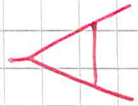
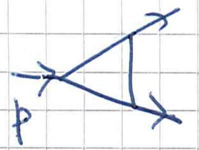
Ex sheet 3 - Discontinuities of  TOP II

Consider the triangle



$p_1^2 = m_1^2$

$p_2^2 = m_2^2$

$$= \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2(k-p_1)^2(k-p_1-p_2)^2}$$

$$= T(p^2, m_1^2, m_2^2)$$

use Feynman parameters

$$T(p^2, m_1^2, m_2^2) = 2\int_0^1 dx \int_0^{1-x} dy \underbrace{\int \frac{d^4k}{(2\pi)^4} \frac{1}{\left[k^2(1-x-y) + (k-p_1)^2 x + (k-p_1-p_2)^2 y\right]^3}}_{A}$$

$$A = \int \frac{d^4k}{(2\pi)^4} \frac{1}{\left[(k-p_1x-p_2y)^2 + m_1^2 x(1-x) + p^2 y(1-y)\right]^3}$$

with $p_{12} = p_1 + p_2$

$$\overset{\text{shift}}{=} \int \frac{d^4\ell}{(2\pi)^4} \frac{1}{\left[\ell^2 + \underbrace{m_1^2 x(1-x) + p^2 y(1-y)}\right]^3}$$

$0 < x < 1$

$0 < y < 1$

EUCLIDEAN:

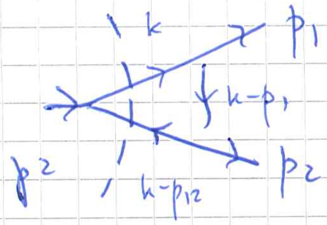
$$\begin{cases} M_1^2 = -m_1^2 > 0 \\ P^2 = -p^2 > 0 \end{cases}$$

$$\xrightarrow[\text{Rotation}]{\text{Wick}} \sim \int \frac{d^4\ell_E}{(2\pi)^4} \frac{1}{\left[\ell_E^2 + M_1^2 x(1-x) + P^2 y(1-y)\right]^3}$$

$$\xrightarrow[M_1^2]{\text{rescaling}} \sim \int \frac{d^4\ell_E}{(2\pi)^4} \frac{1}{\left[\ell_E^2 + x(1-x) + \frac{P^2}{M_1^2} y(1-y)\right]^3} \in \mathrm{Real}$$

if P^2, M_1^2 same SIGN!!!

2. let us use now Cutkosky rule in order to write the discontinuities



cut propagator $\frac{1}{q^2} \to (2\pi)\vartheta(q_0)\,\delta(q^2)$

$$\mathrm{Disc}(p^2, m_1^2, m_2^2) = \int \frac{d^4k}{(2\pi)^4} \frac{(2\pi)^2 \vartheta(k_0)\,\vartheta(E_{12}-k_0)\,\delta(k^2)\,\delta((p_{12}-k)^2)}{(k-p_1)^2}$$

let us work in reference frame where $p^2 \to (p_1+p_2)^2 = s$

$p_1 + p_2 = (\sqrt{s}, \vec{0})$ REST

$$= \int \frac{d^3k}{(2\pi)^2} \int_0^{\sqrt{s}} dk_0 \frac{\delta(k_0^2 - |\vec{k}|^2)\,\delta(s + k^2 - 2p_{12}\cdot k)}{(k^2 + p_1^2 - 2k\cdot p_1)} \qquad p_1^2 = m_1^2$$

in our reference frame we have $\boxed{p_{12}\cdot k = \sqrt{s}\,k_0}$

moreover using the δ function we have $k^2 = 0$

$k\cdot p_1 = k_0 E_1 - |\vec{k}||\vec{p}_1|\cos\vartheta$

$$= \frac{1}{(2\pi)} \int_{-1}^{1} dz \int_0^{\sqrt{s}} dk_0 \int_0^{\infty} d|\vec{k}|\,|\vec{k}|^2 \frac{\delta(k_0^2 - |\vec{k}|^2)\,\delta(s - 2\sqrt{s}\,k_0)}{(m_1^2 - 2k_0E_1 + 2|\vec{k}||\vec{p}_1|z)}$$

integrate in dk_0 $\quad \delta(k_0^2 - |\vec{k}|^2) = \frac{1}{2|\vec{k}|}\delta(k_0 - |\vec{k}|)$ $\quad k_0$ must be positive

