Lecture 9. Gravity as a field theory

Description of gravity, in the form of Einstein equations, is very different from anything we do in quantum field theory. To make the long story short, we start with the Einstein - Hilbert action

$$S_E = \frac{1}{16\pi G}\int d^4x \sqrt{-g}\, R = \int d^4x \sqrt{-g}\, M_p^2 R, \quad (*)$$

where $M_p^2 = \frac{1}{16\pi G}$ is the Planck mass squared, $g = \det(g_{\mu\nu})$, R is the scalar curvature and G is the Newton's constant.

The curvature is calculated from the metric tensor through a complex procedure.

One starts from the Rieman tensor

$$R^\lambda{}_{\mu\nu\varkappa} = \partial_\nu \Gamma^\lambda_{\mu\varkappa} - \partial_\varkappa \Gamma^\lambda_{\mu\nu} + \Gamma^\sigma_{\mu\varkappa}\Gamma^\lambda_{\nu\sigma} - \Gamma^\sigma_{\mu\nu}\Gamma^\lambda_{\varkappa\sigma},$$

where $\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\rho}\left(\partial_\nu g_{\rho\mu} + \partial_\mu g_{\rho\nu} - \partial_\rho g_{\mu\nu}\right)$

are Christofell symbols and then calculates the Ricci tensor $R_{\mu\varkappa} = R^\lambda_{\mu\lambda\varkappa}$ and, finally, the curvature $R = g^{\mu\nu} R_{\mu\nu}$.

Taking a variation of S_E with the metric $g_{\mu\nu}$, one finds the Einstein equation

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0.$$

Adding the matter Lagrangian to $\sqrt{-g}\, R$ in Eq. $(*)$, we

find $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \equiv -8\pi G T_{\mu\nu}$,
where $T_{\mu\nu}$ is the energy-momentum tensor of the matter fields

The Einstein action is unique provided we want it to be invariant under general coordinate transformations $x_\mu \to x'_\mu = f_\mu(x)$ and contain at most two derivatives

On the other hand, S_E is action and we want to study S_E in a field theory manner. The first thing we need to understand is what is a "field" (of our field theory)?
Fields in our field theory will be metric fluctuations (small) around the metric that describes the flat background.
We write $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix};$$

$h_{\mu\nu} \equiv h_{\mu\nu}(x)$ is general function but it is "small", so that the expansion of S_E in $\underline{\underline{h}}$ can be truncated.

Suppose we perform an expansion of S_E in h and find

$$S_E = \frac{1}{16\pi G} \int d^4x \left(\partial h \partial h + h \partial h \partial h + h^2 \partial h \partial h + \dots \right).$$

The action can be interpreted as a field theory of the field $h_{\mu\nu}(x)$. This field is what we call "graviton" field. Of course, our gravitational field theory contains infinitely many terms and this is ~~[illegible]~~ different when compared to renormalizable field theories. However it is not a disaster if the non-renormalizability of S_E is understood in an EFT framework.

Before we study S_E in detail, we will do a simpler study by considering the matter action

$$\boxed{S_M = \frac{1}{2}\int d^4x\,\sqrt{-g}\,\{g^{\mu\nu}\partial_\mu\varphi\,\partial_\nu\varphi - m^2\varphi^2\}}$$

We want to expand S_M in $h_{\mu\nu}$ as well.

Recall that $g_{\mu\nu} \simeq \eta_{\mu\nu} + h_{\mu\nu}$. Then

$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$, where $h^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta}h_{\alpha\beta}$

(This is needed to keep $g^{\mu\alpha}g_{\alpha\nu} \equiv \delta^{\mu}_{\nu}$).

Then $-g = -\det[g_{\mu\nu}]$. Up to terms that are linear in h, we have

$$-g = -\det[g_{\mu\nu}] = -(1+h_{00})(-1+h_{11})(-1+h_{22}) \times (-1+h_{33}) =$$

$$= 1 + h_{00} - h_{11} - h_{22} - h_{33} \Rightarrow$$

$$\boxed{-g = 1 + \eta_{\mu\nu}h^{\mu\nu}} \Rightarrow \boxed{\sqrt{-g} \simeq 1 + \frac{1}{2}\eta_{\mu\nu}h^{\mu\nu}}$$

