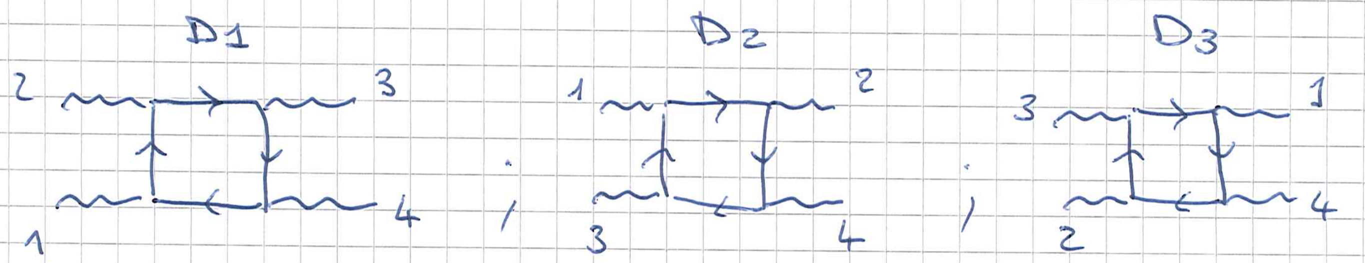
Ex sheet 4

TTP 2

Light-by-Light Scattering

we consider the process $\gamma(p_1)+\gamma(p_2)\to\gamma(p_3)+\gamma(p_4)$
which starts at 1-loop order in QED

$\boxed{1.}$ there are in total $\underline{\underline{6}}$ diagrams



plus the other 3 with loop momentum

of course only these three must be computed, since the other three give exactly the same contribution!

$\boxed{2.}$ the superficial degree of divergence in QED in d dimensions is

$$D = d + \left(\frac{d-4}{2}\right)V - \left(\frac{d-2}{2}\right)N_\gamma - \left(\frac{d-1}{2}\right)N_e$$

n° vertices

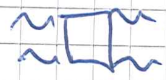
n° external photons

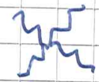
n° external electrons

in our case $V=4$, $N_\gamma=4$, $N_e=0$

$$D^{1loop}_{\gamma\gamma\to\gamma\gamma} = d + 2(d-4) - 2(d-2) = (d-4) \xrightarrow{\text{as } d\to 0} 0$$

D = 0 means potential logarithmic divergence!

of course the 1 loop amplitude  cannot be UV divergent since there is no tree-level amplitude!

In other words there is no vertex  which we might use to renormalize this amplitude!

$\boxed{3.}$ Let us take the FIRST diagram and consider it in the limit for the loop momentum $k \to \infty$. Use a cut-off Λ_{UV} to regulate UV divergence

$$D_1 \overset{k\to\infty}{\simeq} \int_0^{\Lambda_{UV}} dk\, k^3 \; \frac{\mathrm{Tr}\left[\not{k}\, \not{\varepsilon}_1\, \not{k}\, \not{\varepsilon}_2\, \not{k}\, \not{\varepsilon}_3\, \not{k}\, \not{\varepsilon}_4\right]}{k^2\, k^2\, k^2\, k^2}$$

$$\simeq \int_0^{\Lambda_{UV}} dk \frac{k^8}{k^8} \simeq \mathrm{Log}\,(\Lambda_{UV}) \;!$$

$\boxed{4.}$ Let us compute now fully the contribution from the three diagrams D_1, D_2, D_3

$$D_1 = \int^{\Lambda_{UV}} d^4k \; \frac{\mathrm{Tr}\left[\not{k}\, \gamma^{\mu_1}\, \not{k}\, \gamma^{\mu_2}\, \not{k}\, \gamma^{\mu_3}\, \not{k}\, \gamma^{\mu_4}\right]}{k^8} \left[\varepsilon_1^{\mu_1}\, \varepsilon_2^{\mu_2}\, \varepsilon_3^{\mu_3}\, \varepsilon_4^{\mu_4}\right]$$

$$D_2 = \int^{\Lambda_{UV}} d^4k \; \frac{\mathrm{Tr}\left[\not{k}\, \gamma^{\mu_3}\, \not{k}\, \gamma^{\mu_1}\, \not{k}\, \gamma^{\mu_2}\, \not{k}\, \gamma^{\mu_4}\right]}{k^8} \left[\ \cdots\ \right]$$

