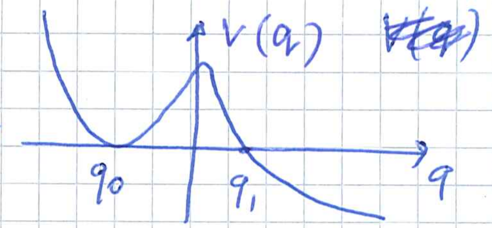
Lecture 6

Tunneling problem in Quantum Mechanics and Schwinger phenomenon

1) The simplest system where the problem of a decay of a meta-stable state arises is that of Quantum Mechanics.

Consider a potential.



There is a quasi-stationary bound state localized at $q = q_0$. However, since the height of the barrier to the right of it is not infinite, this state isn't stable; it tunnels and escapes to $q = \infty$ (the true vacuum of the system).

The lifetime is given by $\tau = 1/\Gamma$

$$\Gamma = A e^{-S_B}, \text{ where}$$

$$S_B = 2 \int_{q_0}^{q_1} \sqrt{2MV(q)}\, dq \,,$$ M is the mass of a particle.

This formula isn't generic – to simplify it, we have choosen $V(q_0) = 0$, so that the total energy of the "bound" state vanishes (we neglect small quantum effects).

An interesting point about S_B is that it can be thought of as the value that Euclidean action takes for a particle moving along the classical trajectory from the point q_0 to the point q_1 and back with zero energy.

Let's see exactly what that means. ~~We take~~ The "normal" action for this type of a problem is

$S = \int \left[\frac{m}{2} \left(\frac{dq}{dt} \right)^2 - V(q) \right] dt$. Let's make a change of variables $t = -i\tau$, so that

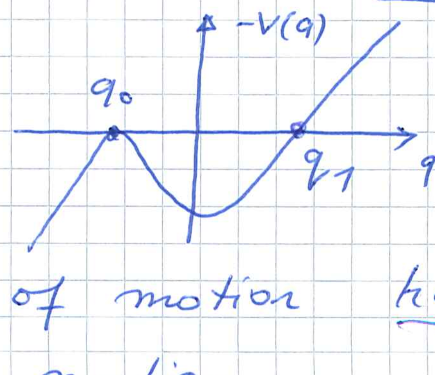
$\frac{dq}{dt} = \frac{1}{-i} \frac{dq}{d\tau}$; $dt = -i\, d\tau$. Then,

$S = +i\, S_E$, where $S_E = \int d\tau \left[\frac{m}{2} \left(\frac{dq}{d\tau} \right)^2 + V(q) \right]$

This action S_E got its name because the Minkowski interval $dt^2 - dq^2$ becomes $-(d\tau^2 + dq^2)$ under $t = -i\tau$ change. and $d\tau^2 + dq^2$ is the Euclidean interval.

Take S_E and find equations of motion:

$$\boxed{+m \frac{d^2 q}{d\tau^2} - \frac{\partial V}{\partial q} = 0},$$ which corresponds to the regular motion in the potential $-V(q)$:



The equation of motion has an integral of motion

$$\frac{m}{2} (\dot{q})^2 - V(q) \equiv E,$$ that we can call "Euclidean" energy.

Consider the $\mathcal{E}=0$ solution, that starts at ~~located at~~ $q=q_0$ at $\tau=-\infty$, reaches q_1 and then returns back to q_0 at $\tau=+\infty$. We can always assume that q_1 is reached at $\tau=\tau_0$. Let's call such a solution "a bounce" and denote it as $q_B(\tau)$.

Then, $\frac{1}{2}m\left(\frac{dq_B}{d\tau}\right)^2 \equiv V(q_B(\tau)) \Rightarrow$

$$S_E[q_B(\tau)] = \int_{-\infty}^{\infty} d\tau \left[\frac{m}{2}\left(\frac{dq_B}{d\tau}\right)^2 + V(q_B(\tau))\right] =$$

$$= 2\int_{-\infty}^{0} d\tau \left[\frac{m}{2}\left(\frac{dq_B}{d\tau}\right)^2 + V(q_B(\tau))\right] =$$

$$= 2\int_{-\infty}^{0} d\tau \left[2V(q_B(\tau))\right].$$

From the Euclidean energy conservation law, we find $dq_B\sqrt{\frac{m}{2V(q_B)}} = d\tau \Rightarrow$

$$S_E[q_B(\tau)] = 2\int_{q_0}^{q_1} dq_B\sqrt{2mV(q_B)} = 2\int_{q_0}^{q_1} dq\sqrt{2mV(q)}$$

which coincides with the ~~exponential~~ exponent of the standard expression for the tunneling rate in Quantum Mechanics.

Our goal, eventually, will be to apply similar formulas to Quantum field theory problems. Let's discuss why this is possible.

