Theoretische Teilchenphysik II

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Exercise Sheet 4

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Problem 1 - Emission of soft photons

In TTP1 we studied the emission of a soft photon. Consider now the cross-section for the emission of n photons. When all photons become soft (i.e. their energies go to zero) the cross-section can be written as

$$d\sigma_n \approx \frac{1}{n!} d\sigma_0 \, T^n(p_i, p_f) \left(\frac{\alpha_{em}}{\pi}\right)^n , \tag{1}$$

where $\alpha_{em} = e^2/(4\,\pi)$, $d\sigma_0$ is the cross-section for the hard process (the cross-section for the processes where all soft photons have been removed) and

$$\left(\frac{\alpha_{em}}{\pi}\right) T(p_i, p_f) = e^2 \int \frac{d^3\,k}{(2\pi)^3 2\omega} \left\{ \frac{2\,p_i \cdot p_f}{(p_f \cdot k)(p_i \cdot k)} - \frac{p_f^2}{(p_f \cdot k)^2} - \frac{p_i^2}{(p_i \cdot k)^2} \right\} . \tag{2}$$

This integral is divergent both for $|k| \to \infty$ and for $|k| \to 0$. The divergence at $|k| \to \infty$ is an artifact of our approximation which is by definition valid only for soft photons. We therefore impose a cutoff $|k| < \omega_{max}$. The residual divergence at $|k| \to 0$ is the infra-red one; it must be properly regularized. In TTP1 this was done by giving a fictitious mass λ to the photons. In this exercise you should repeat the computation by using *dimensional regularization*, in the approximation of back-to-back *massless* emitters.

1. Consider the emission of a soft photon from a *massless electron*, i.e. $p_i^2 = p_f^2 = 0$. In dimensional regularization the quantity in (2) becomes

   $$\left(\frac{\alpha_{em}}{\pi}\right) T(p_i, p_f) = e^2 \int \frac{d^{d-1}\,k}{(2\pi)^{d-1} 2\omega}\, \theta(\omega_{max} - |k|) \left\{ \frac{2\,p_i \cdot p_f}{(p_f \cdot k)(p_i \cdot k)} \right\} , \tag{3}$$

   where, as usual, the physical limit is obtained for $d \to 4$. Is the integral (3) Lorentz-invariant?

2. In order to compute (3), consider the simplifying situation where the emitters are back to back, $p_i = (E, 0, 0, E)$, $p_f = (E, 0, 0, -E)$, and show that

   $$T(p_i, p_f) = \frac{\pi^\epsilon\, \omega_{max}^{-2\epsilon}}{\epsilon^2} \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\,\epsilon)} = \pi^\epsilon\, \omega_{max}^{-2\epsilon}\, \Gamma(1+\epsilon) \left(\frac{1}{\epsilon^2} - \frac{\pi^2}{3} + \mathcal{O}(\epsilon) \right) , \tag{4}$$

   Note that in order to perform the d-dimensional integration you will need the recursive definition of the d-dimensional solid angle

   $$d\Omega_n = d\Omega_{n-1}\, d\cos\theta\, (1 - \cos^2\theta)^{(n-3)/2} . \tag{5}$$

3. Eq (4) contains a $1/\epsilon^2$ pole. What is the physical origin of it? By expanding $\omega_{max}^{-2\epsilon}$ in powers of ϵ, determine the terms that are proportional to $\ln(\omega_{max})$ and $\ln^2(\omega_{max})$.

Problem 2 - Emission of collinear photons

The aim of this exercise is to study the behavior of S-matrix elements (or scattering amplitudes) in the limit when three-momenta of two massless particles become parallel (collinear) to each other[1]. Similar to the situation with soft photons that we considered earlier, the collinear kinematics leads to large (eventually infinite) amplitudes. To this end, consider the amplitude of an *arbitrary* process with n *massless* electrons of momenta $p_1, ..., p_n$ and one photon with momentum k, $\mathcal{M}(p_1, ..., p_n; k)$. We assume that the photon and the electron with momentum p_1 become collinear.

1. Start by dividing the diagrams that contribute to $\mathcal{M}$ into two groups

$$\mathcal{M} = \mathcal{M}_s + \mathcal{M}_{ns}\,, \tag{6}$$

   where $\mathcal{M}_s$ is given by the diagram where the photon k is emitted off the electron p_1, and $\mathcal{M}_{ns}$ contains all other diagrams. Which of the two terms in (6) is singular in the collinear limit?

2. Let us write the two terms as

$$\mathcal{M}_s = e\,\bar{u}(p_1)\,\not{\epsilon}(k)\,\frac{\not{p}_1 + \not{k}}{2p_1 \cdot k}\,\mathcal{M}_a(k + p_1, ..., p_n)\,, \quad \mathcal{M}_{ns} = \bar{u}(p_1)\,\epsilon_\mu(k)\,\mathcal{M}_b^\mu(p_1, ..., p_n; k)\,. \tag{7}$$

   so that potentially singular terms $1/(p_1 \cdot k)$ are shown explicitly.

   We square $\mathcal{M}$ and write the result as

$$|\mathcal{M}|^2 = |\mathcal{M}_s|^2 + |\mathcal{M}_{ns}|^2 + 2\,\mathrm{Re}(\mathcal{M}_s^* \mathcal{M}_{ns})\,. \tag{8}$$

   Use Eq.(7) to determine which of the three terms in Eq.(8) may give non-integrable contributions to scattering cross-section.

3. The above statement depends on the gauge chosen to describe the emitted photon field. Indeed, show that if one uses the axial gauge

$$\sum_\lambda \epsilon_\mu^{(\lambda)}(k)\,\epsilon_\nu^{*(\lambda)}(k) = -g_{\mu\nu} + \frac{\bar{p}_1^\mu k^\nu + \bar{p}_1^\nu k^\mu}{\bar{p}_1 \cdot k}\,, \tag{9}$$

   the only term that gives the divergent contribution to the cross-section is $|\mathcal{M}_s|^2$, i.e. the interference term is integrable, while in the Feynman gauge *both* $|\mathcal{M}_s|^2$ and the interference term give divergent contributions. The momentum $\bar{p}_1$ is introduced through the so-called Sudakov decomposition

$$k^\mu = \alpha p_1^\mu + \beta \bar{p}_1^\mu + k_\perp^\mu \tag{10}$$

   where p_1 is the electron momentum, $\bar{p}_1$ is chosen such that $\bar{p}_1^2 = 0$ and $p_1 \cdot \bar{p}_1 \neq 0$, and $p_1 \cdot k_\perp = \bar{p}_1 \cdot k_\perp = 0$. See additional material to this exercise for details.

4. Use the axial gauge to show that in the collinear limit and up to the terms that give integrable contributions to a cross-section, the amplitude squared $|\mathcal{M}|^2$ reads

$$|\mathcal{M}(p_1, ..., p_n; k)|^2 \approx \left(\frac{4\pi\alpha_{em}}{p_1 \cdot k}\right)\,\frac{1 + x^2}{1 - x}\,|\widetilde{\mathcal{M}}(p, ..., p_n)|^2\,, \tag{11}$$

   where $\alpha_{em} = e^2/(4\,\pi)$ is the fine structure constant, $\widetilde{\mathcal{M}}(p, ..., p_n)$ is the matrix element that only depends on fermion degrees of freedom (photon emission factorizes), $p = k + p_1$, α has been defined in (10) and $x = 1/(1 + \alpha)$. What does α physically represent in the collinear limit? Can you think of a reason why Eq.(11) is important?

[1] For a discussion of collinear kinematics, please refer to additional material for this exercise.