Theoretische Teilchenphysik II

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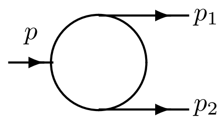
Exercise Sheet 2

Due 2.11.2016

Problem 1 - The imaginary part of a 1-loop triangle

Consider the following 1-loop triangle with massless internal propagators and massive external lines

$\mathcal{T}(p^2, m_1, m_2) =$ 

$$= \int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2 \, (k-p_1)^2 \, (k-p_1-p_2)^2} \,, \qquad p_1^2 = m_1^2 \,, \quad p_2^2 = m_2^2 \,. \tag{1}$$

In the decay kinematics one has $p \to p_1 + p_2$ with $p^2 \geq (m_1 + m_2)^2$. The integral $\mathcal{T}(p^2, m_1, m_2)$ is finite so that no regularization is required.

1. Use Feynman parameters to show that $\mathcal{T}(p^2, m_1, m_2)$ for this choice of the kinematics, cannot develop any imaginary part. *Note: You do not need to compute the integral explicitly, but only to show that it must be real!*

2. We want to compute now this imaginary part explicitly using Cutkosky rules. Start off by computing the discontinuity in p^2, i.e. the one obtained by cutting the two propagators connecting to the vertex with momentum p. Show that this discontinuity reads

$$\text{Disc}(p^2, m_1, m_2) = \frac{1}{8\,\pi\,\sqrt{(p^2-\mu_{12}^2)\,(p^2-\bar{\mu}_{12}^2)}} \ln\left(\frac{p^2 - m_1^2 - m_2^2 - \sqrt{(p^2-\mu_{12}^2)\,(p^2-\bar{\mu}_{12}^2)}}{p^2 - m_1^2 - m_2^2 + \sqrt{(p^2-\mu_{12}^2)\,(p^2-\bar{\mu}_{12}^2)}} \right), \tag{2}$$

   with $\mu_{12} = (m_1 + m_2)$ and $\bar{\mu}_{12} = (m_1 - m_2)$. It is convenient to work in the reference frame where p^μ is at rest, namely $p^\mu = (W, \vec{0}\,)$, where $W = \sqrt{p^2}$ is the total energy of the system.

3. Compute now the discontinuity in $p_1^2 = m_1^2$ by cutting the two propagators connecting to the vertex with momentum p_1. In this case it is convenient to work in the reference frame where p_1 is at rest, namely $p_1^\mu = (m_1, \vec{0}\,)$. Show that in this case one gets

$$\text{Disc}(m_1, p^2, m_2) = \frac{1}{8\,\pi\,\sqrt{(p^2-\mu_{12}^2)\,(p^2-\bar{\mu}_{12}^2)}} \ln\left(\frac{p^2 + m_2^2 - m_1^2 + \sqrt{(p^2-\mu_{12}^2)\,(p^2-\bar{\mu}_{12}^2)}}{p^2 + m_2^2 - m_1^2 - \sqrt{(p^2-\mu_{12}^2)\,(p^2-\bar{\mu}_{12}^2)}} \right), \tag{3}$$

4. Finally the third cut does not need to be computed and can be obtained just permuting m_1 and m_2 in (3)

$$\text{Disc}(m_2, p^2, m_1) = \frac{1}{8\,\pi\,\sqrt{(p^2-\mu_{12}^2)\,(p^2-\bar{\mu}_{12}^2)}} \ln\left(\frac{p^2 + m_1^2 - m_2^2 + \sqrt{(p^2-\mu_{12}^2)\,(p^2-\bar{\mu}_{12}^2)}}{p^2 + m_1^2 - m_2^2 - \sqrt{(p^2-\mu_{12}^2)\,(p^2-\bar{\mu}_{12}^2)}} \right). \tag{4}$$

