Lecture 7

False vacuum decay

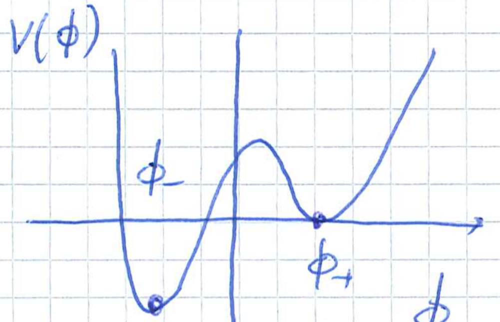
We will now try to apply similar ideas to understand what happens to metastable states in Quantum Field Theory and how does it happen.

To this end, consider a field theory with the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - V(\phi)$$

$$V(\phi) = \frac{\lambda}{4}(\phi^2 - v^2)^2 + \frac{\varepsilon}{2v}\phi + \text{const.} \qquad (*)$$

ε is small, but it splits the values of the two vacua: ϕ_+ and ϕ_-;



$\phi_+ \simeq v$, $\phi_- \simeq -v$ and

$V(\phi_+) - V(\phi_-) \approx \varepsilon$.

The true vacuum state is ϕ_- and the "false" vacuum state is ϕ_+.

Suppose our system is in the ϕ_+ vacuum; we can adjust the constant in Eq. $(*)$ to have $V(\phi_+) = 0$. Thinking about discretized action and neglecting the contribution of $\left(\frac{\partial\phi}{\partial x}\right)^2$ term (that describes mutual dependences of the neighboring points), it is easy to see that for the decay rate is proportional to

$$\Gamma \sim e^{-2\mathcal{V}\int_{\phi_+}^{\phi_-} d\phi \sqrt{2V(\phi)}}$$

, where $\mathcal{V}$ is the spatial volume. In the limit of the spatial volume becoming very large, the ~~decay rate~~ lifetime ~~vanishes~~ of the false vacuum increases exponentially and the decay probability exponentially decreases.

Hence, for a very large system, a decay of a metastable state can not occur homogeneously. The decay of large-volume systems occurs through "bubble creation."

To see how this happens, we imagine the system in the false vacuum. We expect that large field fluctuations are required to go to the "right" vacuum ϕ_-. These large field fluctuations should lead to large action S, that should allow quasiclassical treatment of the problem. Otherwise, we will try to ~~do~~ apply exactly the same construction that we used for the tunneling problems in Quantum Mechanics

Consider the Euclidean action that corresponds to the Lagrangian $\mathcal{L}$:

$$S_E = \int d^4x \left[\frac{1}{2}(\partial_\mu \phi)^2 + V(\phi) \right] \qquad \text{We use the } \cancel{D}\text{-dim. version of space time}$$

Here $(\partial_\mu \phi)^2 = \left(\frac{\partial\phi}{\partial\tau}\right)^2 + \left(\frac{\partial\phi}{\partial x}\right)^2$, so the "metric" is already Euclidean.

The strategy is to look for bounce solutions that approach ϕ_+ in the Euclidian past and ϕ_- at some intermediate time. The equations of motion are

$$\partial_\mu \partial^\mu \phi - \frac{\partial V}{\partial \phi} = 0.$$

~~We will look for radially symmetric solutions, like $\phi(r) = \phi$, with $r = \sqrt{\tau^2 + \sum^D}$~~

We can try to look for solutions that are $SO(4)$ symmetric, i.e. fields that depend on τ and $\vec{x}$ in the following way $\phi \equiv \phi(z)$, $z = \sqrt{\tau^2 + \sum_{i=1}^{3} x_i^2}$.

~~The reason for~~ These solutions - in Minkowski space - turn to proper Lorentz-invariant combinations of time and space for $SO(3,1)$ - symmetric solutions.

We now have a standard QM problem since z will play the role of "time".