$|\vec{k}| < \sqrt{s}$

the other δ gives $k_0 = \frac{s}{2\sqrt{s}}$

$$= \frac{1}{(2\pi)} \int_{-1}^{1} dz \int_{0}^{\sqrt{s}} \frac{d|k|\,|k|^{\cancel{2}}}{2\cancel{|k|}} \frac{\delta(s - 2\sqrt{s}\,|k|)}{(m_1^2 - 2|k|E_1 + 2|k||p_1| z)}$$

now integrate second $\delta \Rightarrow \frac{1}{2\sqrt{s}}$ and $\boxed{|k| = \frac{\sqrt{s}}{2}}$

$$= \frac{1}{4\pi} \int_{-1}^{1} dz \frac{1}{2\sqrt{s}} \left(\frac{\cancel{\sqrt{s}}}{2}\right) \frac{1}{\left(m_1^2 - 2\frac{\sqrt{s}}{\cancel{2}} E_1 + 2\frac{\sqrt{s}}{\cancel{2}}|p_1| z\right)}$$

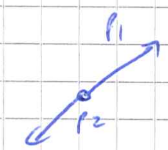
$$= \frac{1}{16\pi} \int_{-1}^{1} dz \frac{1}{\sqrt{s}\,|p_1|} \frac{1}{\left(z + \frac{m_1^2 - \sqrt{s}E_1}{\sqrt{s}\,|p_1|}\right)}$$

$$= \frac{1}{16\pi\sqrt{s}\,|p_1|} \ln\left[\frac{1 + \frac{m_1^2 - \sqrt{s}E_1}{\sqrt{s}\,|p_1|}}{-1 + \frac{m_1^2 - \sqrt{s}E_1}{\sqrt{s}\,|p_1|}}\right]$$ ← this will require a change of sign.

$$= \frac{1}{16\pi\sqrt{s}\,|p_1|} \ln\left[\frac{\sqrt{s}\,|p_1| + m_1^2 - \sqrt{s}E_1}{-\sqrt{s}\,|p_1| + m_1^2 - \sqrt{s}E_1}\right]$$

Some kinematics in the rest frame

$p = p_1 + p_2 = (\sqrt{s}, 0)$



$p^2 = s = m_1^2 + m_2^2 + 2E_1E_2 + 2|p_1||p_2| = (E_1+E_2)^2$ no momentum!

$|p_1| = |p_2|$ momentum conservation

$$s = m_1^2 + m_2^2 + 2|p_1|^2 + 2\sqrt{|p_1|^2 + m_1^2}\,\sqrt{|p_1|^2 + m_2^2}$$

$$\sqrt{s} = E_1 + E_2 \qquad E_1^2 - m_1^2 = E_2^2 - m_2^2 \longleftarrow \text{momentum conservation}$$

$$E_2^2 = E_1^2 - m_1^2 + m_2^2$$

$$(\sqrt{s} - E_1)^2 = E_2^2 = E_1^2 - m_1^2 + m_2^2$$

$$-2\sqrt{s}E_1 + \cancel{E_1^2} + s = \cancel{E_1^2} - m_1^2 + m_2^2 \Rightarrow \begin{cases} E_1 = \dfrac{s + m_1^2 - m_2^2}{2\sqrt{s}} \\[2ex] E_2 = \dfrac{s + m_2^2 - m_1^2}{2\sqrt{s}} \end{cases}$$

$$|\vec{p}_1| = |\vec{p}_2| = \sqrt{E_1^2 - m_1^2} = \sqrt{E_2^2 - m_2^2} = \sqrt{\frac{(s + m_1^2 - m_2^2)^2}{4s} - m_1^2}$$