$$D_3 = \int^{\Lambda_{UV}} d^4k \; \frac{\mathrm{Tr}\left[\not{k}\, \gamma^{\mu_2}\, \not{k}\, \gamma^{\mu_3}\, \not{k}\, \gamma^{\mu_1}\, \not{k}\, \gamma^{\mu_4}\right]}{k^8} \left[\ \cdots\ \right]$$

in order to simplify the traces start off with D1

$$D_1 = \mathrm{Tr}\left[\gamma^{\mu_A}\gamma^{\mu_1}\gamma^{\mu_B}\gamma^{\mu_2}\gamma^{\mu_C}\gamma^{\mu_3}\gamma^{\mu_D}\gamma^{\mu_4}\right]\, \varepsilon_{1\mu_1}\varepsilon_{2\mu_2}\varepsilon_{3\mu_3}\varepsilon_{4\mu_4}$$

$$\times \int^{\Lambda_{UV}} d^4k \;\frac{k^{\mu_A}k^{\mu_B}k^{\mu_C}k^{\mu_D}}{k^8}$$

neglect this for now

now averaging over the solid angle in 4-dimensions gives:

$$\int^{\Lambda_{UV}} d^4k \;\frac{k^{\mu_A}k^{\mu_B}k^{\mu_C}k^{\mu_D}}{k^8} = \int^{\Lambda_{UV}} d^4k\,\frac{k^4}{24}\left(g^{\mu_A\mu_B}g^{\mu_C\mu_D} + g^{\mu_A\mu_C}g^{\mu_B\mu_D} + g^{\mu_A\mu_D}g^{\mu_C\mu_B}\right)$$

such that

$$D_1 = \left[\int^{\Lambda_{UV}} \frac{d^4k}{k^4}\right]\frac{1}{24}\Big\{ \mathrm{Tr}\left[\overbrace{\gamma^{\mu_B}\gamma^{\mu_1}\gamma^{\mu_B}}^{(*)}\gamma^{\mu_2}\overbrace{\gamma^{\mu_D}\gamma^{\mu_3}\gamma^{\mu_D}}^{(*)}\gamma^{\mu_4}\right] \quad (1)$$

$$+ \mathrm{Tr}\left[\overbrace{\gamma^{\mu_C}\gamma^{\mu_1}\gamma^{\mu_D}\gamma^{\mu_2}\gamma^{\mu_C}}^{(**)}\gamma^{\mu_3}\gamma^{\mu_D}\gamma^{\mu_4}\right] \quad (2)$$

$$+ \mathrm{Tr}\left[\underbrace{\gamma^{\mu_D}\gamma^{\mu_1}\overbrace{\gamma^{\mu_C}\gamma^{\mu_2}\gamma^{\mu_C}}^{(*)}\gamma^{\mu_3}\gamma^{\mu_D}}_{(*)}\gamma^{\mu_4}\right] \quad (3) \Big\}$$

use the relations

$$\gamma_\mu\gamma^\nu\gamma^\mu = -2\gamma^\nu \qquad (*)$$

$$\gamma_\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^\mu = -2\gamma^\sigma\gamma^\rho\gamma^\nu \qquad (**)$$

D_1 becomes

$$D_1 = \left[\int^{\Lambda_{UV}} \frac{d^4k}{k^4} \right] \frac{1}{3} \Big\{ \mathrm{Tr}\left[\gamma^{\mu_1} \gamma^{\mu_2} \gamma^{\mu_3} \gamma^{\mu_4} \right] \quad ① + ③$$

$$- \frac{1}{4} \mathrm{Tr}\left[\gamma^{\mu_2} \gamma^{\mu_0} \gamma^{\mu_1} \gamma^{\mu_3} \gamma_{\mu_0} \gamma^{\mu_4} \right] \quad ② \Big\}$$

