Theoretische Teilchenphysik II

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Exercise Sheet 3

Due 9.11.2016

Problem 1 - Photon vacuum polarization

Follow the steps discussed in class in connection with the derivation of the Källen representation for the two-point function of two scalar fields, to derive a similar representation for the correlator of two electromagnetic currents

$$\Pi_{\mu\nu}(q) = i \int \mathrm{d}x e^{iqx} \langle 0|T J_\mu(x) J_\nu(0)|0\rangle \,. \tag{1}$$

Consider only intermediate hadronic states and use the fact that conservation of the electromagnetic current implies

$$\Pi_{\mu\nu}(q) = \Pi(q^2)\,\left(-g_{\mu\nu}\, q^2 + q_\mu q_\nu\right)\,. \tag{2}$$

1. Derive the Källen representation inserting the unity operator in Eq.(1), as we did in class. Use conservation of the vector current $q_\mu J^\mu = 0$ to simplify the dependence of the spectral density on the Lorenz indices.

2. Next, considering the cross-section for the process $e^+e^- \to X$, where X is a generic hadronic state. Note that scattering amplitude for this process can be schematically written as

$$\mathcal{M}_{e^+e^-\to X} = e^2\, \bar{u}_{e^+}\, \gamma^\mu\, u_{e^-}\, \frac{1}{q^2}\, \langle \Omega|J_\mu(0)|X\rangle\,, \tag{3}$$

   where J^μ is the electromagnetic current. Justify Eq.(3).

3. Use Eq.(3) to relate the Källen representation for the correlator of two electromagnetic currents that you derived with the cross-section for $e^+e^- \to X$.

4. Putting everything together show that

$$\Pi(q^2) = \frac{1}{12\,\pi^2} \int \frac{\mathrm{d}s R_h(s)}{s - q^2 + i0}, \tag{4}$$

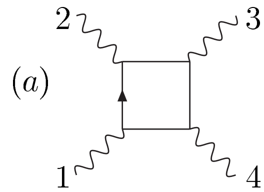
   where $R_h(s) = \sigma_h(s)/\sigma_{\text{point}}$ and σ_h is the cross-section for $e^+e^- \to$ hadrons and σ_{point} is the cross-section for $e^+e^- \to \mu^+\mu^-$ in the approximation of massless muons.

Problem 2 - Light-by-light scattering

Consider the process of light-by-light scattering in QED , i.e. the scattering of two photons into two photons

$$\gamma(p_1) + \gamma(p_2) \to \gamma(p_3) + \gamma(p_4)\,. \tag{5}$$

Since photons do not couple directly to each other this process appears first only at one-loop. The goal of this exercise is to prove that the amplitude for this process is actually finite and does not require any renormalization. An example of a Feynman diagram contributing at one-loop is



1. Draw all Feynman diagrams that contribute to this process at the one-loop order. How many diagrams are there? Which ones are really different from each other and need to be computed?

2. Show that, in the limit of large loop momentum, $k \to \infty$, each diagram has a superficial degree of divergence equal to zero. This implies that in this limit each diagram might diverge logarithmically

$$\mathrm{D}_j \propto \int^{\infty} \frac{dk}{k} \;\propto\; \ln(\infty)\,. \tag{6}$$

   Taking at face value, this seems to imply that $\gamma\gamma$ scattering amplitude in QED is not finite. If this were true, it has unpleasant consequences for the renormalization of QED (explain why).

3. By introducing a UV cutoff Λ_{UV} in order to regulate the UV divergences of the individual diagrams[1], show by explicit computation that every diagram depends logarithmically on Λ_{UV}, i.e.

$$D_j \approx C_j \,\ln(\Lambda_{UV})\,, \tag{7}$$

   where C_j is finite as $\Lambda_{UV} \to \infty$.

4. Consider now all diagrams that contribute to light-by-light scattering. Show explicitly that in the sum, the dependence on $\ln(\Lambda_{UV})$ cancels out, and that the final result is *finite* as $\Lambda_{UV} \to \infty$[2]. The following (*4-dimensional*) identities can be useful for you:

$$\begin{aligned}
\gamma_\mu\,\gamma^\nu\,\gamma_\mu &= -2\,\gamma^\nu\,,\\
\gamma_\mu\,\gamma^\nu\,\gamma^\rho\,\gamma^\sigma\,\gamma_\mu &= -2\,\gamma^\sigma\,\gamma^\rho\,\gamma^\nu\,,\\
\mathrm{Tr}(\gamma^\mu\gamma^\nu) &= 4g^{\mu\nu}\,,\\
\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) &= 4\,(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})\,.
\end{aligned}$$

   Hint: Note that upon averaging over the $4-$*dimensional* solid-angle one gets

$$\int d\Omega_4\, k^\mu\, k^\nu\, k^\rho\, k^\sigma = f(k^2)\,\left(g^{\mu\nu}g^{\rho\sigma} + g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}\right)\,, \tag{8}$$

   where, as indicated explicitly, the function f depends only on k^2.

[1] This implies that every loop integration is restricted from above by the cutoff $\int^{\infty} dk \to \int^{\Lambda_{UV}} dk$.

[2] Since all divergences come from the $k \to \infty$ limit, it is enough to evaluate explicitly the diagrams in this limit.