$$= \frac{1}{2\sqrt{s}} \sqrt{s^2 + (m_1^2 - m_2^2)^2 + 2s(m_1^2 - m_2^2) - 4sm_1^2}$$

$$\mu_{12} = m_1 + m_2$$

$$\tilde{\mu}_{12} = m_1 - m_2$$

$$= \frac{1}{2\sqrt{s}} \sqrt{s^2 + \mu_{12}^2 \tilde{\mu}_{12}^2 - 2s(m_1^2 + m_2^2)}$$

note now $\mu_{12}^2 + \tilde{\mu}_{12}^2 = 2(m_1^2 + m_2^2)$

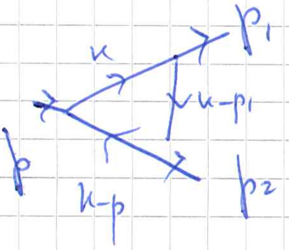
$$= \frac{1}{2\sqrt{s}} \sqrt{s + \mu_{12}^2 \tilde{\mu}_{12}^2 - s\mu_{12}^2 - s\tilde{\mu}_{12}^2} = \frac{1}{2\sqrt{s}} \sqrt{(s - \mu_{12}^2)(s - \tilde{\mu}_{12}^2)}$$

$$\boxed{s = p^2}$$

putting this in our formula we get

$$\text{Disc}(p^2, m_1^2, m_2^2) = \frac{1}{8\pi} \frac{1}{\sqrt{(p^2 - \mu_{12}^2)(p^2 - \tilde{\mu}_{12}^2)}} \ln\left(\frac{p^2 - m_1^2 - m_2^2 - \sqrt{(p^2 - \mu_{12}^2)(p^2 - \tilde{\mu}_{12}^2)}}{p^2 - m_1^2 - m_2^2 + \sqrt{(p^2 - \mu_{12}^2)(p^2 - \tilde{\mu}_{12}^2)}} \right)$$

3. let us now have a look at the other cut



$$\text{Disc}(m_1^2, p_1^2, m_2^2) = \int \frac{d^4k}{(2\pi)^4} (2\pi)^2 \frac{\vartheta(k_0)\,\vartheta(E_1 - k_0)\,\delta(k^2)\,\delta((p_1-k)^2)}{(k-p)^2}$$

use notation $\boxed{p = p_{12} = p_1 + p_2}$

Reference frame $p_1^\mu = (m_1, \vec{0})$

p_1^μ is at rest!

$p_1^\mu = p^\mu - p_2^\mu = (m_2, \vec{0})$!!

as before!

$$\text{Disc}(m_1^2, p_1^2, m_2^2) = \frac{1}{(2\pi)} \int_{-1}^{1} dz \int_0^{m_1} dk_0 \int_0^{\infty} d|\vec{k}|\,|\vec{k}|^2 \frac{\delta(k^2)\,\delta(\cancel{k^2} + m_1^2 - 2k\cdot p_1)}{(\cancel{k^2} + p^2 - 2k\cdot p)}$$

$$= \frac{1}{(2\pi)} \int_{-1}^{+1} dz \int_0^{\infty} d|\vec{k}|\,|\vec{k}|^2 \int_0^{m_1} dk_0 \frac{\frac{1}{2|\vec{k}|}\cancel{\delta(k_0^2 - |\vec{k}|^2)}\,\delta(m_1^2 - 2k_0 m_1)}{(p^2 - 2k_0 p_0 + 2|\vec{k}|\,|\vec{p}|\,z)} \quad \{k_0 = |\vec{k}|$$

$$= \frac{1}{(2\pi)} \int_{-1}^{1} dz \int_0^{\infty} d|\vec{k}|\,\frac{|\vec{k}|}{2} \frac{\cancel{\delta(m_1^2 - 2m_1|\vec{k}|)}}{(p^2 - 2|\vec{k}|p_0 + 2|\vec{k}|\,|\vec{p}|\,z)} \quad \frac{1}{2m_1}; \quad |\vec{k}| = \frac{m_1}{2}$$

