Lecture 5 Monopoles

The simplest model in which monopole solutions appear is the so-called Georgi-Glashow model - an $SU(2)$ gauge theory with a scalar field in adjoint representation.

The Lagrangian is

$$\mathcal{L} = -\frac{1}{4g^2} G^a_{\mu\nu} G^{a,\mu\nu} + \frac{1}{2}(D_\mu \phi^a)(D^\mu \phi^a) - \lambda(\phi^a\phi^a - v^2)^2$$

$$G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \varepsilon^{abc} A^b_\mu A^c_\nu$$

$$(D_\mu \phi)^a = \partial_\mu \phi^a + \varepsilon^{abc} A^b_\mu \phi^c$$

We may also use the notation $\phi = \phi^a \frac{\tau^a}{2}$;

It is useful to first focus on a special case $\lambda \to 0$. This means that the scalar field self-interacting potential will not contribute to field equations but it will enforce the boundary condition for the ϕ-field at $r \to \infty$; $\phi^a\phi^a \equiv v^2$.

A standard treatment of the problem assumes a particular choice of the vacuum: $\phi^a_{vac} = v\delta^{3a}$. The spectrum of the theory then is:

2 massive gauge fields, with the masses $m_V = gv$ and 1 massless gauge field (corresponds to remaining

symmetry of the vacuum state ~~to~~ ~~rotat~~ under rotations around the third axis). The Higgs field has the mass $m_H^2 \sim \lambda v^2$ and so $m_H \to 0$ as $\lambda \to 0$.

We want to look for topologically non-trivial solutions with finite energy. The required boundary condition is $\phi^a \phi^a = v^2$ on $r \to \infty$. At $r = \infty$, the space is a sphere. Hence $\phi^a(\vec{r})\,\phi^a(\vec{r})\big|_{r=\infty} = v^2$ describes a mapping of a spatial sphere to the sphere in the "isotopic" space. This is a topologically non-trivial mapping, similar to the case of a vortex.

Suppose we can choose $A_0^a = 0$ and study static solutions. Then $F_{0i} = \partial_0 A_i = 0$ and the only non-trivial component of the field is G_{ij}^a. The contribution of the ~~energy~~ gauge field to the ~~energy~~ energy functional ~~to~~ ~~gauge~~ is

$$\frac{1}{4g^2} G_{ij}^a G^{aij} \equiv \frac{1}{2g^2} B_i^a \cdot B_i^a, \quad \text{where } B_i^a = -\frac{1}{2}\varepsilon_{ijk} G_{jk}^a$$

The energy functional is then ($\lambda \to 0$!)

$$E = \int d^3\vec{x} \left\{ \frac{1}{2g^2} B_i^a \cdot B_i^a + \frac{1}{2} (\mathcal{D}_i \phi)^a (\mathcal{D}_i \phi)^a \right\}$$

We can now re-write the integrand

$$\frac{1}{2g^2} B_i^a \cdot B_i^a + \frac{1}{2}(\mathcal{D}_i\phi)^a(\mathcal{D}_i\phi)^a = \frac{1}{2}\left(\frac{1}{g}B_i^a - (\mathcal{D}_i\phi)^a\right) \times$$
$$\times \left(\frac{1}{g}B_i^a - (\mathcal{D}_i\phi)^a\right) + \frac{1}{g} B_i^a (\mathcal{D}_i\phi)^a .$$

Now, we take the last term & consider it's contribution to the energy:

$$\int d^3\vec{x}\, B_i^a\left(\partial_i\phi^a + \varepsilon^{abc}A_i^b\phi^c\right) = \int d^3\vec{x}\left(\partial_i(B_i^a\phi^a) - (\partial_i B_i^a)\phi^a + \varepsilon^{abc}A_i^b\phi^c B_i^a\right) =$$

$$= \int d^3\vec{x}\left[\partial_i(B_i^a\phi^a) - (\mathcal{D}_i B_i)^a\phi^a\right]$$

$(\mathcal{D}_i B_i)^a = 0$ by equations of motion for the gauge field. As the result ~~the~~ we find

$$\int d^3\vec{x}\, B_i^a(\mathcal{D}_i\phi)^a = \int d^3\vec{x}\,\partial_i(B_i^a\phi^a) \equiv \int_{|\vec{r}|\to\infty} d^2\vec{S}\,(\vec{B}^a\phi^a)$$