$= +4 g^{\mu_1 \mu_3}$

$$= \left[\int^{\Lambda_{UV}} \frac{d^4k}{k^4} \right] \frac{1}{3} \Big\{ \mathrm{Tr}\left[\gamma^{\mu_1} \gamma^{\mu_2} \gamma^{\mu_3} \gamma^{\mu_4} \right]$$

$$- \mathrm{Tr}\left[\gamma^{\mu_2} \gamma_4 \right] g^{\mu_1 \mu_3} \Big\}$$

perform then the traces

$$\mathrm{Tr}\left[\gamma^{\mu_2} \gamma^{\mu_4} \right] = 4 g^{\mu_2 \mu_4}$$

$$\mathrm{Tr}\left[\gamma^{\mu_1} \gamma^{\mu_2} \gamma^{\mu_3} \gamma^{\mu_4} \right] = 4 \left(g^{\mu_1 \mu_2} g^{\mu_3 \mu_4} - g^{\mu_1 \mu_3} g^{\mu_2 \mu_4} + g^{\mu_1 \mu_4} g^{\mu_2 \mu_3} \right)$$

$$D_1 = \frac{4}{3} \left[\int^{\Lambda_{UV}} \frac{d^4k}{k^4} \right] \Big\{ g^{\mu_1 \mu_2} g^{\mu_3 \mu_4} - 2 g^{\mu_1 \mu_3} g^{\mu_2 \mu_4} + g^{\mu_1 \mu_4} g^{\mu_2 \mu_3} \Big\}$$

Doing some steps we get

$$D_2 = \frac{4}{3} \left[\int \frac{d^4k}{k^4} \right] \Big\{ g^{\mu_1 \mu_2} g^{\mu_3 \mu_4} + g^{\mu_1 \mu_3} g^{\mu_2 \mu_4} - 2 g^{\mu_1 \mu_4} g^{\mu_2 \mu_3} \Big\}$$

$$D_3 = \frac{4}{3} \left[\int \frac{d^4k}{k^4} \right] \Big\{ -2 g^{\mu_1 \mu_2} g^{\mu_3 \mu_4} + g^{\mu_1 \mu_3} g^{\mu_2 \mu_4} + g^{\mu_1 \mu_4} g^{\mu_2 \mu_3} \Big\}$$

Ex sheet 4

TTP II

Photon Vacuum Polarization

1. We start with the expression

$$\Pi_{\mu\nu}(q) = i\int dx\, e^{iq\cdot x} \langle 0|T\{J_\mu(x)J_\nu(0)\}|0\rangle$$

$$= i\int dx\, e^{iq\cdot x}\left[\langle 0|J_\mu(x)J_\nu(0)|0\rangle\theta(x_0) + \langle 0|J_\nu(0)J_\mu(x_0)|0\rangle\theta(-x_0)\right]$$

First, consider just the $\theta(x_0)$ term and insert the unity operator

$$\Rightarrow i\int dx\, e^{iq\cdot x}\sum_n \langle 0|J_\mu(x)|n\rangle\langle n|J_\nu(0)|0\rangle\,\theta(x_0)$$

$$= \sum_n i\int dx\, e^{i(q-p_n)\cdot x}\underbrace{\langle 0|J_\mu(0)|n\rangle\langle n|J_\nu(0)|0\rangle}_{A_{\mu\nu}}\theta(x_0)$$

the $\theta(-x_0)$ term gives

$$\sum_n i\int dx\, e^{i(q+p_n)\cdot x}\underbrace{\langle 0|J_\nu(0)|n\rangle\langle n|J_\mu(0)|0\rangle}_{B_{\nu\mu}}\theta(-x_0)$$

so we have

$$\Pi_{\mu\nu}(q) = i\sum_n\int dx\;A_{\mu\nu}e^{i(q-p_n)\cdot x}\theta(x_0) + B_{\nu\mu}e^{i(q+p_n)\cdot x}\theta(-x_0)$$

Now, we know that

$$\Pi_{\mu\nu}(q) = \Pi(q^2)(-g_{\mu\nu}q^2 + q_\mu q_\nu)$$

For convenience, we define $\Pi_1(q^2) = \Pi(q^2)q^2$ so that

$$\boxed{\Pi_{\mu\nu}(q) = \Pi_1(q^2)\left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right)}$$