We now compute the Laplace operator

$$\frac{\partial \phi(r)}{\partial x_i} = \frac{d\phi}{dr} \times \frac{\partial r}{\partial x_i} = \frac{d\phi}{dr} \times \frac{x_i}{r} \quad (i = 1 .. 4)$$

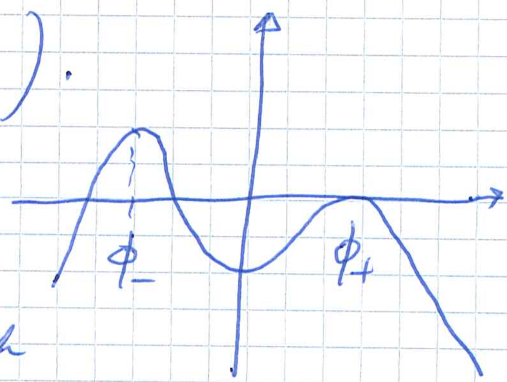
$$\frac{\partial^2 \phi(r)}{\partial x_i^2} = \frac{4}{r}\frac{d\phi}{dr} - \frac{d\phi}{dr}\frac{1}{r} + \frac{d^2\phi}{dr^2} = \frac{d^2\phi}{dr^2} + \frac{3}{r}\frac{d\phi}{dr}$$

Then $\boxed{\frac{d^2\phi}{dr^2} + \frac{3}{r}\frac{d\phi}{dr} = \frac{dV}{d\phi}}$ is the equation of motion. ~~At~~ The solution we look for has $\phi(\tau = -\infty) = \phi_+$. $\tau = -\infty \Rightarrow r = +\infty$

$\Rightarrow \phi(r) \xrightarrow[r\to\infty]{} \phi_+$. The bounce occurs at some time, close to $r = 0$. To have a regular solution we need $\left.\frac{d\phi}{dr}\right|_{r=0} = \phi$ and $\phi(r \to 0) \approx \phi_-$ (this isn't exact equation since ϕ_- is never reached for finite $\mathcal{E}$).

We would like to view the ~~soluti~~ equation for $\phi(r)$ as the Newton's equation for a particle with the mass $m = 1$ that moves in the potential $-V(\phi)$ and is subject to a time-dependent friction force. $\left(\frac{3}{r}\frac{d\phi}{dr}\right)$.

$-V(\phi)$ looks as follows



At $r = 0$ we start it off "close" to ϕ_- and let it reach ϕ_+ at $r = \infty$.

We would like to show that solutions with these properties exist.

To this end, let us consider a situation where $\phi(z=0)$ is close to ϕ_- and let us linearize the equation of motion in $\phi(z)-\phi_- \equiv f$. This will tell us how fast the field rolls at the beginning. We write

$$\frac{\partial V}{\partial \varphi}(\varphi) = \frac{\partial V}{\partial \varphi}(\varphi_-) + \frac{\partial^2 V}{\partial \varphi^2}(\varphi_-)\cdot f \, ;$$

since φ_- is the minimum, $\frac{\partial V}{\partial \varphi}(\varphi_-) = 0$ & $\frac{\partial^2 V}{\partial \varphi^2}(\varphi_-) = \mu^2 > 0$.

The equation becomes

$$\left[\frac{d^2}{dz^2} + \frac{3}{z}\frac{d}{dz} - \mu^2 \right] f = 0. \qquad (*)$$

This equation can be cast into a form of the Bessel equation

$$\boxed{\frac{d^2 w}{dz^2} + \frac{1}{z}\frac{dw}{dz} + \left(1 - \frac{\nu^2}{z^2}\right) w = 0} \qquad (**)$$

To see how the two are connected, write $f(z) = g(z)/z$ & use it in $(*)$.

$$\frac{d^2}{dz^2}\frac{g(z)}{z} = \frac{1}{z}\frac{d^2 g}{dz^2} - 2\frac{1}{z^2}\frac{dg}{dz} - 2\frac{g(z)}{z^3}$$

$$\frac{d}{dz}\left(\frac{g(z)}{z}\right) = \frac{dg/dz}{z} - \frac{g(z)}{z^2} \, ;$$