Next, we write $g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi = \eta^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - h^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi$. As the result

$$S_M = \frac{1}{2}\int d^4x\,\{\underbrace{\partial_\mu\varphi\partial^\mu\varphi - m^2\varphi^2}_{2\mathcal{L}_0}\} + \frac{1}{2}\int d^4x\left[\frac{1}{2}\eta_{\mu\nu}h^{\mu\nu}2\mathcal{L}_0 - h^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi\right] =$$

$$= S^\eta_M[\varphi] - \frac{1}{2}\int d^4x\left\{h^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - \frac{1}{2}h_{\mu\nu}\eta^{\mu\nu}2\mathcal{L}_0\right\} \Rightarrow$$

$$\boxed{S_M = S^\eta_M[\varphi] - \frac{1}{2}\int d^4x\, h^{\mu\nu}T_{\mu\nu}},$$ where

$$\boxed{T_{\mu\nu} = \partial_\mu\varphi\partial_\nu\varphi - \eta_{\mu\nu}\mathcal{L}_0}$$ is the canonical energy momentum tensor of the scalar field

The result is general and applies to other "matter theories" as well (fermions, vector bosons, etc.) The total action of the gravity field interacting with matter reads

$$S_{total} = \frac{1}{16\pi G}\int d^4x\,(\partial h\partial h + h\partial h\partial h + h^2\partial h\partial h + \dots) + \int d^4x\left\{-\frac{1}{2}h^{\mu\nu}T_{\mu\nu}\right\}.$$

This form allows us to determine the strength of interactions – all we need to do is to re-scale the h-field.

The h-field becomes $h \to \sqrt{G}\,h$, and

$$S_{total} = \frac{1}{16\pi}\int d^4x\,(\partial h\partial h + \sqrt{G}\,\partial h\partial h\, h + G\,\partial h\partial h\, h^2 + \dots) + \int d^4x\left\{-\frac{\sqrt{G}}{2}h^{\mu\nu}T_{\mu\nu}\right\}$$

Hence, the strength of the N-graviton vertex is $(\sqrt{G})^{N-2}$. Interaction with matter is proportional to $\sqrt{G}$.

Making the next step requires actual calculation of the gravitational part of the action. It is possible to do this directly, but it is very messy. We will try to do it in a slightly different way.

The first thing to do - is to recall that the original action is invariant under general coordinate transformation: $x_\mu \to x_\mu'$. Such transformation induces ~~the~~ transformations of the metric:

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu = g_{\mu\nu} \frac{\partial x^\mu}{\partial x'^\alpha} \frac{\partial x^\nu}{\partial x'^\beta} dx'^\alpha dx'^\beta$$

$$\Rightarrow \quad \boxed{g'_{\alpha\beta} = g_{\mu\nu} \frac{\partial x^\mu}{\partial x'^\alpha} \frac{\partial x^\nu}{\partial x'^\beta}}$$

The inverse metric $g'^{\alpha\beta}$ is given by

$$\boxed{g'^{\alpha\beta} = g^{\mu\nu} \frac{\partial x'^\alpha}{\partial x^\mu} \frac{\partial x'^\beta}{\partial x^\nu}}$$

The question we would like to understand now is what are the implications of these transformations for $h_{\mu\nu}$?

Let us consider infinitesimal transformations

$$\boxed{x'^\mu = x^\mu - \varepsilon^\mu(x)}$$

$\frac{\partial x'^\alpha}{\partial x^\mu} = \delta^\alpha_\mu - \partial_\mu \varepsilon^\alpha$. Since $g'^{\alpha\beta} = \eta^{\alpha\beta} - h'^{\alpha\beta}$ and $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$, we obtain

$$\eta^{\alpha\beta} - h'^{\alpha\beta} = [\eta^{\mu\nu} - h^{\mu\nu}](\delta^\alpha_\mu - \partial_\mu \varepsilon^\alpha)(\delta^\beta_\nu - \partial_\nu \varepsilon^\beta)$$

$$\Rightarrow \boxed{h'^{\alpha\beta} = h^{\alpha\beta} + \partial^\alpha \varepsilon^\beta + \partial^\beta \varepsilon^\alpha} \quad (**)$$

Now, suppose we write S_E. as a function of h. This function should be such that it does not change under general coordinate transformations Eq. (**). We will discuss how to do this explicitely for up to quadratic terms in the action.