$$= \frac{1}{8\pi} \int_{-1}^{1} dz \cancel{\int_0^{\infty} d|\vec{k}|} \frac{\cancel{m_1}}{2\cancel{m_1}} \frac{1}{(p^2 - m_1 p_0 + m_1|\vec{p}|\,z)}$$

$$= \frac{1}{16\pi} \frac{1}{m_1|\vec{p}|} \int_{-1}^{1} dz \frac{1}{\left(z + \frac{p^2 - m_1 p_0}{m_1|\vec{p}|}\right)}$$

$$= \frac{i}{16\pi m_1 |\vec{p}|} \ln\left(\frac{m_1|\vec{p}| + p^2 - m_1 p_0}{-m_1|\vec{p}| + p^2 - m_1 p_0}\right)$$

now kinematics as before

$$p_1^\mu = (m_1, \vec{0}) = (p - p_2)^\mu = (E - E_2)$$

because total $|\vec{p}_0 - \vec{p}_2|$ must be zero!

$$\boxed{m_1 = E - E_2} \ *$$

plus momentum conservation $|\vec{p}| = |\vec{p}_2| \Rightarrow \boxed{E^2 - p^2 = E_2^2 - m_2^2}$ $**$

using $*$ and $**$ we get $E = \dfrac{p^2 + m_1^2 - m_2^2}{2m_1}$.

$$|\vec{p}| = \sqrt{E^2 - p^2} = \sqrt{\frac{(p^2 + m_1^2 - m_2^2)^2 - 4m_1^2 p^2}{4m_1^2}}$$ as before

$$= \frac{1}{2m_1}\sqrt{(p^2 - \mu_{12}^2)(p^2 - \tilde{\mu}_{12}^2)}$$ identical

$$= \frac{1}{8\pi\sqrt{(p^2 - \mu_{12}^2)(p^2 - \tilde{\mu}_{12}^2)}} \ln\left(\frac{p^2 + m_2^2 - m_1^2 + \sqrt{(p^2 - \mu_{12}^2)(p^2 - \tilde{\mu}_{12}^2)}}{p^2 + m_2^2 - m_1^2 - \sqrt{(p^2 - \mu_{12}^2)(p^2 - \tilde{\mu}_{12}^2)}}\right)$$

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last cut is trivial

$$\mathrm{Disc}(m_2^2, p^2, m_1^2) = \frac{1}{8\pi\sqrt{(\ )(\ )}} \ln\left(\frac{p^2 + m_1^2 - m_2^2 + \sqrt{(\ )(\ )}}{p^2 + m_1^2 - m_1^2 + \sqrt{(\ )(\ )}}\right)$$

taking the sum of the three cuts one gets

$$\mathrm{Disc}_1 + \mathrm{Disc}_2 + \mathrm{Disc}_3 = \frac{1}{8\pi\sqrt{(p^2-\mu_{12}^2)(p^2-\tilde{\mu}_{12}^2)}} \quad (\times)$$

$$\times \ln\left\{\left(\frac{p^2-m_1^2-m_2^2-\sqrt{}}{p^2-m_1^2-m_2^2+\sqrt{}}\right)\left(\frac{p^2+m_2^2-m_1^2+\sqrt{}}{p^2+m_2^2-m_1^2-\sqrt{}}\right)\left(\frac{p^2+m_1^2-m_2^2+\sqrt{}}{p^2+m_1^2-m_2^2-\sqrt{}}\right)\right\}$$

$$\cancel{\ln\frac{(p^2-m_1^2)^2+(m_2^2+\sqrt{})^2}{(p^2-m_1^2)^2-(m_2^2-\sqrt{})^2}\;\frac{p^2+m_1^2-m_2^2+\sqrt{}}{p^2-m_1^2+m_2^2-\sqrt{}}}$$

$$\cancel{\frac{(p^2-a-b-c)(p^2-a+b+c)(p^2+a-b+c)}{(p^2-a-b+c)(p^2-a+b-c)(p^2+a-b-c)}}$$