Combining everything, we find that

Equation (*) becomes

$$\frac{d^2g}{dz^2} + \frac{1}{z}\frac{dg}{dz} + \left(-\mu^2 - \frac{1}{z^2}\right) g(z) = 0$$

Choose $z = i\rho/\mu$; the equation becomes

$$\frac{d^2g}{d\rho^2} + \frac{1}{\rho}\frac{dg}{d\rho} + \left(1 - \frac{1}{\rho^2}\right) g = 0$$

This is the Bessel's equation with the solution

$g = J_1(\rho) = J_1(-iz\mu) \equiv I_1(z\mu)$ (I_1 is the modified Bessel's function).

The function $I_1(\mu z)$ has the following asymptotics :

$$I_1(\mu z) \xrightarrow[z\to\infty]{} \frac{e^{\mu z}}{\sqrt{2\pi z}}$$

$$I_1(\mu z) \xrightarrow[z\to 0]{} \frac{1}{2}\mu z \sum_{k=0}^{\infty} \frac{\left(\frac{1}{4}\mu^2 z^2\right)^k}{k!\,\Gamma(2+k)}.$$

Going back to the equation for f and making sure that the boundary condition at $z=0$ is satisfied, we obtain ; ($f(z) = g(z)/z$)

$$f(z) = 2\left[\phi(0) - \phi_-\right]\left(\frac{I_1(\mu z)}{\mu z}\right).$$

The idea now is to take $\phi(0) - \phi_f$ so small that <u>it takes a lot of time for the system to move</u>

so far away from ~~ϕ_{∞}~~ ϕ_- that the linearized version of the equation breaks down. However, the importance of the friction term $\frac{3}{2}\frac{df}{dr}$ decreases with r. At large enough r, friction can be neglected. Then, the equation becomes $\frac{d^2\phi}{dr^2} = V'(\phi)$ and now the energy is conserved. Suppose this moment is denoted by r_0. Then

$$\frac{1}{2}\left(\frac{d\phi}{dr}\right)^2 - V(\phi)\Big|_{r \geq r_0} = \text{const}.$$

At $r \to \infty$, we want ~~d~~ $\phi \underset{r\to\infty}{\to} \phi_+$ with no kinetic energy $\Rightarrow$ const $= -V(\phi_+)$

$$\Rightarrow \frac{1}{2}\left(\frac{d\phi}{dr}\right)^2\Big|_{r=r_0} = -V(\phi_+) + V(\phi)\Big|_{r=r_0} \Rightarrow$$

if $V(\phi)\big|_{r=r_0} < V(\phi_+)$, the solution is impossible. (particle undershoots). If $V(\phi)\big|_{r=r_0} > V(\phi_+)$, then the particle will overshoot, in general. For our solution to be possible, $\phi(r=r_0)$ should be just right but the existence of this solution should be clear from the continuity of undershooting/overshooting scenarios.

Now, suppose we know that a bounce solution exists. We would like to show that it gives the maximum of the action. To this end, consider a generalized action

$$S_E[\phi_B, v] = \frac{1}{2} v^2 \int d^4x \, (\partial_\mu \phi_B)^2 + v^4 \int d^4x \, V(\phi_B)$$

Change variables to ~~ϕ~~ $x = y/v$; $\frac{\partial}{\partial x_\mu} = v \frac{\partial}{\partial y_\mu}$

$d^4x = \frac{d^4y}{v^4}$ $\Rightarrow$

$$S_E[\phi_B, v] = \frac{1}{2} \int d^4y \left\{ \left[\frac{\partial}{\partial y_\mu} \phi_B\left(\frac{y}{v}\right) \right]^2 + V\left(\phi_B\left(\frac{y}{v}\right)\right) \right\}$$

Hence, if a bounce solution contains ~~the~~ for the scale that fixes a typical ~~bubble~~ size of a bubble that is nucleated during the transition, v is a parameter that changes it. The question we would like to explore is whether the action increases or decreases when the bubble size changes.