There are 4 possible Lorentz-scalars that we can write down: $[h_{\mu\nu} = h_{\nu\mu}]$

1 $\partial_\lambda h^{\mu\nu} \partial^\lambda h_{\mu\nu}$

2 $\partial_\mu h^{\mu\nu} \partial_\nu h^\alpha{}_\alpha$

3 $\partial_\mu h^{\mu\nu} \partial_\alpha h^{\alpha}{}_{\nu}$

4 $\partial_\lambda h^\mu{}_\mu \partial^\lambda h^\nu{}_\nu$.

There is nothing else that can be written down (of course, we do not put two derivatives on a single h since this can be rewritten back to 1 .. 4 using integration-by-parts).

We therefore proceed with writing the most general action for quadratic h-terms with 2 derivatives

$$S_E = \int d^4x \left[a\, \partial_\lambda h^{\mu\nu} \partial^\lambda h_{\mu\nu} + b\, \partial_\mu h^{\mu\nu} \partial_\nu h^\alpha_\alpha + c\, \partial_\mu h^{\mu\nu} \partial_\alpha h^{\alpha\nu} + d\, \partial_\lambda h^\mu_\mu \partial^\lambda h^\nu_\nu \right], \quad 20$$

that

$$\delta S_E = \int d^4x \Big[2a\, \partial_\lambda [\delta h^{\mu\nu}] (\partial^\lambda h_{\mu\nu}) + b \big(\partial_\mu \delta h^{\mu\nu} \partial_\nu h^\alpha_\alpha + \partial_\mu h^{\mu\nu} \partial_\nu \delta h^\alpha_\alpha \big) + 2c \big(\partial_\mu \delta h^{\mu\nu} \big) \big(\partial_\alpha h^{\alpha\nu} \big) + 2d\, \partial_\lambda (\delta h^\mu_\mu)(\partial^\lambda h^\nu_\nu) \Big].$$

The individual terms are:

1) $\partial_\lambda(\delta h^{\mu\nu})\, \partial^\lambda h_{\mu\nu} = -\partial^2 [h_{\mu\nu}]\, \delta h^{\mu\nu} =$
$= -2\, \partial^2 h_{\mu\nu}\, \partial^\mu \varepsilon^\nu = 2 \varepsilon^\nu \partial_\nu \partial^2 h_{\mu\nu}$,

where we used integration-by-parts to remove derivatives that act on "ε".

2) $\partial_\mu \delta h^{\mu\nu} \partial_\nu h^\alpha_\alpha = -[\partial_\mu \partial_\nu h^\alpha_\alpha][\partial^\mu \epsilon^\nu + \partial^\nu \varepsilon^\mu]$
$= -2(\partial^\mu \epsilon^\nu)\, \partial_\mu \partial_\nu h^\alpha_\alpha = 2\varepsilon^\nu \partial^2 \partial_\nu h^\alpha_\alpha$

3) $\partial_\mu h^{\mu\nu} \partial_\nu \delta h^\alpha_\alpha = -(\partial_\mu \partial_\nu h^{\mu\nu})\, 2\, \partial^\alpha \varepsilon^\alpha =$
$= 2\varepsilon^\alpha \partial_\alpha (\partial_\mu \partial_\nu h^{\mu\nu})$

4) $(\partial_\mu \delta h^{\mu\nu})(\partial_\alpha h^{\alpha\nu}) = -\partial_\mu \partial_\alpha h^{\alpha\nu} (\partial^\mu \epsilon^\nu + \partial^\nu \varepsilon^\mu)$
$= \varepsilon^\nu \partial_\alpha \partial^2 h^{\alpha\nu} + \varepsilon^\mu \partial_\mu \partial_\nu \partial_\alpha h^{\alpha\nu}$

5) $\partial_\lambda(\delta h^\mu_\mu)(\partial^\lambda h^\nu_\nu) = 2(\partial^\mu \epsilon^\mu)(-\partial^2 h^\nu_\nu) =$
$= 2\varepsilon^\mu \partial_\mu \partial^2 h^\nu_\nu$.

Next we need to write the variation of the action, combining similar terms. We find

$$\delta S_E = \int d^4x \Big\{ \varepsilon^\nu [\partial^2 \partial^\mu h_{\mu\nu}](4a+2c) + \varepsilon^\nu [\partial^2 \partial^\nu h^\mu_\mu](4B+2d) + \varepsilon^\nu [\partial_\lambda \partial_\mu \partial_\nu h^{\lambda\mu}] \times (2c+2d) \Big\}.$$

Since we want $\delta S_E = 0$ under these transformations, we need $4a+2c=0$; $4b+2d=0$; $2c+2d=0$.