$$(p^2-m_1^2-m_2^2-\sqrt{})(p^2-m_1^2-m_2^2+\sqrt{}) = 4m_1^2m_2^2$$

$$(p^2-m_1^2+m_2^2-\sqrt{})(p^2-m_1^2+m_2^2+\sqrt{}) = 4\,m_2^2\,p^2$$

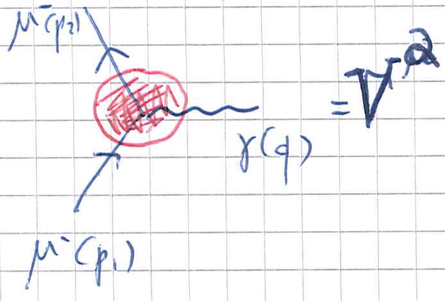
$$(p^2+m_1^2-m_2^2-\sqrt{})(p^2+m_1^2-m_2^2-\sqrt{}) = 4\,m_1^2p^2$$

$$(\quad -\sqrt{})(\quad +\sqrt{})(\quad +\sqrt{}) = 8\,m_1^2m_2^2p^2$$ numerator

$$(\quad +\sqrt{})(\quad -\sqrt{})(\cdots -\sqrt{}) = 8\,m_1^2m_2^2p^2$$ denominator

Ex sheet 3 – Leading Hadronic contribution to a_μ

In this exercise we will show how to estimate the hadronic contributions to the muon anomalous magnetic moment by exploiting <u>Dispersion Relations</u>.



$= V^\alpha$ $\quad \mu\mu\gamma^*$ interaction vertex

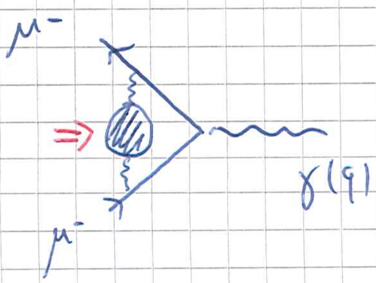
Lorentz symmetry and gauge invariance imply that

$$V^\alpha(q) = -ie\,\bar{u}(p_2)\Big[\underbrace{F_D(q^2)}_{\text{Dirac Form Factor}}\gamma^\alpha + \underbrace{F_P(q^2)}_{\text{Pauli Form Factor}}\frac{i\sigma^{\alpha\beta}q_\beta}{2m}\Big]u(p_1)$$

The anomalous magnetic moment is defined as

$$\boxed{a_\mu = F_P(q^2=0)}$$

Hadronic contribution can be depicted as:



where

$$\underbrace{\frac{i}{k^2}\;\frac{i}{k^2} = \frac{ie^2}{(k^2)^2}\Pi(k^2)\left[g^{\mu\nu}k^2 - k^\mu k^\nu\right]}_{\text{Because of Gauge invariance}}$$

$\rightarrow$ TRANSVERSALITY

we will see in next ex-sheet that $\Pi(k^2)$ satisfies a DISPERSION RELATION

$$\Pi(k^2) = \frac{k^2}{\pi}\int_{s_0}^{\infty} ds \, \frac{Im(\Pi(s))}{s(s-k^2-i0)} = \frac{k^2}{12\pi^2}\int_{s_0}^{\infty} ds \, \frac{R^{hadr}(s)}{s(s-k^2-i0)}$$

where by definition $\boxed{R^{hadr}(s) = \frac{\sigma_{e^+e^-\to hadrons}(s)}{\sigma_{point}}}$

$$\sigma_{point} = \frac{4\pi\alpha^2}{3s}$$

$\sigma_{e^+e^-\to hadr}(s)$ = cross-section to produce hadrons in e^+e^- annihilation