To find an expression for $\Pi_1(q^2)$, we multiply both sides by $g^{\mu\nu}$

$$\Rightarrow \Pi_1(q^2)[-4+1] = i\sum_n\int dx\;Ae^{i(q-p_n)\cdot x}\theta(x_0) + Be^{i(q+p_n)\cdot x}\theta(-x_0)$$

where

$$\left.\begin{aligned}A &= \langle 0|J_\mu(0)|n\rangle\langle n|J^\mu(0)|0\rangle\\ B &= \langle 0|J^\mu(0)|n\rangle\langle n|J_\mu(0)|0\rangle\end{aligned}\right\} = |\langle 0|J^\mu(0)|n\rangle|^2$$

so

$$\Pi_1(q^2) = -\frac{i}{3}\sum_n|\langle 0|J^\mu(0)|n\rangle|^2 \times \int dx\, e^{i(q-p_n)\cdot x}\theta(x_0) + e^{i(q+p_n)\cdot x}\theta(-x_0)$$

Now, we need to simplify $\Pi_1(q^2)$. Just like in class, we introduce a δ-function

$$1 = \int\frac{d^4p}{(2\pi)^4}(2\pi)^4\delta^{(4)}(p-p_n)$$

so

$$\Pi_1(q^2) = -\frac{i}{3}\int\frac{d^4p}{(2\pi)^4}\sum_n|\langle 0|J^\mu(0)|n\rangle|^2(2\pi)^4\delta^{(4)}(p-p_n)$$
$$\times\int dx\, e^{i(q-p)\cdot x}\theta(x_0) + e^{i(q+p)\cdot x}\theta(-x_0)$$

The term $\sum_n|\langle 0|J^\mu(0)|n\rangle|^2(2\pi)^4\delta^{(4)}(p-p_n)$ is a function of p^2 that is non-vanishing for $p_0>0$. so we write

$$\sum_n |\langle 0|J^\mu(0)|n\rangle|^2 (2\pi)^4 \delta^4(p-p_n) \equiv \rho(p^2)\theta(p_0)$$

so we have

$$\Pi_1(q^2) = -\frac{i}{3}\int \frac{d^4p}{(2\pi)^4}\, \rho(p^2)\theta(p_0) \int dx \left\{ \theta(x_0) e^{i(q-p)\cdot x} + \theta(-x_0) e^{i(q+p)\cdot x} \right\}$$

The integral over x is done using

$$\int d^3\vec{x}\, e^{i\vec{k}\cdot\vec{x}} = (2\pi)^3 \delta^{(3)}(\vec{k}) \quad \text{and}$$

$$\int_{-\infty}^{0} dx_0\, e^{i\omega x_0} = \frac{i}{\omega + i0} \;; \quad \int_0^\infty dx_0\, e^{i\omega x_0} = \frac{-i}{\omega - i0}$$

so

$$\Pi_1(q^2) = -\frac{i}{3}\int \frac{d^4p}{(2\pi)^4}\, \rho(p^2)\theta(p_0) \left[(2\pi)^3 \delta^{(3)}(\vec{q}-\vec{p}) \frac{i}{q_0 - p_0 + i0} + (2\pi)^3 \delta^{(3)}(\vec{q}+\vec{p}) \frac{-i}{q_0 + p_0 - i0} \right]$$

note that $\rho(p^2)$ does not depend on the sign of $\vec{p}$ so we can change $\vec{p} \to -\vec{p}$ in the second term.

$$\Pi_1(q^2) = \frac{1}{3}\int \frac{d^4p}{(2\pi)^4}\, \rho(p^2)\theta(p_0) (2\pi)^3 \delta^{(3)}(\vec{q}-\vec{p}) \frac{2p_0}{q_0^2 - p_0^2 + i0}$$

$$= \frac{1}{3}\int \frac{d^4p}{(2\pi)^4}\, \rho(p^2)\theta(p_0) (2\pi)^3 \delta^{(3)}(\vec{q}-\vec{p}) \frac{2p_0}{q^2 - p^2 + i0}$$

Now, we can write $d^4p\,\theta(p_0)$ as

$$\frac{d^4p\,\theta(p_0)}{(2\pi)^4} = \frac{dp^2}{(2\pi)} \frac{d^3\vec{p}}{(2\pi)^3 2p_0}$$

