Theoretische Teilchenphysik II

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Exercise Sheet 11

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Problem 1 - Path integral for scalar fields

In the lecture we have seen that the correlation function of real-valued scalar fields can be extracted from the generating functional by taking functional derivatives (see footnote[1] on the next page),

$$\langle\Omega|T\{\varphi(x_1)\cdots\varphi(x_n)\}|\Omega\rangle = \frac{1}{Z[0]}\frac{\delta}{i\,\delta J(x_1)}\cdots\frac{\delta}{i\,\delta J(x_n)}Z[J]\bigg|_{J=0}\ , \tag{1}$$

where the generating functional is given by

$$Z[J] = \int \mathcal{D}\varphi(x)\,\exp\left[i\int d^4x\,[\mathcal{L}+J(x)\varphi(x)]\right]\ . \tag{2}$$

Let us consider a non-interacting scalar field theory with the Lagrangian $\mathcal{L}_0 \equiv \frac{1}{2}\varphi(x)(-\partial^2 - m^2 + i\epsilon)\varphi(x)$. Before applying eq. (1) it is convenient to first rewrite the generating functional in such a way as to extract its dependence on the source fields outside the path integral. In order to do so, it is convenient to Fourier transform all fields to momentum space, $\varphi(k)$ and $J(k)$, which are related to the position-space fields via

$$\varphi(x) = \int \frac{d^4k}{(2\pi)^4}\,e^{-ik\cdot x}\,\varphi(k)\ ,\quad \varphi(k) = \int d^4x\,e^{ik\cdot x}\,\varphi(x)\ . \tag{3}$$

The same relations hold between $J(x)$ and $J(k)$.

1. Show that the Fourier transform changes the exponent in eq. (2) into

$$\text{exponent} = i\int\frac{d^4k}{(2\pi)^4}\left[\tfrac{1}{2}\varphi(k)\left(k^2 - m^2 + i\epsilon\right)\varphi(-k) + J(k)\varphi(-k)\right]\ . \tag{4}$$

2. Shift the fields in order to complete the square. That is, find $\lambda(k)$ such that $\varphi'(k) = \varphi(k) + \lambda(k)$ and

$$\text{exponent} = i\int\frac{d^4k}{(2\pi)^4}\left[\tfrac{1}{2}\varphi'(k)\left(k^2 - m^2 + i\epsilon\right)\varphi'(-k) - \tfrac{1}{2}J(k)\frac{1}{k^2-m^2+i\epsilon}J(-k)\right]\ . \tag{5}$$

3. A Fourier transform back to position space subsequently leads to

$$Z[J] = Z[0]\exp\left[-\tfrac{1}{2}\int d^4x\,d^4y\,J(x)D_F(x-y)J(y)\right]\ . \tag{6}$$

   Give the explicit form of $D_F(x-y)$ as the Fourier transform of a scalar Feynman propagator.

The generating functional in eq. (6) can be used to compute correlation functions.

4. What can you say about $\langle\Omega|T\{\varphi(x_1)\cdots\varphi(x_n)\}|\Omega\rangle$ for n odd? What is the underlying reason?
5. Calculate $\langle\Omega|T\{\varphi(x_1)\varphi(x_2)\}|\Omega\rangle$ using eqs. (1) and (6). Can you recognise this Green's function?
6. Calculate $\langle\Omega|T\{\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)\}|\Omega\rangle$ using eqs. (1) and (6) and show that the result coincides with Wick's theorem for non-interacting scalar fields, which was discussed in TTP1.
7. The generating functional in eq. (2) also applies to interacting field theories, such as scalar field theory with a quartic coupling in the Lagrangian $\mathcal{L} = \mathcal{L}_0 - \frac{\lambda}{4!}\varphi^4(x)$, but in that case the path integral is non-Gaussian. Without actually doing it, can you guess how to rewrite the path integral to make a systematic expansion in λ possible? [Hint: what do you get by applying $\delta/\delta J(x)$ to eq. (2)?]

Problem 2 - Path integral for fermionic fields

The functional methods used in the previous problem are devised for fields that obey canonical commutation relations. Fermionic fields obey anti-commutation relations instead. In order to ensure that this property carries over to correlation functions in the appropriate way, one introduces anti-commuting fields (Grassmann fields) in the path integral. After decomposing Grassmann fields into ordinary fields times anti-commuting numbers (Grassmann numbers), this naturally leads to Gaussian integrals over Grassmann numbers. In this problem we investigate how this impacts the generating functional method.

The crucial property of Grassmann numbers is that they anti-commute, $\theta\eta = -\eta\theta$.

1. Show that an arbitrary function f of a Grassmann number can be written as $f(\theta) = c_0 + c_1\theta$, with undetermined (ordinary numbers) c_0 and c_1. [Hint: think about Taylor expansion.]
2. Use invariance of integrals under shifts of the (Grassmann) integration variable to argue that $\int d\theta f(\theta) = c_1$. (In this sense, integration over Grassmann numbers mimicks differentiation of ordinary numbers.)

Since fermionic fields are complex-valued we consider complex Grassmann numbers. A complex Grassmann number θ is composed of a real and an imaginary part $\theta = \frac{1}{\sqrt{2}}(\theta_1 + i\theta_2)$, just like ordinary numbers, and thus has two degrees of freedom. One may choose to integrate over the variables θ_1 and θ_2, or equivalently over the variables θ and θ^*. We proceed to compute Gaussian integrals over the latter variables.

3. Show that

$$\int d\theta^* d\theta \exp\left(-\theta^* c\, \theta\right) = c\ , \tag{7}$$

for some real constant c.

4. In order to generalise the previous result to the multi-dimensional case, analogous to

$$\int dx\ \exp\left(-\tfrac{1}{2}ax^2\right) = \sqrt{\frac{2\pi}{a}} \quad \longrightarrow \quad \int d^n x\ \exp\left(-\tfrac{1}{2}x \cdot A \cdot x\right) = \sqrt{\frac{(2\pi)^n}{\det A}}\ , \tag{8}$$

show that a multi-dimensional integral over complex Grassmann variables is invariant under unitary transformations. [Hint: first prove that $\prod_i (U_{ij}\theta_j) = (\det U)(\prod_i \theta_i)$.]

5. Using the previous result, calculate the integral

$$\left(\prod_{i=1}^{n} \int d\theta_i^* d\theta_i\right) \exp\left(-\theta_i^* C_{ij}\, \theta_j\right)\ . \tag{9}$$

Apart from the difference between eq. (8) on one hand, and eqs. (7) and (9) on the other hand, the construction of path integrals for fermionic fields is quite similar to the construction of path integrals for scalar fields. One further difference is that sources are Grassmann fields as well, for instance the Dirac generating functional depends on Grassmann source fields $\overline{\eta}$ and η,

$$Z[\overline{\eta}, \eta] = \int \mathcal{D}\overline{\psi}(x)\mathcal{D}\psi(x) \exp\left[i \int d^4x\ \left[\mathcal{L} + \overline{\eta}(x)\psi(x) + \overline{\psi}(x)\eta(x)\right]\right]\ , \tag{10}$$

which may be contrasted against eq. (2).

[1]Recall that functional derivatives satisfy $\frac{\delta}{\delta J(x)} J(y) = \delta^{(4)}(x - y)$. Apart from that, functional derivatives behave rather similar to ordinary derivatives. For instance, $\frac{\delta}{\delta J(x)} \exp\left[\int d^4y J(y) f(y)\right] = f(x) \exp\left[\int d^4y J(y) f(y)\right]$.