To see this, note that since the ~~bubble~~ bounce solution extremizes the action, $\left.\frac{\partial S_E}{\partial v}\right|_{v=1} = 0$. This implies

$$0 = 2\frac{1}{2}\int d^4x \, (\partial_\mu \phi_B)^2 + 4\int d^4x \, V(\phi_B) \Rightarrow$$

$$\boxed{\int d^4x \, (\partial_\mu \phi_B)^2 = -4 \int d^4x \, V(\phi_B)}$$

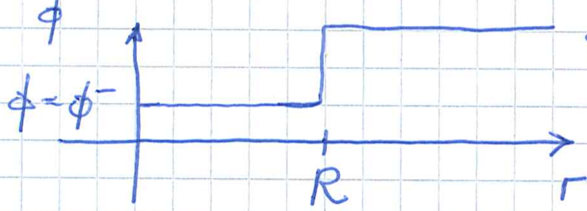
Now, $$\left.\frac{d^2 S_E}{dv^2}\right|_{v=1} = \int d^4x\,(\partial_\mu \phi_B)^2 + 12 \int d^4x\, V(\phi_B) =$$
$$= \int d^4x\,(\partial_\mu \phi_B)^2 - 3 \int d^4x\,(\partial_\mu \phi_B)^2 =$$
$$= -2 \int d^4x\,(\partial_\mu \phi_B)^2 < 0 \quad \Rightarrow$$

~~Then~~ Inflating/deflating the size of the bubble decreases the action. $S_E[\phi_B]$ is the <u>maximum</u>.

Analytic ~~solution~~ ~~of the~~ form of the bounce solution $\phi_B(r)$ is not known. However, there is an approximation that allows us to find the action and the lifetime of the false vacuum. This approximation is the <u>thin wall</u> approximation for the bubble.

for general $V(\phi)$

We consider the following profile of a bubble



The idea is that if a typical size of ~~the~~ a bubble is large, then the curvature of the wall is small and we can treat the field profile as a 1-dimensional (r-variable) kink.

Energy densities of this field configuration are: $V(\phi_+) = 0 \qquad V(\phi_-) = -\mathcal{E}$.

The energy of the transition wall

We can be read of from the kink solution. The energy density of the kink (wall) $T = \frac{4gv^3}{3\sqrt{2}}$;

the Higgs mass is $m_H = \sqrt{2}\, g v$

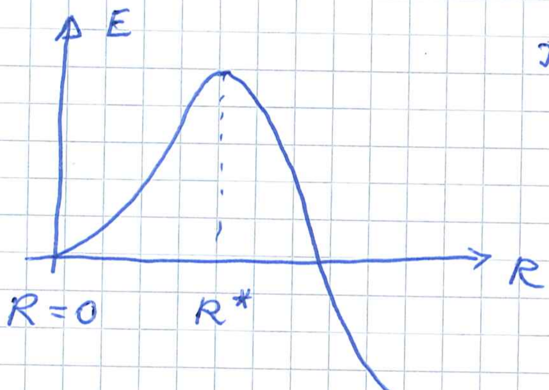
$$\Rightarrow \boxed{T = \frac{4(gv)^3}{3\sqrt{2}\, g^2} = \frac{4\, m_H^3}{3\sqrt{2}\, 2\sqrt{2}\, g^2} = \frac{m_H^3}{3g^2}}$$

The area of the transition wall is an area of $(4-1)$-dim. sphere; it is given by $2\pi^2 R^3$. Hence, the Energy functional is

$$E[R] = -\varepsilon \cdot \text{Volume}_{D=4}(R) + \frac{m_H^3}{3g^2}\,\text{Volume}_{D-1}(R) \Rightarrow$$

$$\boxed{E(R) = -\varepsilon \frac{\pi^2}{2} R^4 + \frac{m_H^3}{3g^2}\, 2\pi^2 R^3 .}$$

We therefore see that the dependence of the energy of the produced bubble on its radius is as follows:



first, it costs energy to produce bubbles but after some critical radius, it becomes energetically favorable.

