Theoretische Teilchenphysik II

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Exercise Sheet 1

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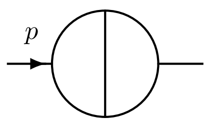
Problem 1 - A two-loop massless bubble

The goal of this exercise is to get some familiarity with the use of the Integration-By-Parts identities (IBP's). We will use them to calculate analytically the following two-loop (!) integral in dimensional regularization ($d = 4 - 2\epsilon$)

$\mathcal{I}(p^2) =$ 

$$= \int \frac{d^d q_1}{(2\pi)^d} \int \frac{d^d q_2}{(2\pi)^d} \, \frac{1}{q_1^2\,(q_1-p)^2\,(q_1-q_2)^2\,q_2^2\,(q_2-p)^2} \,. \tag{1}$$

This integral is finite for $\epsilon = 0$ and is given by the very simple expression

$$\mathcal{I}(p^2) = -\frac{6\,\zeta_3}{(4\,\pi)^4\,p^2} \; + \mathcal{O}(\epsilon)\,.$$

However, deriving this result by means of direct integration over Feynman parameters is difficult. IBP's provide instead a much more elegant way.

1. Start off by performing a Wick rotation in order to go to the euclidean region

$$q_1^0 = -i\,k^0\,, \qquad q_2^0 = -i\,l^0\,, \qquad p_0 = -i\,p_E^0 \tag{2}$$

   such that the integral becomes

$$\mathcal{I}(p^2) = \int \frac{d^d k}{(2\pi)^d} \int \frac{d^d l}{(2\pi)^d} \, \frac{1}{k^2\,(k-p_E)^2\,(k-l)^2\,l^2\,(l-p_E)^2} \; = \mathcal{I}_E(p_E^2)\,, \tag{3}$$

   where the vectors $k,\ l$ are Euclidean, i.e. $k^2 = k_0^2 + \vec{k}^2\,,\ l^2 = l_0^2 + \vec{l}^2$ and $p_E^2 = -p^2$.

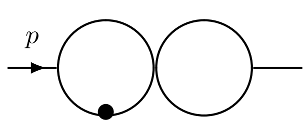
2. Let us focus now on the euclidean integral $\mathcal{I}_E(p_E^2)$. Starting from the following IBP identity

$$\int \frac{d^d k}{(2\pi)^d} \int \frac{d^d l}{(2\pi)^d} \, \left[\frac{\partial}{\partial k_\mu}\,(k_\mu - l_\mu)\,\frac{1}{k^2\,(k-p_E)^2\,(k-l)^2\,l^2\,(l-p_E)^2} \right] = 0\,, \tag{4}$$

   prove that the integral $\mathcal{I}_E(p_E^2)$ can be *reduced* as

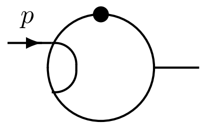
$$\mathcal{I}_E(p_E^2) = \frac{2}{d-4}\left(\mathcal{I}_1(p_E^2) - \mathcal{I}_2(p_E^2)\right)\,, \tag{5}$$

   where:

$\mathcal{I}_1(p_E^2) =$ 

$$= \int \frac{d^d k}{(2\pi)^d} \int \frac{d^d l}{(2\pi)^d} \, \frac{1}{k^4\,(k-p_E)^2\,l^2\,(l-p_E)^2}\,, \tag{6}$$

$$\mathcal{I}_2(p_E^2) =$$



$$= \int \frac{d^d k}{(2\pi)^d} \int \frac{d^d l}{(2\pi)^d} \, \frac{1}{k^4\,(k-p_E)^2\,(k-l)^2\,(l-p_E)^2}\,. \tag{7}$$

3. You need now to compute the integrals $\mathcal{I}_1(p_E^2)$ and $\mathcal{I}_2(p_E^2)$. Start off by defining the Euclidean one-loop bubble with arbitrary powers of propagators

$$\mathcal{B}(q_E^2; a, b) = \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2)^a\,\left((k-q_E)^2\right)^b}\,. \tag{8}$$

   Using Feynman parameters prove that

$$\mathcal{B}(q_E^2; a, b) = \frac{(4\pi)^\epsilon}{16\pi^2} \, \frac{\Gamma(2-\epsilon-a)\,\Gamma(2-\epsilon-b)\,\Gamma(a+b-2+\epsilon)}{\Gamma(a)\,\Gamma(b)\,\Gamma(4-2\epsilon-a-b)} \, \left(q_E^2\right)^{2-\epsilon-a-b}\,, \tag{9}$$

   where as usual $d = 4 - 2\epsilon$.

4. Using only eq. (9) and defining

$$S_\epsilon = \frac{(4\pi)^\epsilon\,\Gamma(1+\epsilon)\,\Gamma(1-\epsilon)^2}{\Gamma(1-2\epsilon)}\,,$$

   prove that[1]:

$$\mathcal{I}_1(p_E^2) = \left(\frac{S_\epsilon}{16\pi^2}\right)^2 \left(-\frac{1}{\epsilon^2(1-2\epsilon)}\right) \left(p_E^2\right)^{-1-2\epsilon}\,, \tag{10}$$

$$\mathcal{I}_2(p_E^2) = \left(\frac{S_\epsilon}{16\pi^2}\right)^2 \left(-\frac{1}{\epsilon^2(1-2\epsilon)}\right) \frac{\Gamma(1-2\epsilon)^2\,\Gamma(1+2\epsilon)}{\Gamma(1-\epsilon)\,\Gamma(1+\epsilon)^2\,\Gamma(1-3\epsilon)} \left(p_E^2\right)^{-1-2\epsilon}\,. \tag{11}$$

