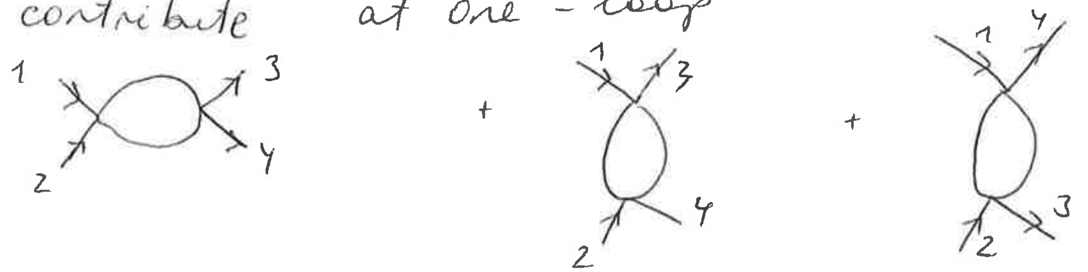
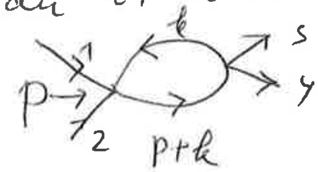
Lecture 5 Optical theorem, Cutkosky rules, unstable particles

Consider a theory of a scalar field with the interaction Lagrangian $\mathcal{L}_{int} = -\frac{\lambda \varphi^4}{4!}$. We will study a $2 \to 2$ scattering process: $\varphi_1 + \varphi_2 \to \varphi_3 + \varphi_4$

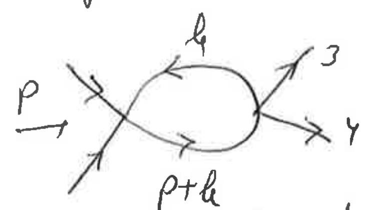
There are three Feynman diagrams that contribute at one-loop



We will take the first one & consider it as a function of the incoming momentum



We write



$$= -i\Sigma(p) = \int \frac{d^dk}{(2\pi)^d} (-i\lambda)^2 \frac{i}{k^2-m^2} \frac{1}{(p+k)^2-m^2}$$

$$\equiv \lambda^2 \int \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2-m^2)((p+k)^2-m^2)} \Bigg|^{*}$$

To calculate this loop integral, combine propagators using Feynman parameters

$$\frac{1}{[k^2-m^2][(k+p)^2-m^2]} \equiv \int_0^1 dx \frac{1}{[k^2+2pkx+p^2x-m^2]^2}$$

$$= \int_0^1 \frac{dx}{[(k+px)^2+p^2x(1-x)-m^2+i0]^2}$$

*) No symmetry factor included.

Integrating over k, we find

$$-i\Sigma = \frac{i\lambda^2}{(4\pi)^{d/2}}\Gamma(\varepsilon)\int_0^1 dx \, \frac{1}{\left[m^2 - p^2x(1-x) - i0\right]^{\varepsilon}} \Rightarrow$$

$$\boxed{\Sigma(p) = -\frac{\lambda^2\Gamma(1+\varepsilon)}{(4\pi)^{d/2}}\int_0^1 dx \left[\frac{1}{\varepsilon} - \ln\left(m^2 - p^2x(1-x) - i0\right)\right]}$$

We would like to discuss properties of Σ as a function of p^2. First, we can imagine that p^2 is either positive or negative. If p^2 is negative, $m^2 - p^2x(1-x) > 0$ and so $\Sigma(p)$ is real. If, on the other hand, $p^2 > 0$, $m^2 - p^2x(1-x)$ is not sign-definite. It is easy to see that for $m^2 - p^2x(1-x)$ to become negative for $0 < x < 1$, p^2 must be larger than $4m^2$.

Indeed, $m^2 - p^2x(1-x) = 0$ implies $(p^2 > 0)$

$$x_{\pm}^* = \frac{1}{2}\left(1 \pm \sqrt{1 - \frac{4m^2}{p^2}}\right).$$

Therefore, if $p^2 < 4m^2$, there are no solutions on the real axis. For $p^2 > 4m^2$, we have

$0 < x_{\pm}^* < 1$ and $m^2 - p^2x(1-x) < 0$ for $x_-^* < x < x_+^*$.

Hence, for $p^2 > 4m^2$, $m^2 - p^2x(1-x) - i0$ becomes negative on the interval of x embedded in the integration region.