5. Show then that the total imaginary part must is, *as expected* from 1., zero

$$\frac{1}{\pi}\,\mathrm{Im}\left(\mathcal{T}(p^2, m_1, m_2)\right) = \mathrm{Disc}(p^2, m_1, m_2) + \mathrm{Disc}(m_1, p^2, m_2) + \mathrm{Disc}(m_2, p^2, m_1) = 0\,. \qquad (5)$$

Problem 2 - Leading Order Hadronic Contribution to a_μ

The muon anomalous magnetic moment shows one of the most intriguing discrepancies between its measured value and the prediction within the Standard Model; numerically, the discrepancy is 3.6σ which, in principle, implies that there is only one chance in 10 thousand (roughly) that the Standard Model is a valid description of Nature.

The hadronic contributions to the muon anomalous magnetic moment are currently the source of the largest error in the theoretical determination. In this problem we will compute an approximation of the leading hadronic contribution, shown in Figure 1.

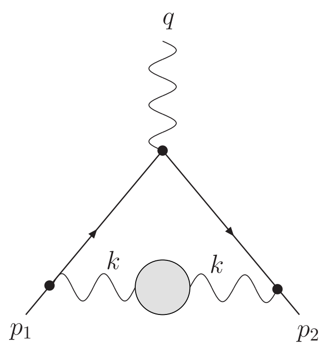


Figure 1: The leading order hadronic contribution to the muon anomalous magnetic moment.

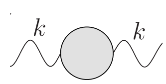
Let us remind ourselves how the anomalous magnetic moment of the muon is defined. Consider $\mu\mu\gamma^*$ interaction vertex that represents muon scattering in the external electromagnetic field. Using Lorenz symmetry and gauge invariance one can show that this vertex can be written as

$$V^\alpha(q) = -ie\overline{u}(p_2)\left[F_D(q^2)\,\gamma^\alpha + F_P(q^2)\,\frac{i\,\sigma^{\alpha\beta}\,q_\beta}{2\,m}\right]u(p_1)\,,$$

where $q = p_2 - p_1$, m is the muon mass and the two *form factors*, Dirac F_D and Pauli F_P, are scalar functions of q^2, the momentum transfer[1]. The anomalous magnetic moment of the muon is then defined as the value of the Pauli form factor at zero momentum transfer

$$a_\mu = F_P(0). \qquad (6)$$

Our goal is to calculate the contribution of Figure 1 to a_μ. Since we do not have an analytic form for the hadronic loop, we will use a dispersion representation for the photon propagator to express this contribution through the cross-section of e^+e^- annihilation to hadrons. To this end, we write



$$= \frac{ie^2}{(k^2)^2}\Pi(k^2)(g^{\mu\nu}k^2 - k^\mu k^\nu), \qquad (7)$$

where k is the momentum of the photons and $\Pi(k^2)$ satisfies the subtracted dispersion relation

$$\Pi(k^2) = \frac{k^2}{\pi}\int_{s_0}^{\infty} ds\frac{\mathrm{Im}(\Pi(s))}{s(s-k^2-i0)} = \frac{k^2}{12\pi^2}\int_{s_0}^{\infty} ds\frac{\mathcal{R}^{hadr}(s)}{s(s-k^2-i0)}. \qquad (8)$$

[1] Note that, since this decomposition is based only on Lorentz and gauge invariance, it is valid independently on the perturbative order and it is not restricted to QED only!

The quantity $R^{hadr}(s)$ is defined as the ratio

$$R^{hadr}(s) = \frac{\sigma_{e^+e^-\to\text{hadrons}}(s)}{\sigma_{\text{point}}}, \tag{9}$$

where $\sigma_{e^+e^-\to\text{hadrons}}(s)$ is the cross-section to produce hadrons in e^+e^- annihilation at the energy $\sqrt{s}$ and $\sigma_{\text{point}} = (4\pi\alpha^2)/3/s$ is an auxiliary normalization factor.

1. To calculate the hadronic vacuum polarization contribution to a_μ, start by writing the expression for the diagram in Figure 1 using Eqs. (7) and (8).