Hence, the energy functional reads

$$\boxed{E = \int d^3\vec{x}\left\{\frac{1}{2}\left[\frac{1}{g}B_i^a - \mathcal{D}_i\phi^a\right]\left[\frac{1}{g}B_i^a - \mathcal{D}_i\phi^a\right]\right\} + \frac{1}{g}\int_{|\vec{r}|\to\infty} d\vec{S}\,(\vec{B}^a\phi^a).}$$

Now, we will take the field ϕ^a ~~to be~~ at $r=\infty$ to be $\phi^a = v\dfrac{\vec{r}^{\,a}}{|\vec{r}|}$ (this corresponds to winding number 1). Next, we compute $\mathcal{D}_i\phi^a = \partial_i\phi^a + \varepsilon^{abc}A_i^b\phi^c$.

$$\partial_i\phi^a = \frac{v}{r}\left(\delta^{ia} - n^a n^i\right),$$ so that

$$\mathcal{D}_i\phi^a = \frac{v}{r}\left(\delta^{ia} - n^a n^i\right) + \varepsilon^{abc}A_i^b\phi^c.$$

$$= \frac{v}{r}\left(\delta^{ia} - n^a n^i\right) + \varepsilon^{abc} A_i^b \frac{v}{r} \cdot r^c =$$

$$= \frac{v}{r}\left(\delta^{ia} - n^a n^i + \varepsilon^{abc} A_i^b r^c\right).$$ We want $D_i \phi^a$ to vanish faster than $1/r$ $\Rightarrow$

$$\delta^{ia} - n^a n^i + \varepsilon^{abc} A_i^b r^c = 0 \quad \Rightarrow \quad \boxed{A_i^b \xrightarrow[r\to\infty]{} \frac{\varepsilon^{bik} r^k}{r^2}}$$

It is then easy to compute the magnetic field: $B_i^a = -\frac{1}{2}\varepsilon_{ijk} G_{jk}^a$; $G_{jk}^a = \partial_j A_k^a - \partial_k A_j^a + \varepsilon^{abc} A_j^b A_k^c$

Now $$\partial_j A_k^a \Big|_{r\to\infty} = \partial_j \varepsilon^{akm} \frac{r^m}{r^2} = \frac{\varepsilon^{akj}}{r^2} - 2\varepsilon^{akm}\frac{r^m r^j}{r^4}$$

$$\Rightarrow \varepsilon_{ijk}\, \partial_j A_k^a = \varepsilon_{ijk}\left\{\frac{\varepsilon^{akj}}{r^2} - 2\varepsilon^{akm}\frac{r^m r^j}{r^4}\right\} =$$

$$\equiv -\frac{2\delta^{ia}}{r^2} + 2\left(\delta^{ia}\delta^{jm} - \delta^{im}\delta^{aj}\right)\frac{r^m r^j}{r^4} =$$

$$= -\frac{2\delta^{ia}}{r^2} + 2\left(\frac{\delta^{ia}}{r^2} - \frac{r^i r^a}{r^4}\right) = -\frac{2 r^i r^a}{r^4};$$ calculating the remaining terms (hw!), we find

$$\boxed{B_i^a \longrightarrow \frac{1}{r^2} n^a n_i, \quad \vec{n} = \frac{\vec{r}}{r}}$$

Now, from our construction of the energy functional, $\frac{1}{g} B_i^a - D_i \phi^a = 0$ ~~gives~~ the energy will be minimal ~~potentially~~ ~~finite~~ for field ~~contribution~~ ~~to energy~~, ~~but~~ ~~the~~ configurations.

The minimal energy is

$$E_{min} = \frac{1}{g} \int\limits_{|\vec{r}|\to\infty} d\vec{S} \cdot \left(\vec{B}^a \cdot \phi^a\right).$$

Let us denote $\vec{B}^a \cdot \hat{\phi}^a \Big|_{vac} = \vec{B}$; ~~call~~ ~~called~~

then $$\vec{B}^i \Big|_{r\to\infty} \to \frac{1}{r^2} n^a \cdot n_i \cdot v \cdot n^a = \frac{n_i}{r^2}$$

Therefore $\boxed{E_{min} = \frac{1}{g}\int r^2 d\Omega^{4\pi} \frac{1}{r^2} = \frac{4\pi}{g}}$

We have just found that the magnetic field has a non-vanishing flux through a 3-d sphere. According to Gauss' law, this means that the sphere encloses magnetic charge – the monopole.