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Let us write down the expression for the diagram:

$$\Gamma^\alpha = \int \frac{d^dk}{(2\pi)^d}\, \bar{u}(p_2)(-ie\gamma_\nu)\left(i\frac{(\not p_2+\not k+m)}{(p_2+k)^2-m^2}\right)\gamma^\alpha\left(i\frac{\not p_1+\not k+m}{(p_1+k)^2-m^2}\right)(-ie\gamma_\mu)\,u(p_1)$$

$$\times\left[i\frac{e^2}{(k^2)^2}\Pi(k^2)\,(g^{\mu\nu}k^2-k^\mu k^\nu)\right]$$

and substituting Disp Relation

$$= -i\frac{e^4}{12\pi^2}\int\frac{d^dk}{(2\pi)^d}\int_{s_0}^{\infty}\frac{ds}{s}\,\frac{\bar{u}_{p_2}\gamma_\nu(\not p_2+\not k+m)\gamma^\alpha(\not p_1+\not k+m)\gamma_\mu u_{p_1}(g^{\mu\nu}k^2-k^\mu k^\nu)}{k^2(k^2-s)((p_2+k)^2-m^2)((p_1+k)^2-m^2)}R^{hadr}(s)$$

we want to simplify numerator and to extract the FORM FACTORS F_P and F_D

Let us consider the $k^\mu k^\nu$ term FIRST.

$$\bar{u}(p_2)\, \not{k}\, (\not{p}_2 + \not{k} + m)\, \gamma^\alpha\, (\not{p}_1 + \not{k} + m)\, \not{k}\, u(p_1)$$

anticommuting and using Dirac equation $\begin{cases} \bar{u}(p_2)(\not{p}_2 - m) = 0 \\ (\not{p}_1 - m)\, u(p_1) = 0 \end{cases}$

$\Downarrow$

$$\bar{u}(p_2)\left[(k^2 + 2k\cdot p_2)\,\gamma^\alpha\,(k^2 + 2k\cdot p_1)\right] u(p_1) \sim C\, \bar{u}_{p_2}\gamma^\alpha u_{p_1}$$

such that this term contributes only to $F_D(q^2)$ and we can throw it away!

Now the term in $g^{\mu\nu}k^2$ becomes

$$\bar{u}(p_2)\,\gamma_\nu\,(\not{p}_2 + m + \not{k})\,\gamma^\alpha\,(\not{p}_1 + \not{k} + m)\,\gamma_\nu\, u(p_1)\, k^2 =$$

$$= \bar{u}_{p_2}\Big\{ \gamma_\nu (\not{p}_2 + m)\gamma^\alpha(\not{p}_1 + m)\gamma^\nu \quad (1)$$

$$+ \gamma_\nu \not{k} \gamma^\alpha (\not{p}_1 + m)\gamma^\nu \quad (2)$$

$$+ \gamma_\nu (\not{p}_2 + m)\gamma^\alpha \not{k} \gamma^\nu \quad (3)$$

$$+ \gamma_\nu \not{k} \gamma^\alpha \not{k} \gamma^\nu \quad (4) \Big\}\, u_{p_1}\, k^2$$

Let us consider the 4 pieces separately, again anticommuting the γ^μ and using Dirac equation repeatedly anticommute

① $\bar{u}(p_2)\gamma_\nu(\not{p}_2+m)\gamma_\alpha(\not{p}_1+m)\gamma^\nu u(p_1)\, k^2$

$= 4\, p_1\cdot p_2\, k^2\, \bar{u}(p_2)\gamma^\alpha u(p_1) \Rightarrow F_D(q^2)$ THROW AWAY!

anticommute

② $\bar{u}(p_2)\gamma_\nu \not{k}\gamma^\alpha(\not{p}_1+m)\gamma^\nu u(p_1)$

+ Dirac Equation
$(\not{p}_1 - m)u(p_1) = 0$

anticommute

$= 2k^2\, \bar{u}(p_2)\,[\not{p}_1 \not{k}\gamma^\alpha]\, u(p_1)$

$= \dots = 2k^2\, \bar{u}(p_2)[\underbrace{2k\cdot p_1\gamma^\alpha}_{F_D(q^2),\ \text{throw away}} \underbrace{- 2\not{k}p_1^\alpha + \not{k}m\gamma^\alpha}_{\text{these stay!}}]u(p_1)$

$= 2k^2\, \bar{u}(p_2)\not{k}\,[m\gamma^\alpha - 2p_1^\alpha]\, u(p_1)$

③ Same thing but on $(\not{p}_2+m)$ on the left

$\bar{u}(p_2)[\gamma_\nu(\not{p}_2+m)\gamma^\alpha\not{k}\gamma^\nu]u(p_1)\, k^2$

$= \dots = 2k^2\, \bar{u}(p_2)[m\gamma^\alpha - 2p_2^\alpha]u(p_1)$] again throwing away pieces that contribute to $F_D(q^2)$ only!