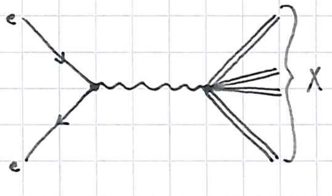
This allows us to integrate out the δ-function so

$$\Pi_1(q^2) = \frac{1}{3}\int \frac{dp^2}{(2\pi)} \frac{\rho(p^2)}{q^2 - p^2 + i0}$$

Finally, we have the Källen representation

$$\boxed{\Pi_{\mu\nu}(q) = \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right) \times \frac{1}{3}\int \frac{dp^2}{(2\pi)} \frac{\rho(p^2)}{q^2 - p^2 + i0}}$$

2. The process we consider is given by



The matrix element is

$$i\mathcal{M} = \bar{u}_{e^+} (-ie\gamma^\mu) u \left(\frac{-ig_{\mu\nu}}{q^2}\right) (-ie) \langle 0|J^\nu(0)|X\rangle$$

$$\Rightarrow \mathcal{M} = e^2 \bar{u}_{e^+} \gamma^\mu u_e \frac{1}{q^2} \langle 0|J_\mu(0)|X\rangle$$

3) The cross-section looks like

average over spin-$\frac{1}{2}$ initial states

$$\sigma_{e^+e^-\to X} = \frac{1}{4E_{e^+}E_{e^-}|v_{e^+}-v_{e^-}|} \sum_X |M(e^+e^-\to X)|^2 \delta^{(4)}(p-p^X)(2\pi)^4 \cdot \left(\frac{1}{2}\right)^2$$

$$= \frac{e^4}{4E_{e^+}E_{e^-}|v_{e^+}-v_{e^-}|} \sum_X \frac{L^{\mu\nu}(p_1,p_2)}{p^4} \langle 0|j^\mu(0)|X\rangle\langle X|j^\nu(0)|0\rangle \delta^{(4)}(p-p^X)\frac{(2\pi)^4}{4}$$

In c.o.m. frame $E_{e^+}=E_{e^-}=\frac{\sqrt{p^2}}{2}$, $|v_{e^+}-v_{e^-}|=2$. Sum Σ_X represents integration over final state phase space $\sum_X = \sum_{x_i}\int \frac{dp_i^x}{2E_i(2\pi)^3}$.

We can re-write $\sigma_{e^+e^-\to X}$ as:

$$\sigma_{e^+e^-\to X} = \frac{e^4}{2p^6\cdot 4} L^{\mu\nu}(p_1,p_2)\, A(p^2)\left(-g^{\mu\nu}p^2+p^\mu p^\nu\right)$$

now, $L^{\mu\nu}p_\mu = L^{*\nu\mu}p_\mu = 0$ so $\sigma = \frac{e^4}{2p^6\cdot 4}L^{\mu\nu}(p_1,p_2)A(p^2)(-g^{\mu\nu}p^2)$ (*)

we used $g_{\mu\nu}\,\Big|\; A(p^2)(-g^{\mu\nu}p^2+p^\mu p^\nu) = \sum_X \langle 0|j^\mu(0)|X\rangle\langle X|j^\nu(0)|0\rangle(2\pi)^4\delta^{(4)}(p-p^X)$

$\Rightarrow A(p^2)(-3p^2)=f(p^2)$ since r.h.s. corresponds to $f(p^2)$ with $X\to n$.

Solving (*) gives $A(p^2) = \frac{2\sigma p^6\cdot 4}{L^{\mu\nu}(-g_{\mu\nu}p^2)e^4} \Rightarrow f(p^2) = \sigma_{e^+e^-\to X}\frac{6p^6\cdot 4}{(L^{\mu\nu}g_{\mu\nu})e^4}$

This gives: $\Pi_{\mu\nu}(q) = \left(-g_{\mu\nu}+\frac{q_\mu q_\nu}{q^2}\right)\frac{1}{3}\int\frac{dp^2}{(2\pi)}\frac{4\cdot 6p^6}{(L^{\mu\nu}g_{\mu\nu})e^4}\frac{\sigma_{e^+e^-\to X}}{q^2-p^2+i0}$ (**)