We find $a = -\frac{1}{2}c$; $d=-c$ $B = -\frac{d}{2} = \frac{c}{2}$.

The effective action therefore becomes:

$$S_E = -c \int d^4x \Big\{ \frac{1}{2} \partial_\lambda h^{\mu\nu} \partial^\lambda h_{\mu\nu} - \frac{1}{2} \partial_\lambda h^\mu_\mu \partial^\lambda h^\nu_\nu - \partial_\lambda h^{\lambda\nu} \partial^\mu h_{\mu\nu} + \partial^\mu h^\lambda_\lambda \partial^\nu h_{\mu\nu} \Big\}.$$

We see that the invariance under general coordinate transformation is able to fix the action up to a single overall factor.

A proper action S_E is found by requiring that Newton's gravity is recovered in the small h limit. We find (we'll explain how later)

$$S_E = \int d^4x \Big\{ \frac{1}{32\pi G} I - \frac{1}{2} h_{\mu\nu} T^{\mu\nu} \Big\}, \text{ where}$$

$$I = \frac{1}{2} \partial_\lambda h^{\mu\nu} \partial^\lambda h_{\mu\nu} - \frac{1}{2} \partial_\lambda h^\mu_\mu \partial^\lambda h^\nu_\nu - \partial_\lambda h^{\lambda\nu} \partial^\mu h_{\mu\nu} + \partial^\mu h^\lambda_\lambda \partial^\nu h_{\mu\nu}.$$

If S_E is the QFT action, I is a "free" action & so should give us a propagator of the h-field. We'll start with writing I as $I = h^{\mu\nu}[P_{\mu\nu,\alpha\beta}]h^{\alpha\beta}$, where

$$\boxed{P_{\mu\nu,\alpha\beta} = -\frac{1}{2}\eta_{\mu\alpha}\eta_{\nu\beta}\partial^2 + \frac{1}{2}\eta_{\mu\nu}\eta_{\alpha\beta}\partial^2 + \partial_\mu\partial_\alpha\eta_{\nu\beta} - \eta_{\mu\nu}\partial_\alpha\partial_\beta}$$

We can go to the momentum space, by writing $h_{\mu\nu} \to h_{\mu\nu}(k)e^{ix\cdot k}$ $\Rightarrow$

$\partial_\mu \to ik_\mu$, $\partial^2 \to -k^2$. $\Rightarrow$

$$P_{\mu\nu,\alpha\beta} \to \frac{1}{2}\eta_{\mu\alpha}\eta_{\nu\beta}k^2 - \frac{1}{2}\eta_{\mu\nu}\eta_{\alpha\beta}k^2 - k_\mu k_\alpha\eta_{\nu\beta} + k_\alpha k_\beta\eta_{\mu\nu}.$$

To find the propagator, we need to find an inverse. However, similar to the cause of gauge field theories, the inverse <u>does not exist</u>. To see this note that e.g. $k_\mu P_{\mu\nu,\alpha\beta}k_\beta = 0$, so there are eigenvectors with vanishing eigenvalues, and this should prevent us from finding the inverse.

To get around this problem, we need to "fix the gauge": we require that $h_{\mu\nu}$ satisfies the following condition:

$$\boxed{\partial_\mu h^\mu{}_\nu = \frac{1}{2}\partial_\nu h^\lambda{}_\lambda.}$$

This is called <u>harmonic gauge</u>.

Let us prove that satisfying this condition is possible. Suppose we have $h_{\mu\nu}$ that doesn't satisfy harmonic gauge condition. Then, we perform a transformation $h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu$. The measure of non-conformity with harmonic gauge condition is:

$$\Delta_\nu = \partial_\mu h^\mu_\nu - \frac{1}{2} \partial_\nu h^\lambda_\lambda \to \Delta_\nu + \partial_\mu (\partial^\mu \varepsilon_\nu + \partial_\nu \varepsilon^\mu) - \frac{1}{2} \partial_\nu (2 \partial \cdot \varepsilon) =$$

$$= \Delta_\nu + \partial^2 \epsilon_\nu + \partial_\nu (\partial \cdot \varepsilon) - \partial_\nu (\partial \cdot \varepsilon) = \Delta_\nu + \partial^2 \varepsilon_\nu .$$

Clearly, we can make $\Delta_\nu + \partial^2 \varepsilon_\nu = 0$ for any given Δ_ν.