To find R^*: $\left.\frac{\partial E(R)}{\partial R}\right|_{R=R^*} = 0 \Rightarrow$

$$\frac{\partial E}{\partial R}\; -2\varepsilon\pi^2 R^3 + \frac{m_H^3}{\cancel{3}g^2}\, 2\pi^2 R^2 \Rightarrow$$

$$\cancel{R_* = \frac{m_H^3}{g^2\varepsilon}}$$

$$\boxed{R_* = \frac{m_H^3}{g^2 \varepsilon}}$$

We see that as the two vacua become degenerate ($\mathcal{E} \to 0$), the size of a critical bubble grows and in the limit $\mathcal{E} \to 0$ it becomes infinite.

To calculate the lifetime of a metastable vacuum we require the Eucledian action of the bounce solution (in the thin wall approximation). The action is

$$S_B = \int d^4x \left[(\partial_\mu \phi_B)^2 + V(\phi_B) \right] \Rightarrow$$

$$S_B = 2\pi^2 \int_0^\infty r^3 dr \left[\left(\frac{d\phi}{dr}\right)^2 + V(\phi_B) \right] =$$

$$\equiv -\frac{2\pi^2 R_*^4}{4} \mathcal{E} + 2\pi^2 R_*^3 \cdot T \equiv E(R_*)$$

Note that $S_B = E(R^*)$ and $\frac{\partial E}{\partial R^*} = 0$ implies that bounce with R^* is the extremum

Hence, the lifetime is $\tau = 1/\Gamma_{vac}$, where

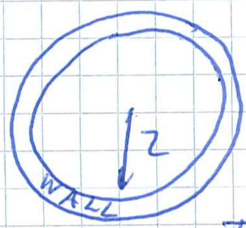
$$\boxed{\Gamma_{vac} \sim e^{-S_B} \sim e^{-\frac{27}{2}\pi^2 T^4/\mathcal{E}^3}}$$, where

$T = \frac{m_H^3}{3g^2}$ is the "wall" tension.

In the limit $\mathcal{E} \to 0$, the transition rate becomes ~~very~~ vanishingly small.

We have seen now the construction of a bounce solution in Euclidean space. It would be interesting to see how it works in Minkowski space as well. The picture is, essentially, similar.

At a time t, we have a bubble of a radius r. Inside we have a true vacuum. Outside – the metastable one.



The energy balance is

$$E_{inside} = \frac{4}{3}\pi r^3(-\mathcal{E}), \qquad E_{wall} = T \cdot 4\pi r^2,$$

$$E_{outside} = \emptyset$$

where T – tension.

Hence, the "potential" energy of the bubble is

$$E_{inside} + E_{wall} = \frac{4}{3}\pi r^3(-\mathcal{E}) + T \cdot 4\pi r^2$$

$$\equiv -\frac{\mathcal{E}\cdot 4\pi r^2}{3}(r - r_*), \text{ where } r_* = \frac{3T}{\mathcal{E}},$$

the critical size of the bubble.

~~Again,~~ For the bubble to expand, it must have kinetic energy $E_{kin} > 0$.

The initial energy of the world (false vacuum) is $E = \emptyset$. $\Rightarrow$ Energy is conserved.

Therefore for $E_{kin} + E_{inside} + E_{wall} = 0$, one needs $\underline{r > r_*}$.

The dynamic of the expanding bubble is described by the Lagrangian

$$L = -\underbrace{4\pi r^2 T}_{\text{"mass of the wall"}} \sqrt{1-\dot{r}^2} + \underbrace{\frac{4}{3}\pi r^3 \mathcal{E}}_{\text{["$-$" potential energy inside the bubble]}}$$

The first term is the standard Lagrangian for the motion of a relativistic particle along the radius. We can treat this Lagrangian using standard methods of classical mechanics.