To understand this, we imagine that we have to study Quantum field theory with the action $S = \int dt\, d^3\vec{x} \left[\frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - V(\varphi) \right]$

We now assume that the space time is discretized and the spatial variable x is defined on a lattice, with a lattice spacing a. $\vec{x} = a\vec{n}$ $n \in$ integers

$$\partial_\mu \phi \partial^\mu \phi = \left(\frac{\partial \varphi_{\vec{i}}}{\partial t}\right)^2 - \frac{\partial \varphi_{\vec{i}}}{\partial \vec{x}} \frac{\partial \varphi_{\vec{i}}}{\partial \vec{x}} = \left(\frac{\partial \varphi_i}{\partial t}\right)^2 - \frac{1}{a^2}\left(\varphi_{\vec{i}+\vec{e}} - \varphi_i\right)^2$$

and $d^3\vec{x} \equiv a^3 \sum_{\vec{i}}$. Hence,

$$S = a^3 \sum_{\vec{i}} \int dt \left[\left(\frac{d\varphi_{\vec{i}}}{dt}\right)^2 - \frac{1}{a^2}\left(\varphi_{\vec{i}+\vec{e}} - \varphi_i\right)^2 - V(\varphi_i) \right]$$

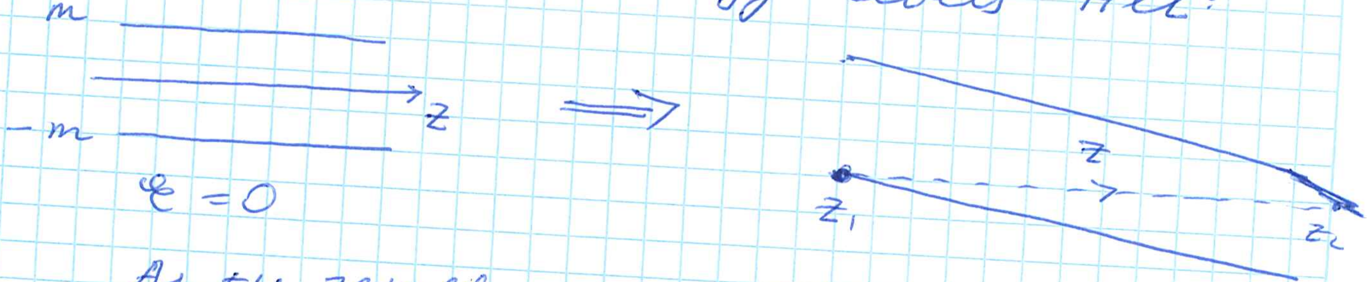
For any <u>finite</u> <u>volume</u>, this is an action of a classical problem with very large but finite number degrees of freedom. Essentially one can view $\varphi(t, \vec{i}) \equiv \varphi_{\vec{i}}(t)$ as "<u>coordinates</u>" and "<u>$\vec{i}$</u>"'s as labels. This classical action can be ~~[illegible]~~ studied using ordinary methods of Quantum Mechanics — <u>and this includes</u> ~~[illegible] using regular formulas~~ <u>issues of vacuum stability.</u> $\boxed{\text{Lattice method in general: see later}}$

We have <u>not</u> discussed how to deal with vacuum stability issues in Quantum Mechanical problems with large number of degrees of freedom. However, the QFT problems we will be dealing with will have a high degree of symmetry that will allow us to focus on the non-trivial dependence of the action on just one "label". The problem then will be very similar to what we have in Quantum mechanics.

We will now try to apply this understanding to compute the rate for producing electron-positron pairs in a constant electric field in a vacuum. (Schwinger)

Consider an empty space with a constant electric field (along the z-axis) $\mathcal{E}$. The potential energy of a particle with the charge e is $U = -e\mathcal{E}z$. When a particle moves in such a field, the total energy $E = \sqrt{\vec{p}^2c^2 + m^2c^4} - e\mathcal{E}z$ and the transverse momentum $\vec{p}_\perp = (p_x, p_y, 0)$ are conserved.

We will assume that the system is in a vacuum state and that the vacuum wave function is given by by a Dirac sea: all levels with energies $E < -m$ filled with electrons and all levels with energies $E > m$ empty. If the constant electric field is applied the energy levels tilt:



As the result, any of the filled energy levels (at a point z_1) can be made free by electron tunnelling from z_1 to z_2. If this happens, electron and a hole will be created with an interpretation that production of an e^+e^- pair occured.

Let $E = \pm\sqrt{m^2c^4 + p^2c^2} - e\mathcal{E}z$ be the energy of a particle in the Dirac sea. The longitudinal momentum

$$p_z(z) = \frac{1}{c}\sqrt{(e\mathcal{E}z + E)^2 - m^2c^4 - p_\perp^2c^2}.$$

$p_z(z) = 0$ for $z_{1,2} = \dfrac{-E}{e\mathcal{E}} \mp \dfrac{\sqrt{m^2c^4 + p_\perp^2c^2}}{e\mathcal{E}}.$

A particle from the Dirac sea enters a barrier at $z = z_1$ and leaves at $z = z_2$.