Now, we can do the following: similar to R_ξ-gauges, we can add a gauge-fixing term to the action

$$S \to S_\xi = \int d^4x \left\{ \frac{1}{32\pi G} I - \frac{1}{2} h_{\mu\nu} T^{\mu\nu} - \frac{\xi}{32\pi G} \left(\partial_\mu h^\mu_\nu - \frac{1}{2} \partial_\nu h^\lambda_\lambda \right)^2 \right\}$$

If we view ξ as the Lagrange parameter, the eq. of. motion $\frac{\delta S}{\delta \xi} = 0$ implies the gauge condition. In principle, we can solve for the graviton propagator for the case of general ξ, but we can also do it in such a way that ξ is fixed in a clever way. It turns out that if $\xi = 1$ is chosen, the last two terms in I (bottom of page 8) drop out. We find

$$S_{\xi=1} = \int d^4x \left\{ \frac{1}{32\pi G} \left(\frac{1}{2} \partial_\lambda h^{\mu\nu} \partial^\lambda h_{\mu\nu} - \frac{1}{2} \partial_\lambda h^\mu_\mu \partial^\lambda h^\nu_\nu + \frac{1}{4} \partial_\nu h^\lambda_\lambda \partial^\nu h^\beta_\beta \right) - \frac{1}{2} h_{\mu\nu} T^{\mu\nu} \right\} =$$

$$S_{\xi=1} = \int d^4x \left[\frac{1}{32\pi G} \left(\frac{1}{2} \partial_\lambda h^{\mu\nu} \partial^\lambda h_{\mu\nu} - \frac{1}{4} \partial_\lambda h^\mu_\mu \partial^\lambda h^\nu_\nu \right) - 1/2\, h_{\mu\nu} T^{\mu\nu} \right].$$

Writing $S^{\xi=1}_{kin} = \int \frac{d^4x}{32\pi G} \frac{1}{2} \left[h^{\mu\nu} K_{\mu\nu,\lambda\sigma} (-\partial^2) h^{\lambda\sigma} \right]$,

where $K_{\mu\nu,\lambda\sigma} = \eta_{\mu\lambda}\eta_{\nu\sigma} - \frac{1}{2}\eta_{\mu\nu}\eta_{\lambda\sigma}$.

To simplify finding the inverse, we first symmetrize $K_{\mu\nu,\lambda\sigma}$, using the symmetry of $h^{\mu\nu}$,

$$K_{\mu\nu,\lambda\sigma} \equiv \frac{1}{2}\left[\eta_{\mu\lambda}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\lambda} \right] - \frac{1}{2}\eta_{\mu\nu}\eta_{\lambda\sigma} =$$

$$= \frac{1}{2}\left[\eta_{\mu\lambda}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\lambda} - \eta_{\mu\nu}\eta_{\lambda\sigma} \right]$$

The inverse should satisfy

$$K_{\mu\nu,\lambda\sigma} (K^{-1})^{\lambda\sigma,\alpha\beta} \equiv \frac{1}{2}\left[\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha} \right] \quad (**)$$

Let's write the inverse as (x_1, x_2 ÷ arbitrary)

$$\boxed{(K^{-1})^{\lambda\sigma,\alpha\beta} = x_1 \eta^{\lambda\sigma}\eta^{\alpha\beta} + x_2 \left(\eta^{\lambda\alpha}\eta^{\sigma\beta} + \eta^{\lambda\beta}\eta^{\sigma\alpha} \right)}$$

It is ~~then~~ a matter of simple algebra to show that Eq. (**) is satisfied provided that $x_1 = -x_2$ and $x_2 = 1/2$. This gives

$$(K^{-1})^{\lambda\sigma,\alpha\beta} = \frac{1}{2}\left(\eta^{\lambda\alpha}\eta^{\sigma\beta} + \eta^{\lambda\beta}\eta^{\alpha\sigma} - \eta^{\lambda\sigma}\eta^{\alpha\beta} \right)$$

$$\equiv K^{\lambda\sigma,\alpha\beta} \Rightarrow$$

$$\boxed{(K^{-1})^{\lambda\sigma,\alpha\beta} = K^{\lambda\sigma,\alpha\beta}}$$

This allows us to find the graviton propagator in harmonic gauge.