The logarithmic function $\ln(z)$ has a branch-cut for negative z. Using this, we compute the imaginary part of $\Sigma(p)$ for $p^2 > 4m^2$:

$$\mathrm{Im}\left(\Sigma(p^2)\right) = \theta(p^2-4m^2)\frac{(-\lambda^2)\Gamma(1+\varepsilon)}{(4\pi)^{d/2}}\int_0^1 dx\,(-1)\,\mathrm{Im}\left[\ln\left(m^2-p^2x(1-x)-i0\right)\right]$$

$$= \theta(p^2-4m^2)\frac{(-\lambda^2)}{(4\pi)^2}\int_{x_-^*}^{x_+^*} dx\,(-1)(-i\pi) = \theta(p^2-4m^2)\frac{\lambda^2}{(4\pi)^2}i\pi\left(x_-^*-x_+^*\right).$$

Hence, we find

$$\boxed{\mathrm{Im}\,\Sigma(p^2) = \theta(p^2-4m^2)\frac{\lambda^2}{(4\pi)^2}(-i\pi)\sqrt{1-\frac{4m^2}{p^2}}}$$

We will try to interpret this result from a different perspective. Let's go back to the f-la for $\Sigma(p^2)$ on page 2 & rewrite the logarithm as

$$\ln\left[\underline{m^2-p^2x(1-x)-i0}\right] = \ln\left[m^2-sx(1-x)\right], \text{ where } s = p^2+i0.$$

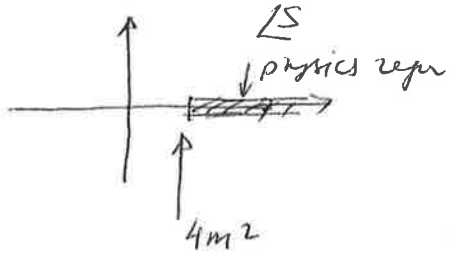
Then we write

$$\boxed{\Sigma(s) = \frac{-\lambda^2\Gamma(1+\varepsilon)}{(4\pi)^{d/2}}\int_0^1 dx\left\{\frac{1}{\varepsilon} - \ln\left(m^2-sx(1-x)\right)\right\}}$$

and study $\Sigma(s)$ as a function of <u>complex</u> variable s!

Analytic properties of ~~the~~ $\Sigma(s)$ follow from analytic properties of the $\log(z)$. Since $\log(z)$ is made analytic function by cutting e.g. from 0 to $-\infty$ along the real axis,

$\Sigma(s)$ is made analytic if a cut is made in the complex s-plane, from $s = 4m^2$ to $s = \infty$



The physical values of Σ are obtained by computing $\Sigma(s)$ for s at the upper side of the cut ($s = p^2 + i0$).

At this point, we introduce another useful quantity — the discontinuity across the cut, $\mathrm{Disc}(\Sigma(s))$.

It is defined as follows: consider a point $s = s_0$, at the cut and study the difference

$$\boxed{\lim_{\delta \to 0} \left[\Sigma(s_0 + i\delta) - \Sigma(s_0 - i\delta) \right] \equiv \mathrm{Disc}\left(\Sigma(s_0)\right)}$$

Since $\ln(z) \equiv \ln|z| + i\,\mathrm{Im}\left(\ln(z)\right)$ and since $|s_0 + i\delta| = |s_0 - i\delta|$, the discontinuity can only come from the imaginary part of the logarithm.

Moreover, because $\mathrm{Im}\left[\ln\left(m^2 - (s_0 - i\delta)x(1-x)\right)\right] =$
$= -\mathrm{Im}\left[\ln\left(m^2 - (s_0 + i\delta)x(1-x)\right)\right]$, we find that $\mathrm{Disc}\left[\Sigma_2(s_0)\right] \equiv 2\,\mathrm{Im}\left(\Sigma_2(s_0 + i\delta)\right)$

Hence, we conclude that

$$\boxed{\mathrm{Disc}\left(\Sigma(s_0)\right) = \theta(s_0 - 4m^2)\,\frac{\lambda^2}{(4\pi)^2}\left(-2i\pi\right)\sqrt{1 - \frac{4m^2}{s_0}}}$$

We will now generalize the above considerations. ~~addition~~ ~~then~~ The first point we want to consider is the unitarity of the S-matrix.