4) Consider first cross-section $\sigma_{e^+e^-\to\mu^+\mu^-}$ massless $e^\pm,\mu^\pm$: $\frac{d\sigma}{d\Omega} = \frac{(\frac{1}{2})^2|M|^2}{64\pi^2p^2} = \frac{e^4}{64\pi^2p^2}\left\{L^{\mu\nu}\frac{1}{p^4}\mathcal{L}_{\mu\nu}\right\}\cdot\left(\frac{1}{2}\right)^2$. For $2\to2$ process with where $(\frac{1}{2})^2$ is the averaging over spin-$\frac{1}{2}$ initial (unpolarized) states and

$\mathcal{L}_{\mu\nu} = (\bar{u}_{\mu^-}(p_3)\gamma_\mu u_{\mu^+}(p_4))(\bar{u}_{\mu^+}(p_4)\gamma_\nu u_{\mu^-}(p_3)) = \mathrm{Tr}[\not{p}_3\gamma_\mu\not{p}_4\gamma_\nu] = 4(p_{3\mu}p_{4\nu}+p_{3\nu}p_{4\mu}\underbrace{\;}_{-2g_{\mu\nu}p_3\cdot p_4}$

We can play the same trick as we did for $\sigma_{e^+e^-\to X}$, namely we write $\int d\Omega\,\mathcal{L}_{\mu\nu} = \int d\Omega\, B(p^2)(p^\mu p^\nu - p^2g^{\mu\nu}) \Rightarrow \mathcal{L}_{\mu\nu} = \frac{4}{3}((p_{3\mu}+p_{4\mu})(p_{3\nu}+p_{4\nu})-(p_3+p_4)^2g_{\mu\nu})$ w.l.o.g.

Current conserv. implies $(p_3+p_4)_\mu L^{\mu\nu} = (p_1+p_2)_\mu L^{\mu\nu}=0 \Rightarrow \frac{d\sigma}{d\Omega} = \frac{e^4(4/3)}{4\cdot 64\pi^2p^2}\frac{(-g^{\mu\nu}L_{\mu\nu})p^2}{p^4}$

$\Rightarrow \sigma_{e^+e^-\to\mu^+\mu^-} = \frac{e^4(-g^{\mu\nu}L_{\mu\nu})\frac{4}{3}}{4\cdot 64\pi^2p^4}\int d\Omega = \frac{e^4(-g^{\mu\nu}L_{\mu\nu})}{64\pi p^4}\cdot\frac{4}{3}$. The factor $\frac{4}{3}$ follows same as above from: $\mathcal{L}^{\mu\nu} = B(p^2)(p^\mu p^\nu - p^2g^{\mu\nu}) \Rightarrow B(p^2) = \frac{1}{-3p^2}\mathrm{Tr}[\not{p}_3\gamma_\mu\not{p}_4\gamma^\mu] = \frac{4}{3}$.

We find $e^4\mathcal{L}^{\mu\nu}g_{\mu\nu} = -16\pi\cdot 3p^4\sigma_{\text{point}} \Rightarrow$ plugging into (**):

$$\Pi_{\mu\nu}(q) = \left(-g_{\mu\nu}+\frac{q_\mu q_\nu}{q^2}\right)\cdot\frac{1}{3}\int\frac{dp^2}{(2\pi)}\frac{p^2}{(2\pi)}\frac{R(p^2)}{p^2-q^2-i0}$$

So we have $\Pi_1(q^2) = \int \frac{dp^2}{12\pi^2} \frac{p^2 R(p^2)}{p^2 - q^2 - i.0}$

Note that $\Pi_1(q^2 \to 0) = 0$ otherwise the photon would acquire a mass. This is not true for $\Pi_1(q^2)$. Therefore we define

$$\Pi_1^S(q^2) = \Pi_1(q^2) - \Pi_1(0) = \int \frac{dp^2}{12\pi^2} p^2 R(p^2) \left[\frac{1}{p^2 - q^2 - i.0} - \frac{1}{p^2 - i.0} \right]$$

$$= q^2 \int \frac{dp^2}{12\pi^2} \frac{R(p^2)}{p^2 - q^2 - i.0}$$

This is now suitable for $\Pi_{\mu\nu}(q)$. We have finally:

$$\Pi_{\mu\nu}(q) = (-g_{\mu\nu} q^2 + q_\mu q_\nu) \int \frac{dp^2}{12\pi^2} \frac{R(p^2)}{p^2 - q^2 - i.0}$$