First, we compute the Hamiltonian.

$$p = \frac{\partial L}{\partial \dot{r}} = \frac{4\pi r^2 T \dot{r}}{\sqrt{1-\dot{r}^2}} \qquad H = \frac{\partial L}{\partial \dot{r}}\dot{r} - L =$$

$$= \frac{4\pi r^2 T \dot{r}^2}{\sqrt{1-\dot{r}^2}} + 4\pi r^2 T \sqrt{1-\dot{r}^2} - \frac{4}{3}\pi r^3 \mathcal{E} \Rightarrow$$

$$\boxed{H = \frac{4\pi r^2 T}{\sqrt{1-\dot{r}^2}} - \frac{4}{3}\pi r^3 \mathcal{E}}$$

At $\dot{r} = 0$, we have the expression for the energy that we already had.

To proceed further, we write H through p.

$$p^2 = \frac{(4\pi r^2 T)^2 \dot{r}^2}{(1-\dot{r}^2)} = -(4\pi r^2 T)^2 + \frac{(4\pi r^2 T^2)^2}{1-\dot{r}^2}$$

$$\Rightarrow \left(H + \frac{4}{3}\pi r^3 \mathcal{E}\right)^2 = \frac{(4\pi r^2 T)^2}{1-\dot{r}^2} = \left(p^2 + (4\pi r^2 T)^2\right)$$

So that:

$$\left(H + \frac{4\pi}{3} r^3 \mathcal{E}\right)^2 - P^2 = \left(4\pi r^2 T\right)^2$$

Since the energy of spontaneously nucleated bubble should vanish, we find $(H \to 0)$

$$P = 4\pi r^2 T \sqrt{\frac{r^2}{r_*^2} - 1}$$

From $H \to 0$ condition, we find the equation for $\frac{dr}{dt}$: $\dot{r} = \sqrt{1 - \left(\frac{r_*}{r}\right)^2}$

It's solution is $r = \sqrt{r_*^2 + t^2}$ or $r^2 - t^2 = r_*^2$. We see that the bubble expands with the speed of light.

Given the classical solution for the equation of motion, we can calculate the life-time of the ~~va~~ false vacuum in a more familiar way by considering

$$\Gamma \sim \exp\left[-2 \int_0^{r_*} dr \left|P_E(r)\right|\right]$$, so that

the integration goes over the range of r which are classically forbiden $0 < r < r^*$.

$$P_E(r) = 4\pi r^2 T \sqrt{1 - \frac{r^2}{r_*^2}} \Rightarrow$$

$$4\pi T \int_0^{r_*} dr\, r^2 \sqrt{1 - \frac{r^2}{r_*^2}} = 4\pi T r_*^3 \int_0^1 dx\, x^2 \sqrt{1 - x^2} =$$

$$= 4\pi T r_*^3 \int_0^{\pi/2} d\theta \cos^2\theta \sin^2\theta = \frac{4\pi T r^{*3}}{4} \int_0^{\pi/2} d\theta \sin^2(2\theta) =$$

$$= \frac{\pi T r^{*3}}{2} \int_0^{\pi} d\bar{\theta} \sin^2\bar{\theta} = \frac{\pi T r^{*3}}{2} \int_0^{\pi} d\bar{\theta} \frac{1-\cos(2\bar{\theta})}{2} =$$

$$= \frac{\pi^2 T r^{*3}}{4}.$$ Therefore

$$\Gamma_{vac} \sim e^{-2\int_0^{r^*} p(r)\,dr} = e^{-\frac{\pi T r^{*3}}{2}} = e^{-\frac{27}{2}\pi^2 \frac{T^4}{\varepsilon^3}}.$$

The result is the same as what one obtains using calculations ~~with~~ in Euclidean space.