The unitarity of the S-matrix is the statement that the total probability is conserved $S \cdot S^{+} = 1$.

Hence, $\sum\limits_{f} \langle i | S | f \rangle \langle f | S^{+} | i \rangle = \langle i | i \rangle = 1$.

Using representation for S through the T-matrix $\hat{S} = \hat{1} + i \hat{T}$, we find

$$1 = S^{+} S = \left(\hat{1} - i \hat{T}^{+} \right) \left(\hat{1} + i \hat{T} \right) = i \left(\hat{T} - \hat{T}^{+} \right) + \hat{T}^{+} \hat{T} + 1$$

$$\Rightarrow \boxed{-i \left(\hat{T} - \hat{T}^{+} \right) = \hat{T}^{+} \hat{T}}$$

Let us now see why this equation is useful.

To this end, we take the matrix element of the left-hand side and the right-hand side of that equation between two 2-particle states.

We have

$$\langle \vec{p}_1, \vec{p}_2 | T^{+} T | \vec{k}_1, \vec{k}_2 \rangle = \sum_f \langle \vec{p}_1 \vec{p}_2 | T^{+} | f \rangle \langle f | T | \vec{k}_1, \vec{k}_2 \rangle$$

where $| f \rangle \equiv | \{ q_i \} \rangle$ is the collection of single-part. states and $\sum\limits_f \equiv \sum\limits_n \prod\limits_{i=1}^{n} \int \frac{d^3 \vec{q}_i}{(2\pi)^3} \frac{1}{2 q_i^0}$

If we use representation of ~~the~~ matrix elements of $\hat{T}$ in terms of scattering amplitudes, e.g.

$$\langle \{ \vec{q} \} | \hat{T} | \vec{k}_1, \vec{k}_2 \rangle = (2\pi)^4 \delta^{(4)} \left(\sum_i q_i - k_1 - k_2 \right) M(k_1, k_2 \to \{ q \})$$

we can extract the total energy-momentum-conserving δ-function to obtain:

$$-i\left[\mathcal{M}(k_1k_2\to p_1p_2)-\mathcal{M}^*(p_1p_2\to k_1k_2)\right]$$
$$\equiv\sum_n\left(\prod_{i=1}^n\int\frac{d^3\vec{q}_i}{(2\pi)^3 2q_i^0}\ \mathcal{M}^*(p_1p_2\to\{q_i\})\,\mathcal{M}(k_1k_2\to\{q_i\})\right)$$
$$\times(2\pi)^4\,\delta^{(4)}\Big(k_1+k_2-\sum_i q_i\Big).$$

The overall energy-momentum conservation requires $k_1+k_2\equiv p_1+p_2$.

This formula becomes useful if we take the two states $|\vec{k}_1,\vec{k}_2\rangle$ and $|\vec{p}_1,\vec{p}_2\rangle$ to be the same. Then we find:

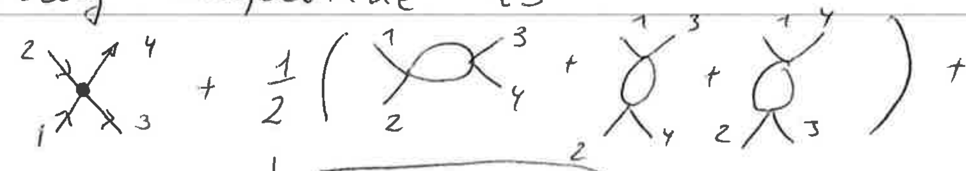
$$\boxed{\begin{aligned}&-i(2i)\,\mathrm{Im}\left[\mathcal{M}(k_1k_2\to k_1k_2)\right]\equiv\\&\equiv\sum_n\prod_{i=1}^n\int\frac{d^3\vec{q}_i}{(2\pi)^3 2q_i^0}\left|\mathcal{M}(k_1k_2\to\{q_i\})\right|^2(2\pi)^4\delta^{(4)}\Big(k_1+k_2-\sum q_i\Big)\end{aligned}}$$

We recognize that the r.h.s. is almost the total cross-section that describes scattering of k_1 & k_2 into arbitrary final state. Restoring normalization, we find

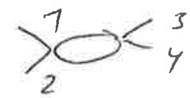
$$\boxed{\mathrm{Im}\left[\mathcal{M}(k_1k_2\to k_1k_2)\right]=2E_{cm}|\vec{k}_{cm}|\,\sigma_{tot}(k_1,k_2\to\text{anything})}$$

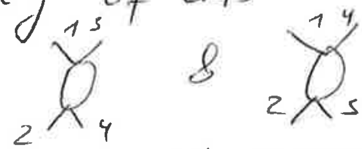
Here E_{cm} is the $|k_1,k_2\rangle$ center-of-mass energy and $|\vec{k}_{cm}|$ is the momentum of either $\vec{k}_1$ or $\vec{k}_2$ in the center-of-mass frame. The above relation is known as the "optical theorem".