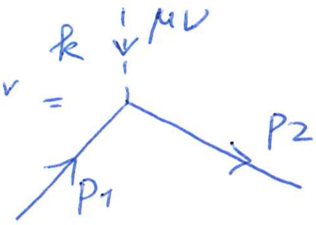
It reads:

$$\boxed{D_{\mu\nu,\lambda\sigma}(k) = \frac{i}{2}\,\frac{\eta_{\mu\lambda}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\lambda} - \eta_{\mu\nu}\eta_{\lambda\sigma}}{k^2 + i0}}\,.$$

Let us now discuss what we can do with that. One thing we can try to do is to see how the Newton's law arises from our field theory. We need to find the interaction vertex of a graviton with the matter fields. The vertex is proportional to the energy momentum tensor $T_{\mu\nu}$ that we take in the scalar theory to be

$$T_{\mu\nu} = \partial_\mu\varphi\,\partial_\nu\varphi - \frac{\eta_{\mu\nu}}{2}\left(\partial_\alpha\varphi\,\partial^\alpha\varphi - m^2\varphi^2\right)$$

The Feynman rule for the vertex V is then given by

$V^{\mu\nu} =$ 

$$= \frac{i}{2}\left(-p_1^\mu p_2^\nu - p_1^\nu p_2^\mu + \eta^{\mu\nu}(p_1\cdot p_2 - m^2)\right)\sqrt{32\pi G}$$

For on-shell scalar particles that scatter off the graviton field, we have

$$k_\mu V^{\mu\nu} = \frac{i}{2}\left(-k\cdot p_1\, p_2^\nu - k\cdot p_2\, p_1^\nu + k^\nu(p_1\cdot p_2 - m^2)\right)\sqrt{32\pi G}$$

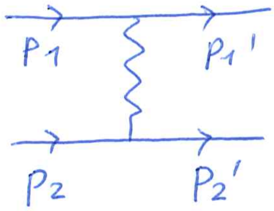
Next, $k = p_2 - p_1 \Rightarrow$ $\begin{array}{l} k\cdot p_1 = -k^2/2 \\ k\cdot p_2 = k^2/2 \end{array}$ $\Rightarrow$

$\frac{1}{2}k^2 = m^2 - p_1\cdot p_2 \Rightarrow$

$$k_\mu V^{\mu\nu} = \frac{i}{2}\left[\frac{k^2}{2}(p_2 - p_1)^\nu + k^\nu\left(-\frac{k^2}{2}\right)\right] \equiv \phi,$$

which is the Ward identity (simplest) for gravity interactions and, incidentally, the conservation property of the energy-momentum tensor.

Now, consider the scattering amplitude of two φ- particles, mediated by the graviton exchange.



In the non-relativistic approximation

$$V^{\mu\nu}(p_1', p_1) = \frac{i}{2}\left(-p_1^{\mu} p_1'^{\nu} - p_1^{\nu} p_1'^{\mu} + \eta^{\mu\nu}(p_1 \cdot p_1' - m^2)\right)\sqrt{32\pi G}$$

$$(p_1 \to (m, \vec{0}),\ p_1' \to (m, 0)) \to -i m^2 g^{\mu 0} g^{\nu 0} \sqrt{32\pi G}.$$

The graviton propagator $[(p_1' - p_1)^{\mu} = (0, \vec{k})]$:

$$\mathcal{D}_{00,00} = \frac{1}{2}\frac{1}{-\vec{k}^2 + i0} \Rightarrow$$

$$iM = (-im_1^2)(-im_2^2)\frac{1}{2}\frac{i}{-\vec{k}^2} 32\pi G \Rightarrow$$

$$\boxed{iM = i\, 16\pi G \frac{m_1^2 m_2^2}{\vec{k}^2},}$$

this is the result of the scattering amplitude of 2 scalar particles caused by the graviton exchange in the non-relativistic approximation.

The interaction potential is obtained by computing a Fourier transform of $M/2m_1 2m_2$

$$\frac{M}{2m_1 2m_2} = 4\pi G \frac{m_1 m_2}{\vec{k}^2} \longrightarrow \frac{G m_1 m_2}{r}. \Rightarrow$$

$$\boxed{U(r) = \frac{G m_1 m_2}{r},}$$

the Newton's gravitational potential.