2. Observe that the integration over the loop momentum k can be performed in a standard way (introducing Feynman parameters, shifting the momentum, etc.). In general, the integration over the loop momentum gives you both the Dirac and the Pauli form factor; the Dirac form factor is the one that is more difficult to compute but it is of no interest for us since it does not contribute to a_μ. Therefore, when you perform the algebraic manipulations, make sure to disregard all terms that can contribute only to the Dirac form factor; this will simplify the calculation quite a bit. Also, you may find the Gordon identity

$$\overline{u}(p_2)\gamma^\alpha u(p_1) = \overline{u}(p_2)\left[\frac{(p_1+p_2)^\alpha}{2m} + \frac{i\sigma^{\alpha\beta}q_\beta}{2m}\right]u(p_1), \tag{10}$$

   useful to get rid of the Lorenz structures p_1^α and p_2^α in favor of γ^α and $\sigma^{\alpha\beta}q_\beta$.

3. Show that the anomalous magnetic moment a_μ can be written in the following form

$$a_\mu = F_P(0) = \frac{\alpha^2 m^2}{3\pi^2}\int \frac{\text{ds}}{s} R^{hadr}(s)\int_0^1 \frac{x^2(1-x)}{s(1-x)+m^2x^2}, \tag{11}$$

   where x is one of the Feynman parameters.

4. Usually, the integral in Eq.(11) is performed numerically, using experimental data for $R(s)$. We will not do that but, instead, we will evaluate the integral in Eq.(11) approximately. At low energies, the cross-section for e^+e^- annihilation to hadrons is dominated by the contribution of the ρ-meson (see Figure 2.).

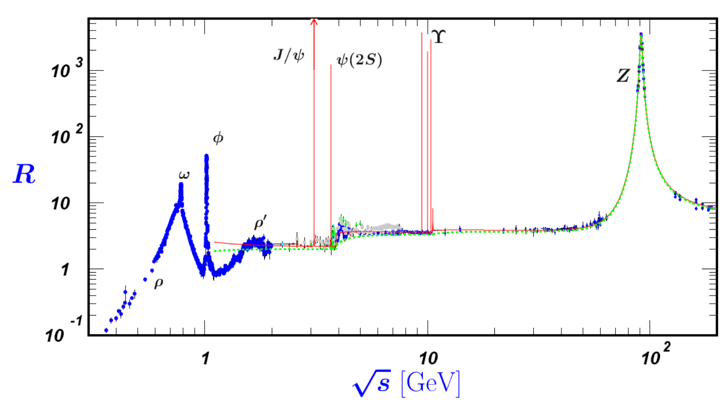


Figure 2: Dependence of $R^{hadr}(s)$ on $\sqrt{s}$.

The cross-section for $e^+e^- \to$ hadrons at the ρ-peak can be written as

$$\sigma_{e^+e^-\to\rho}(s) = \frac{12\,\pi^2\,\Gamma_{\rho\to e^+e^-}}{m_\rho}\,\delta(s-m_\rho^2)\,, \tag{12}$$

where $\Gamma_{\rho \to e^+e^-}$ is the partial decay width of the ρ into e^+e^-.[2]

Use Eq.(12) and Eq.(11) to calculate the ρ-meson contribution to a_μ. Use $\Gamma_{\rho \to e^+e^-} = 7.02$ keV, $m_\mu = 105$ MeV, $m_\rho = 770$ MeV to derive numerical value of $a_\mu(\rho)$. To simplify the integration over x in Eq.(11) use the fact that $m_\rho \gg m_\mu$.

5. Detailed analysis of $\sigma(e^+e^- \to \text{hadrons})$ data gives $a_\mu^{\text{hpv}} = 6923(42)(3) \times 10^{-11}$, where the first error is experimental and the second is due to theory. How does the contribution of the ρ-meson computed by you compare with the result of a more complete analysis? Can it be used as a rought estimate?

[2] The origin of this formula is explained in Lecture 4, see notes.