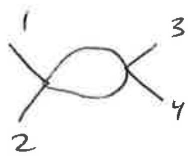
We will now investigate how the optical theorem applies to our example of $2 \to 2$ scattering in $\lambda \varphi^4$ theory. The scattering amplitude is

$$-iM(k_1+k_2 \to k_3+k_4) =$$  $$+ \ldots$$

↳ symmetry factor

The imaginary part appears first at one-loop and it comes from the diagram  that we investigated at the beginning of this lecture.

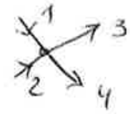
The other two diagrams  do not have imaginary parts because the momentum transfer squared $(k_1-k_3)^2$ & $(k_1-k_4)^2$ are both <u>negative</u> for $(k_1+k_2)^2 > 4m^2$.

We have calculated $i\,\mathrm{Im}\Big[$  $\Big]$ and found it to be $\theta(s-4m^2)\,\dfrac{\lambda^2}{(4\pi)^2}(-i\pi)\sqrt{1-\dfrac{4m^2}{s}}$.

Hence

$$\boxed{\mathrm{Im}\Big[M(k_1k_2 \to k_1k_2)\Big] = \frac{\pi}{2}\frac{\lambda^2}{(4\pi)^2}\,\theta(s-4m^2)\sqrt{1-\frac{4m^2}{s}}}$$

According to the optical theorem this imaginary part should be proportional to the scattering cross-section of k_1 & k_2 into anything.

The first contribution to this cross-section comes from leading order diagram , so that $M(k_1k_2\to 3,4)=\lambda$, $|M|^2=\lambda^2$, the phase space is

$$\frac{1}{2!}\frac{d^3\vec{p}_3}{(2\pi)^3 2E_3}\frac{d^3\vec{p}_4}{(2\pi)^3 2E_4}(2\pi)^4\delta^{(4)}(k_1+k_2-p_3-p_4) = (\text{in } k_1+k_2 \text{ CM frame})$$

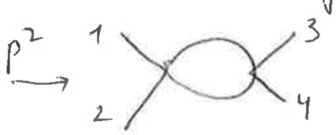
$$=\frac{1}{2!}\frac{1}{(2\pi)^2}\frac{1}{4E_3E_4}4\pi\, p_3^2\,\frac{E_3}{p_3}\,\frac{1}{2} = \frac{p_3}{2!\,8\pi E_3} = \frac{1}{16\pi}\sqrt{1-\frac{4m^2}{s}}$$

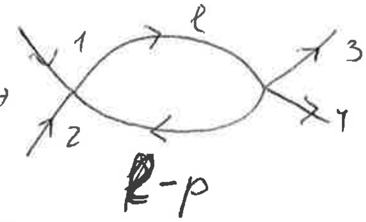
$$\Rightarrow \quad \sigma = \frac{1}{4\sqrt{(k_1\cdot k_2)^2-m^4}}\,\frac{\lambda^2}{16\pi}\sqrt{1-\frac{4m^2}{s}}$$

We find $\sqrt{(k_1\cdot k_2)^2-m^4} = E_{cm}\,|\vec{k}_{cm}| \Rightarrow$

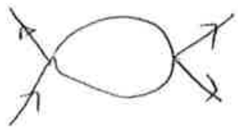
$$2E_{cm}\cdot|\vec{k}_{cm}|\,\sigma = \frac{\lambda^2}{32\pi}\sqrt{1-\frac{4m^2}{s}},$$ to be compared to

$$\mathrm{Im}\left[M(k_1,k_2\to k_1,k_2)\right] = \frac{\lambda^2}{32\pi}\sqrt{1-\frac{4m^2}{s}} \Rightarrow$$ We find agreement with the optical theorem expectation.

