu + A^c_\mu \delta A^d_\nu\right)\Big\}$$

We can use antisymmetry of $F_{\mu\nu}$ and $f^{bcd}$ to rewrite the above equation as

$$\delta S = -\int d^4x\, F^b_{\mu\nu}\left(\partial^\mu \delta A^{b,\nu} - g f^{bcd} A^c_\mu \delta A^d_\nu\right)$$

$$= -\int d^4x\left[-\partial_\mu F^{b\mu\nu}\delta A^b_\nu - g F^b_{\mu\nu} f^{bcd} A^c_\mu \delta A^d_\nu\right]$$

Since $\delta S = 0$ for any variation $\delta A_\nu^d$,
we find

$$- \partial_\mu F^{a\mu\nu} - g F^b_{\mu\nu} f^{bca} A^c_\mu = 0 \Rightarrow$$

$$\boxed{\partial_\mu F^{a\mu\nu} - g f^{acb} A^c_\mu F^b_{\mu\nu} = 0}$$

We can again write it as

$$\boxed{[D_\mu, \hat{F}^{\mu\nu}] = 0.}$$

Note that the equation of motion for the field $A_\mu$ is <u>non-linear</u>, for a generic non-abelian group ($f^{acb} \neq 0$) and <u>linear</u> for the abelian group.