Theoretische Teilchenphysik II

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Exercise Sheet 6

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Problem 1 - Ward-Takahashi identities in QED

The goal of this problem is to study constraints on QED matrix elements that follow from Ward-Takahashi identities (and other symmetries such as Bose symmetry).

1. We will start with a familiar example of the electron scattering in the external electromagnetic field. A scattering is described by an amplitude $M_\mu = \bar{u}(p_2)\Gamma^\mu u(p_1)$, that should be a four-vector under Lorentz transformations.

   a) Write the most general parametrization of Γ^μ. Recall that any four-by-four matrix can be represented as a linear combination of sixteen γ-matrices. Do not ignore contributions related to γ_5 or the Levi-Cevita tensor $\epsilon_{\mu\nu\alpha\beta}$.

   b) The Ward-Takahashi identity for M_μ is $q_\mu M^\mu = 0$, where $q = p_2 - p_1$ is the momentum transfer to the electron line from the electromagnetic field. Use this constraint to determine the minimal number of Lorentz-invariant structures that are needed to describe an electromagnetic interaction of leptons.

2. Consider a decay of a neutral massive particle (the Higgs boson) to two photons. The photons are described by their polarization vector $\epsilon_{1,2}$. The decay ampltiude is given by $M_{\mu\nu}\epsilon_1^\mu\epsilon_2^\nu$. Wite the most general expression for $M_{\mu\nu}$ consistent with Lorentz invariance. Impose the Ward-Takahashi identity and the Bose symmetry for the two photons and determine the minimal parametrization of the amplitude for $H \to \gamma\gamma$.

3. Repeat the calculation described in Part 2 for the decay of a neutral *spin-one* particle to two photons. The polarization of a neutral spin-one particle with momentum q_μ is described by a four-vector ϵ_μ such that $q \cdot \epsilon = 0$. If you solve this problem, you will discover the so-called Landau-Yang theorem, named after two Nobel Prize winners, L.D. Landau and C.N. Yang.

Problem 2 - Isospin symmetry of strong interactions

Consider a theory that consists of three pions, a proton and a neutron. Assume that the proton and the neutron have equal masses. Protons and neutrons are fermions, while pions are scalar bosons. The proton and the neutron form a doublet of a fundamental representation of $SU(2)$ and three pions make a triplet of the adjoint representation.

1. Give explicit transformation rules for the fermion doublet under infinitesimal transformations of $SU(2)$.

2. In order to construct an adjoint representation, take a vector space as $\hat{\pi} = \vec{\pi} \cdot \vec{\sigma}$, where $\vec{\pi} \in \mathbb{R}^3$ and $\vec{\sigma}$ are the Pauli matrices, given by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1}$$

Define the action of the group in this vector space by a transformation $\hat{\pi} \to \hat{\pi}' = U\hat{\pi}U^{-1}$, where $U \in SU(2)$. Prove that this is indeed a representation. In particular show that $\hat{\pi}'$ can again be written as $\hat{\pi} = \vec{\pi}' \cdot \vec{\sigma}$ where $\vec{\pi}'$ is real. Use the fact that any matrix $U \in SU(2)$ can be written as a linear combination of an identity matrix and three Pauli matrices.

3. Write the kinetic term in the Lagrangian for both the fermion doublet and the pion triplet that makes invariance under $SU(2)$ explicit. Why is it important that proton and neutron have identical masses?

4. Write the term in the Lagrangian that describes an interaction of pions and the fermion doublet. The term should be linear in the pion field and bi-linear in fermion fields and should be invariant under $SU(2)$. It should contain no derivatives or masses. How many different terms one can write down?

5. The $SU(2)$ symmetry of the Lagrangian connects amplitudes of various processes that involve pions and fermions. In particular, use your Lagrangian to derive relative strengths of the $\pi_0 nn$ and $\pi^{\pm} pn$ interaction vertices.