We now go back to the discussion of the diagram . We found that this diagram is an analytic function of p^2 and computed its discontinuity across the cut from the Feynman parameter representation. We will now discuss how to obtain the same result without explicit integration over the loop momentum.

Consider $p \rightarrow$ 

$p = p_1 + p_2$; we will work in the center-of-mass frame $p^\mu = (p_0, \vec{0})$ and take $p_0 > 0$. We write


$$= -i\Sigma = \frac{\lambda^2}{2} \int \frac{d^d\ell}{(2\pi)^d} \frac{1}{(\ell^2 - m^2)((\ell \mp p)^2 - m^2)}$$

and investigate the analytic properties of Σ.

To this end, we imagine that first integration to be performed is the ℓ_0-integration & we study poles that we encounter when doing so.

$$0 = \ell^2 - m^2 + i0 \Rightarrow \ell_0 = \pm\sqrt{E_{\vec{\ell}}^2 - i0} = \pm E_{\vec{\ell}} \mp i0,$$

$$0 = (\ell - p)^2 - m^2 + i0 \Rightarrow \ell_0 = p_0 \pm E_{\vec{\ell}} \mp i0, \quad \text{where}$$

$$\boxed{E_{\vec{\ell}} = \sqrt{\vec{\ell}^2 + m^2}.}$$

Let us denote these poles as

a) $\ell_0 = +E_{\vec{\ell}} - i0$ a') $\ell_0 = -E_{\vec{\ell}} + i0$

b) $\ell_0 = p_0 + E_{\vec{\ell}} - i0$ b') $\ell_0 = p_0 - E_{\vec{\ell}} + i0$

and draw them in the complex-ℓ_0 plane (we assume $p_0 > 0$)

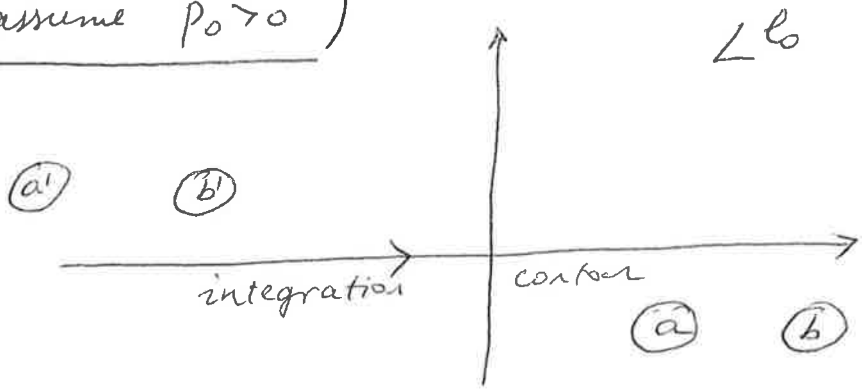
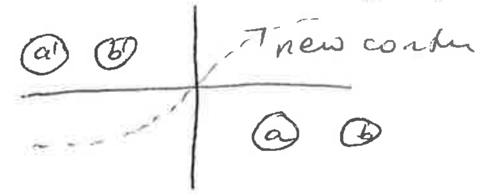


Fig. 1

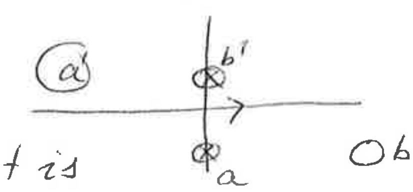
We note that if the poles are located indeed as shown, it is easy to argue that no discontinuity can not arise if we change the value of p_0 by a tiny complex amount $p_0 \to p_0 \pm i\delta$.

Indeed, for poles as in Fig. 1 we can always deform the integration contour without changing the integral, as shown here



Changes in p_0, $p_0 \to p_0 \pm i\delta$ move poles b and b' by a tiny amount ~~be~~ and, since the integration contour is far away, this doesn't change the integral. So, if we have poles lined up as in Fig. 1, ~~the~~ the integral doesn't have a discontinuity.

Let us now discuss how discontinuity can arise. For this to happen, we should not be able to move the contour. This ~~which~~ happens if b' and a are on top of each other



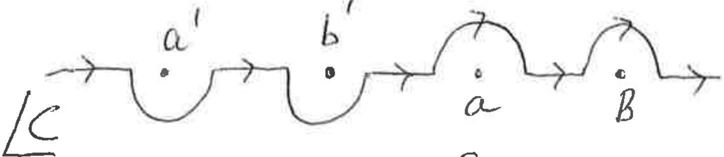
We single out b' & a because, as it is easy to check, these are the only poles that may end up sitting on top of each other.