The region $z_1 < z < z_2$ is kinematically forbidden, so to get from z_1 to z_2 requires tunneling.

So, the underbarrier action is

$$S_B = 2\int_{z_1}^{z_2} |P(z)|\, dz = \frac{2}{c}\int_{z_1}^{z_2} \sqrt{e^2 \mathcal{E}^2 (z-z_1)(z_2-z)}\, dz$$

$$= \frac{2}{c}\, e\mathcal{E}\,(z_2-z_1)^2 \int_0^1 d\xi\, \sqrt{\xi(1-\xi)} =$$

$$= \frac{2}{c}\, e\mathcal{E}\,(z_2-z_1)^2 \cdot \frac{\pi}{8} = \frac{2}{c}\,\frac{(m^2c^4 + p_\perp^2 c^2)}{e\mathcal{E}} \cdot \frac{\pi}{2}$$

$$\Rightarrow S_B = \pi \frac{(m^2c^2 + p_\perp^2)\,c}{e\mathcal{E}}.$$

The transition rate is given by

$$W = A e^{-S_B/\hbar} = A \exp\left[-\frac{\pi (m^2c^2 + p_\perp^2)\,c}{e\mathcal{E}\cdot\hbar}\right]$$

We can now estimate the coefficient A – the pre-exponential factor.

To this end, consider a particle in the element of momentum space $d^3\vec{p} = d^2\vec{p}_\perp\, dp_z$. Their <u>space</u> density is

$dn = \dfrac{2\, d^3\vec{p}}{(2\pi\hbar)^3}$, where a factor 2 corresponds to two spin orientations of the electron. The number of particles passing through an elementary area $dx\,dy$ to the left of the barrier is

$dN = dj_z(z)\,dx\,dy$ with $dj_z(z) = v_z(z)\,dn_p$

This includes $v_z(z)\,dp_z = \frac{\partial E_{kin}}{\partial p_z} dp_z = dE_{kin}$

Hence $dN \equiv \frac{2 d^2\vec{p}_\perp}{(2\pi\hbar)^3} dE_{kin}\,dx\,dy \equiv \frac{2 d^2\vec{p}_\perp}{(2\pi\hbar)^3} e\mathcal{E}\,dz\,dx\,dy$

Hence, the ~~totatal~~ total number of pairs created in a given volume $dV = dx\,dy\,dz$, we need to multiply $e^{-S_b/\hbar}$ with dN. Hence $A \sim dN$.

We find for the total number of pairs created per unit time in a unit volume

$$P_{1/2} = \frac{dW}{dt\,dV} = 2e\mathcal{E}\int_0^\infty \frac{d^2\vec{p}_\perp}{(2\pi\hbar)^3}\, e^{-\frac{\pi(m^2c^2+\vec{p}_\perp^{\,2})c}{e\mathcal{E}\hbar}} \Rightarrow$$

$$\boxed{P_{1/2} = \frac{e^2\mathcal{E}^2}{4\pi^3\hbar^2 c}\exp\left[-\frac{\pi m^2 c^3}{e\mathcal{E}\hbar}\right]}$$

Our calculation is valid in the <u>quasi-classical approximation</u>, where the exponent is ~~small~~ large

$\frac{m^2c^3}{e\mathcal{E}\hbar} \gg 1$. What does this mean physically? To see this, note that ~~to~~ the length of the tunneling path $|z_2 - z_1|$ is $l \sim |z_2 - z_1| \sim \frac{mc^2}{e\mathcal{E}}$. The Compton wave length of the electron is $\lambda = \frac{\hbar}{mc}$.

Hence $\frac{\ell}{\lambda} = \frac{m^2 c^3}{e \mathcal{E} \hbar}$. Therefore, the applicability of the quasi-classical approximation implies $\ell/\lambda \gg 1$.

The tunneling probability is very small. Take the hydrogen atom: The Bohr radius is $a_B \sim \frac{1}{\alpha m}$; the electric field is $eE_{at} \simeq \frac{\alpha}{a_B^2} \sim m^2 \alpha^3$; Then (in $c \to 1$, $\hbar \to 1$ units)

$$\frac{m^2}{e \mathcal{E}} = \frac{m^2}{m^2 \alpha^3} \simeq \frac{1}{\alpha^3}$$

$$= 1/\alpha^3 \sim 10^7 \Rightarrow$$

Probability of spontaneous ~~vacuum~~ pair creation is $\sim \exp(-\pi \times 10^7) \ll 1$.