④ $\bar{u}(p_2)\gamma_\nu\not{k}\gamma^\alpha\not{k}\gamma^\nu u(p_1)\, k^2$

here using $\gamma_\nu\gamma_\delta\gamma^\alpha\gamma_\sigma\gamma^\nu = -2\gamma_\sigma\gamma^\alpha\gamma_\delta + (4-d)\gamma_\delta\gamma^\alpha\gamma_\sigma$

$\to = \dots = (2-d)\,\bar{u}(p_2)[2k^\alpha\not{k} - \cancel{\gamma^\alpha k^2}]u(p_1)\, k^2$ $\xrightarrow{F_D(q^2)}$ throw away in total summing everything we are left with

$$\Gamma^{\alpha} = -\frac{ie^4}{6\pi^2}\int_{s_0}^{\infty}\frac{ds}{s}R^{hadr}(s)\int\frac{d^dk}{(2\pi)^d}\underbrace{\frac{\bar{u}_{p2}\left[(2-d)k^{\alpha}\not{k}+2mk^{\alpha}-2\not{k}(p_1+p_2)^{\alpha}\right]u_{p1}}{(k^2-s)\left[k^2+2p_1\cdot k\right]\left[k^2+2p_2\cdot k\right]}}_{\text{use here Feynman Parameters}}$$

notice that in the limit $q^2 \to 0$ $\quad p_2^{\alpha} \to p_1^{\alpha}$ such that

$$\sim \int\frac{d^dk}{(2\pi)^d}\frac{N^{\alpha}}{(k^2-s)\left[k^2+2p_1\cdot k\right]^2} =$$

$$= \int_0^1 dx\int\frac{d^dk}{(2\pi)^d}\frac{2x\,N^{\alpha}}{\left[k^2+2p_1\cdot k\,x-s(1-x)\right]^3}$$

with usual $\boxed{l^{\mu} = k^{\mu}+p_1^{\mu}x}$

$$= \int_0^1 dx\int\frac{d^dl}{(2\pi)^d}\frac{2x\,N^{\alpha}}{\left[l^2-\Delta\right]^3}$$

$\Delta = s(1-x)+m^2x^2$

Rewrite now numerator in l^{μ}!

$$\bar{u}(p_2)\left[(2-d)(l^{\alpha}-p_1^{\alpha}x)(\not{l}-\not{p}_1x)+2m(l^{\alpha}-p_1^{\alpha}x)-2(\not{l}-\not{p}_1x)(p_1+p_2)^{\alpha}\right]u(p_1)$$

(*) ODD POWERS OF l^{μ} integrate to zero by symmetry

$$\int\frac{d^dl}{(2\pi)^d}\frac{l^{\mu}}{(l^2-\Delta)^3} = 0 \quad !$$

(*)(*) use also

$$\int\frac{d^dl}{(2\pi)^d}\frac{l^{\mu}l^{\nu}}{(l^2-\Delta)^3} = \frac{1}{d}g^{\mu\nu}\int\frac{d^dl}{(2\pi)^d}\frac{l^2}{(l^2-\Delta)^3}$$

this gives for numerator

$$N^\alpha = \bar{u}(p_2)\left[\underbrace{\cancel{\frac{2-d}{d}\ell^2\gamma^\alpha}} + (p_1+p_2)^\alpha\, m x\left(\frac{2-d}{2}x+1\right)\right] u(p_1)$$

throw away $\to F_D(q^2)$!