If b' is just above a, shifting p_0 by $\pm i\delta$ changes the value of the integral because b' depends on p_0 and, upon changing it, may cross the integration contour. This may lead to discontinuity (i.e. the integral for $p_0 + i\delta$ may differ from the integral for $p_0 - i\delta$)

To extract the discontinuity, write the integral

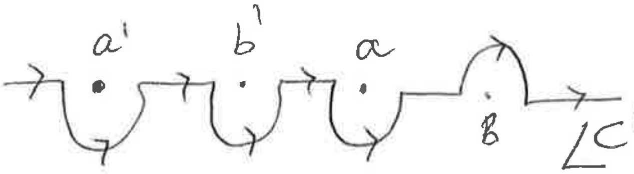
$$\Sigma = \frac{i\lambda^2}{2} \int \frac{d^3\vec{l}}{(2\pi)^3} \int_C \frac{dl_0}{2\pi} \frac{1}{\left(l_0^2 - E_{\vec{l}}^2\right)\left((l_0 - p_0)^2 - E_{\vec{l}}^2\right)}, \text{ where}$$

p_0 is real and the integration contour C looks like



We write

$$\int_C \frac{dl_0}{2\pi} = \int_{C'} \frac{dl_0}{2\pi} + \int_\gamma \frac{dl_0}{2\pi}, \quad \text{where}$$



and $\gamma =$ 

By the previous argument, there can't be any discontinuity for the integral over C' and therefore the discontinuity can only come from the integral over γ, which is easily computed by the residue theorem.

In fact, this integral can be written as as

$$\boxed{\frac{1}{l_0^2 - E_{\vec{l}}^2} \to -2\pi i\, \delta\left(l_0^2 - E_{\vec{l}}^2\right) \theta(l_0).}$$

We write the discontinuity of a diagram

$$\mathcal{D}isc\left(\Sigma(p_0)\right) = \Sigma(p_0+i\delta) - \Sigma(p_0-i\delta) =$$

$$= \frac{i\lambda^2}{2}\int \frac{d^3\vec{\ell}}{2\pi}\,\frac{d\ell_0}{2\pi}\,(2\pi i)\,\delta(\ell_0^2-E_{\vec{\ell}}^2)\,\theta(\ell_0)\ \times$$

$$\times\left[\frac{1}{(\ell_0-p_0-i0)^2-E_{\vec{\ell}}^2} - \frac{1}{(\ell_0-p_0+i0)^2-E_{\vec{\ell}}^2}\right]$$

We rewrite $(i0)$ dependent denominators

$$\frac{1}{(\ell_0-p_0\mp i0)^2-E_{\vec{\ell}}^2} = \frac{1}{(\ell_0-p_0)^2-E_{\vec{\ell}}^2\mp i0(\ell_0-p_0)} \qquad (*)$$

Now, since $\ell_0^2=E_{\vec{\ell}}^2$, $(\ell_0-p_0)^2-E_{\vec{\ell}}^2 = p_0^2-2p_0\ell_0 = p_0(p_0-2E_{\vec{\ell}})$.

Hence, as long as $p_0<2E_{\vec{\ell}}$, the denominator is sign-definite, $i0$ prescription in $(*)$ isn't needed and $\mathcal{D}isc(\Sigma(p_0))\equiv 0$. However, if $p_0>2E_{\vec{\ell}}$, which happens if $\underline{p_0>2m}$, the $\pm i0$ prescription is important. For such values of ℓ_0 & p_0, $\mp i0(\ell_0-p_0)\equiv \pm i0$, & so

$$\frac{1}{(\ell_0-p_0\mp i0)^2-E_{\vec{\ell}}^2} \equiv \frac{1}{(\ell-p)^2-m^2\pm i0},$$ the usual Feynman propagator