5. Using the series expansion

$$\Gamma(1+n\epsilon)\,e^{n\,\gamma\,\epsilon} = 1 + \frac{\pi^2}{12} n^2 \epsilon^2 - \frac{\zeta_3}{3} n^3 \epsilon^3 + \mathcal{O}(\epsilon^4)\,,$$

   where γ is the Euler-Mascheroni constant, expand all Γ functions around $\epsilon = 0$ and prove that

$$\frac{\Gamma(1-2\epsilon)^2\,\Gamma(1+2\epsilon)}{\Gamma(1-\epsilon)\,\Gamma(1+\epsilon)^2\,\Gamma(1-3\epsilon)} = 1 - 6\,\zeta_3\,\epsilon^3 + \mathcal{O}(\epsilon^4)\,. \tag{12}$$

6. Finally putting everything together show that

$$\mathcal{I}_E(p_E^2) = \left(\frac{S_\epsilon}{16\pi^2}\right)^2 \left(6\,\zeta_3 + \mathcal{O}(\epsilon)\right) \left(p_E^2\right)^{-1-2\epsilon} \tag{13}$$

   such that in the minkowskian, physical, region we have:

$$\mathcal{I}(p^2) = \left(\frac{S_\epsilon}{16\pi^2}\right)^2 \left(6\,\zeta_3 + \mathcal{O}(\epsilon)\right) \left(-p^2 - i\,\delta\right)^{-1-2\epsilon} = -\frac{6\,\zeta_3}{(4\,\pi)^4\,p^2} + \mathcal{O}(\epsilon)\,, \tag{14}$$

   where $0 < \delta \ll 1$ comes from Feynman's prescription.

[1] Make use, where necessary, of the functional identity $\Gamma(1+x) = x\,\Gamma(x)$ in order to extract explicitly all poles in $1/\epsilon$.

Problem 2 - Gamma matrices in d dimensions

In this problem we want to show how one can build a d-dimensional representation of the γ-matrices by an iterative procedure in the number of dimensions. Let us suppose we work in an even number of dimensions $d = 2\,\omega$, with $\omega \in \mathbb{N}$. We are looking for a set of $2\,\omega$ matrices $\gamma^\mu_{(\omega)}$ and a matrix $\hat{\gamma}_{(\omega)}$ such that

$$\begin{aligned} &\{\gamma^\mu_{(\omega)}, \gamma^\nu_{(\omega)}\} = 2\, g^{\mu\nu}_{(\omega)}\,, \\ &\{\gamma^\mu_{(\omega)}, \hat{\gamma}_{(\omega)}\} = 0 \\ &\left(\hat{\gamma}_{(\omega)}\right)^2 = 1\,. \end{aligned} \tag{15}$$

where $g^{\mu\nu}_{(\omega)}$ is the metric tensor in $2\,\omega$ dimensions.

1. Let us start for $\omega = 1$, $d = 2$. Define the following two (2×2) matrices

$$\gamma^0_{(1)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_3\,, \qquad \gamma^1_{(1)} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\,\sigma_2\,, \tag{16}$$

   plus a third matrix defined as

$$\hat{\gamma}_{(1)} = \gamma^0_{(1)}\,\gamma^1_{(1)}\,,$$

   and show that they indeed respect the algebra (15).

2. Let us consider now the $\omega = 2$, $d = 4$ case. Define the following four (4×4) matrices

$$\begin{aligned} &\gamma^\mu_{(2)} = \begin{pmatrix} \gamma^\mu_{(1)} & 0 \\ 0 & \gamma^\mu_{(1)} \end{pmatrix}, \quad \text{for} \quad \mu = 0, 1 \\ &\gamma^2_{(2)} = \begin{pmatrix} 0 & \hat{\gamma}_{(1)} \\ -\hat{\gamma}_{(1)} & 0 \end{pmatrix}, \qquad \gamma^3_{(2)} = \begin{pmatrix} 0 & i\,\hat{\gamma}_{(1)} \\ i\,\hat{\gamma}_{(1)} & 0 \end{pmatrix}, \end{aligned} \tag{17}$$

   and the fifth matrix

$$\hat{\gamma}_{(2)} = i\,\gamma^0_{(2)}\gamma^1_{(2)}\gamma^2_{(2)}\gamma^3_{(2)}\,.$$

   Show that they indeed respect the algebra (15).

3. Compare the $\gamma^\mu_{(2)}$ matrices found in point 2. with the γ-matrices in Dirac representation. Why are they different? Do you know any other representation?

4. Following the steps above, write down a representation of the γ matrices valid in $\omega = 3$, $d = 6$ dimensions and verify that it fulfils the algebra (15)[2]. What dimensionality will the matrices have?

5. Assume now that a set of matrices $\{\gamma^\mu_{(\omega)}\,,\ \hat{\gamma}_{(\omega)}\}$ provides a valid representation for $d = 2\,\omega$, with $\omega \in \mathbb{N}$. Construct then a representation valid for $d = 2(\omega+1)$. What dimensionality will the matrices in this representation have?

[2] Note that in defining $\hat{\gamma}_{(3)}$ you will have to properly normalize it by multiplying it with the imaginary unit i raised to an appropriate power!