For second term use Gordon identity.

$$\underbrace{\bar{u}(p_2)\gamma^\alpha u(p_1)} = \bar{u}(p_2)\left[\frac{(p_1+p_2)^\alpha}{2m} + \frac{i\sigma^{\alpha\beta}q_\beta}{2m}\right]u(p_1)$$

this can be thrown away again!

$$N^\alpha = -\bar{u}(p_2)\left[\frac{i\sigma^{\alpha\beta}q_\beta}{2m}(2m^2)\,x\left(\frac{2-d}{2}x+1\right)\right]u(p_1)$$

$\Uparrow$

this is precisely the structure that multiplies PAULI FORM FACTOR!!! we can read:

$$F_P(q^2=0) = i\,\frac{2e^4m^2}{3\pi^2}\int_{s_0}^{\infty}\frac{ds}{s}R^{hadr}(s)\int_0^1 dx\int\frac{d^d\ell}{(2\pi)^d}\frac{x^2\left(1-\frac{d-2}{2}x\right)}{(\ell^2-\Delta)^3}$$

$$= i\,\frac{2e^4m^2}{3\pi^2}\int_{s_0}^{\infty}\frac{ds}{s}R^{hadr}(s)\int_0^1 dx\left(\frac{-i}{(4\pi)^{d/2}}\right)\frac{\Gamma(3-d/2)}{\Gamma(3)}\left(\frac{1}{\Delta}\right)^{3-\frac{d}{2}}x^2\left(1-\frac{d-2}{2}x\right)$$

we can put $d=4$ $\Longleftrightarrow$ EVERYTHING IS FINITE!!!

$$F_P(0) = \frac{\alpha^2m^2}{3\pi^2}\int_{s_0}^{\infty}\frac{ds}{s}R^{hadr}(s)\int_0^1 dx\,\frac{x^2(1-x)}{\left[s(1-x)+m^2x^2\right]}$$

what do we do with this integral ?

(*) if Energy is low $\sigma_{e^+e^- \to hadrons}$ is dominated by the contribution of $\boxed{\rho \text{ mesons}}$ $m_\rho \simeq 770\,MeV$

$\rho^+, \rho^-, \rho^0 \simeq 770\,MeV$ $\quad \rho^+ = u\bar{d}$ $\quad \rho^- = \bar{u}d$ $\quad \rho^0 = \frac{u\bar{u} - d\bar{d}}{\sqrt{2}}$

$\rho^0 \to e^+e^-$ with branching ratio $\boxed{\Gamma_{\rho \to e^+e^-} \simeq 7.02\,keV}$

in such "narrow-width" situation we can use narrow-width approximation

$$\sigma_{e^+e^- \to \rho} = \frac{12\pi^2 \Gamma_{\rho \to e^+e^-}}{m_\rho} \delta(s - m_\rho) \Leftarrow$$

Peaked on the resonance !

$$R^{hadr}_{(s)} = \frac{9\pi s}{m_\rho \alpha^2} \Gamma_{\rho \to e^+e^-} \delta(s - m_\rho)$$

and putting everything together

$$a_\mu = \frac{3m^2}{\pi m_\rho} \Gamma_{\rho \to e^+e^-} \int_0^1 dx \int_{s_0}^{\infty} ds \frac{x^2(1-x)}{s(1-x) + m^2 x^2} \delta(s - m_\rho^2)$$

$s_0 = m_\pi^2 \leftarrow$ you will see it next ex-sheet.

$$= \frac{3m^2}{\pi m_\rho} \Gamma_{\rho \to e^+e^-} \int_0^1 dx \frac{x^2(1-x)}{m_\rho^2(1-x) + m^2 x^2}$$

ρ mass $\quad$ muon mass $\quad$ $\boxed{\frac{m^2}{m_\rho^2} \ll 1}$

Expanding in $\frac{m^2}{m_\rho^2}$

$$\simeq \frac{3m^2}{\pi m_\rho^3} \int_0^1 dx\, x^2 \Rightarrow \frac{m^2}{\pi m_\rho^3} \Gamma_{\rho \to e^+e^-} = \underline{5396 \cdot 10^{-11}}$$

Against $6923(42)(3) \cdot 10^{-11}$