Hence

$$\mathcal{D}isc\left[\Sigma(p^2)\right] = \frac{i\lambda^2}{2}\int\frac{d^3\vec{\ell}}{(2\pi)^3}\,\frac{d\ell_0}{2\pi}\,(-2\pi i)\,\delta(\ell^2-m^2)\,\theta(\ell_0)$$

$$\times\left[\frac{1}{(\ell-p)^2-m^2+i0} - \frac{1}{(\ell-p)^2-m^2-i0}\right]$$

Using the following identity: $\frac{1}{x+i0} = \mathcal{P}\left(\frac{1}{x}\right) - i\pi\,\delta(x)$,

we find

$$\boxed{\begin{aligned}\mathrm{Disc}\left[\Sigma(p^2)\right] &= \frac{+i\lambda^2}{2}\int \frac{d^3\vec{l}}{(2\pi)^3}\,\frac{dl_0}{2\pi}\left[-2\pi i\,\delta_+(l^2-m^2)\right]\times \\ &\quad\times\left[-2\pi i\,\delta_+\left((l-p)^2-m^2\right)\right], \\ \text{where}\quad \delta_+(l^2-m^2) &\equiv \delta(l^2-m^2)\,\theta(l_0)\end{aligned}}$$

We see that the discontinuity of a Feynman diagram is obtained by replacing its propagators with $-2\pi i\,\delta_+(p^2-m^2)$, so that 2 propagators go on the mass-shell at the same time. This result is general. The algorithm for getting discontinuity of any Feynman diagram is known as Cutkosky rules. They read:

1) cut through the diagram in all possible ways such that cut propagators can go on the mass-shell simultaneously.

2) for each cut propagator replace $\frac{1}{p^2-m^2+i0}$ with $-2\pi i\,\delta(p^2-m^2)$; perform loop integral

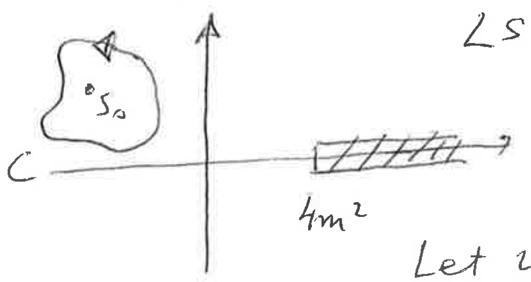
3) sum contributions of all possible cuts

The result should give you the discontinuity of a particular diagram.

Let us now discuss <u>why</u> we might be interested in these discontinuities. <u>One reason is that, with some reservations, knowledge of discontinuities makes it relatively simple to compute the entire Feynman diagram.</u> This is so because, being <u>analytic functions</u> of external kinematic invariants, Feynman diagrams satisfy dispersion relations

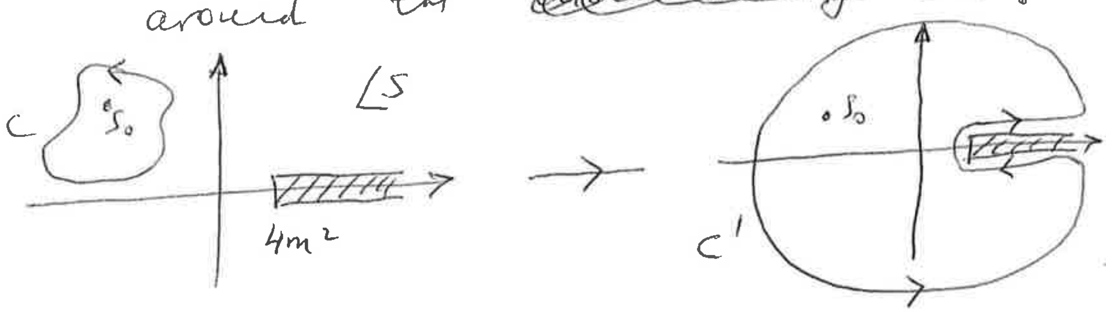
To give you an example, consider again $\overset{p}{\rightarrow}\!\!\!\!>\!\!\!\bigcirc\!\!\!\!< \;= -i\Sigma(p^2)$

We found that $\Sigma(s)$ is an analytic function of s, in the complex s-plane with a cut from $s = 4m^2$ to $s = \infty$. Take any point $s = s_0$ in the complex s-plane and use Cauchy theorem to write



$$\Sigma(s_0) = \frac{1}{2\pi i} \oint_C \frac{ds}{s - s_0} \Sigma(s)$$

Let us now move the integration contour to $s = \infty$; in doing so we must wrap it around the ~~discontinuity~~ cut. Since we do not cross any singularity by deforming $C \to C'$, the integral is still the same, so



that
$$\Sigma(s_0) = \frac{1}{2\pi i} \oint_{C'} \frac{ds\ \Sigma(s)}{s - s_0}$$

We now split it into three integrals:

$$\Sigma(s_0) = \frac{1}{2\pi i}\int_{C_\infty}\frac{ds\,\Sigma(s)}{s-s_0} + \frac{1}{2\pi i}\int_{4m^2+i0}^{\infty}\frac{ds\,\Sigma(s)}{s-s_0} - \frac{1}{2\pi i}\int_{4m^2-i0}^{\infty}\frac{ds\,\Sigma(s)}{s-s_0}$$

$$= \frac{1}{2\pi i}\int_{C_\infty}\frac{ds\,\Sigma(s)}{s-s_0} + \frac{1}{2\pi i}\int_{4m^2}^{\infty}\frac{ds}{s-s_0}\,2i\,\mathrm{Im}\left(\Sigma(s)\right) =$$

$$= \frac{1}{2\pi i}\int_{C_\infty}\frac{ds\,\Sigma(s)}{s-s_0} + \frac{1}{\pi}\int_{4m^2}^{\infty}\frac{ds}{s-s_0}\,\mathrm{Im}\left[\Sigma(s)\right].$$

Note that $\mathrm{Im}\left(\Sigma(s)\right)$ on the cut is the discontinuity that can be obtained from Cutcosky rules and that $\int_{C_\infty}\frac{ds\,\Sigma(s)}{s-s_0}$ provides a polynomial in s_0. Hence, we conclude that

$$\boxed{\Sigma(s) \equiv (\text{polynomial in } s) + \frac{1}{\pi}\int_{4m^2}^{\infty}\frac{ds'}{s'-s}\,\mathrm{Im}\left[\Sigma_2(s')\right]}$$

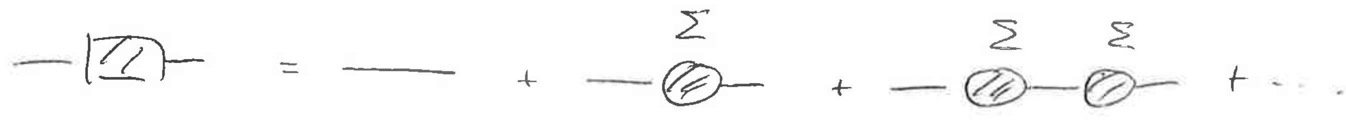
so that Feynman diagram (or an amplitude in certain cases) can be restored from its discontinuity up to a polynomial in external kinematic invariants.

Finally, we are in position to discuss one other usefull application of the optical theorem — the width of unstable particles.

Consider a two-point function  =

$= \langle 0|T\varphi(x)\varphi(0)|0\rangle$ in the interacting theory.

We can isolate "one-particle irreducible" contributions and write



$$\Rightarrow \int \langle 0 | T\varphi(x)\varphi(0) | 0 \rangle \, d^4x \, e^{ipx} \equiv \frac{i}{p^2 - m^2 - \Sigma(p^2)}$$

We will discuss what exactly is "m" in the full theory later on. For now, let's imagine that

$$\mathrm{Re}\left[\Sigma(m^2)\right] = 0 \quad \text{and} \quad \mathrm{Im}\left[\Sigma(m^2)\right] \neq 0.$$

Note that $\Sigma(p)$ for $p^2 = m^2$ can be thought of as a matrix element for $1_{\vec p} \to 1_{\vec p}$ scattering. $M(\vec p \to \vec p) = -\Sigma(p^2)$

~~By~~ Let's write $i\,\mathrm{Im}\left[\Sigma(p^2)\right]\Big|_{p^2=m^2} \equiv -im\Gamma \Rightarrow$

$$\Gamma = -\frac{1}{m}\,\mathrm{Im}\left[\Sigma(p^2)\right]\Big|_{p^2 \approx m^2} = -\frac{1}{2m}$$

$$\boxed{\Gamma = \frac{1}{m}\,\mathrm{Im}\left(M(\vec p \to \vec p)\right) \equiv \frac{1}{2m}\sum_f \int d\Pi_f \, |M(p \to f)|^2},$$

where the last step follows from the unitarity of the S-matrix. Hence, for $p^2 \approx m^2$, the two point function has a Breit-Wigner form $\dfrac{i}{p^2 - m^2 + im\Gamma}$ and Γ is what we call the "particle width". It is related to the lifetime of ~~an~~ unstable particle by $\tau = 1/\Gamma$.