We can now try to find the field configurations that give the monopole solution. For this we need to have

$\frac{1}{g} B_i^a - \mathcal{D}_i \phi^a = \emptyset$ & the proper asymptotic behaviour of the fields at $r=\infty$

Since $\phi^a = v\frac{r^a}{|\vec{r}|}$ at $r\to\infty$, we require

$\boxed{\phi^a = v n^a H(r)}$ for general r.

Similarly, $\boxed{A_i^a = \varepsilon^{aij} \frac{1}{r} n^j F(r).}$

Then $H(r) \xrightarrow[r\to\infty]{} 1$, $F(r) \xrightarrow[r\to\infty]{} 1$, for the monopole solution.

Let us check that this Anzats is consistent with the equations. For the magnetic field we get

$$B_i^a = (\delta^{ai} - n^a n^i)\frac{1}{r}\frac{dF}{dr} + n^a n^i \frac{1}{r^2}(2F - F^2)$$

$$\mathcal{D}_i \phi^a = v(\delta^{ai} - n^a n^i)\frac{1}{r} H(1-F) + v n^a n^i \frac{dH}{dr}$$

Requiring $\mathcal{D}_i\phi^a - \frac{1}{g}B_i^a = 0$, we get

two equations:

$$\begin{cases} \frac{1}{g}\cdot\frac{1}{r}\frac{dF}{dr} - \frac{v}{r}H(1-F) = 0 \\ \frac{1}{g}\frac{1}{r^2}(2F-F^2) - \frac{dH}{dr} = 0 \end{cases} \Rightarrow \begin{cases} \frac{dF}{dr} = gvH(1-F); \\ \frac{dH}{dr} = \frac{1}{gvr^2}(2F-F^2). \end{cases}$$

It is convenient to switch to new variables $\rho = rgv$; ρ is dimensionless $\Rightarrow$

$$\frac{dF}{d\rho} = H(1-F) \quad \& \quad \frac{dH}{d\rho} = \frac{1}{\rho^2}(2F-F^2)$$

The solutions ~~becomes~~ to these equations are known in analytic form

$$\boxed{F(\rho) = 1 - \frac{\rho}{sh(\rho)} \; ; \; H(\rho) = \frac{ch(\rho)}{sh(\rho)} - \frac{1}{\rho}}.$$

Since, $\rho \gg 1$ corresponds $r \gg \frac{1}{gv}$, we find that $\boxed{F(\rho) \xrightarrow[\rho\to\infty]{} 1 - rgve^{-rgv}}$ & $\boxed{H(\rho) \to 1 - \frac{1}{rgv}}$

Hence, A_i^a reaches its asymptotic form much faster than the ϕ-field (exponential, vs. power-like). The long-range nature ($1/r$) of the Higgs field is related to the fact that we work in $\lambda \to 0$ limit, where the mass of the Higgs boson vanishes

— Let us discuss what might change if we do not consider critical ($\lambda = 0$) monopoles? There are a few points that need to be noted. The important one is that ~~thing~~ analytic solution for the Higgs and the gauge fields are out of question. Nevertheless, the general features of the construction are:

1) We look for finite energy static solutions. The energy functional is

$$E = \int d^3 r \left[\frac{1}{4g^2} G^a_{ij} G^{aij} + \frac{1}{2} (D_i\phi)^a (D_i\phi)^a + \lambda(\varphi^a\varphi^a - v^2)^2 \right].$$

The Higgs field asymptotic at $r \to \infty$ $\varphi^a\varphi_a = v^2$. We choose it, again, to be $\varphi^a = v n^a$, at $r \to \infty$, for the monopole solution.

2) The asymptotic form of the gauge field is again $A^a_i = \varepsilon^{aij} \frac{1}{r} n^j$. This asymptotic makes $(D_i\phi)^a$ vanish, at $r \to \infty$, faster than $1/r$.

3) One can write down the field equations for static configurations of the fields. The solutions are searched for, using the same Anzats as before:

$$\phi^a = v n^a H(r) \qquad A^a_i = \varepsilon^{aij} \frac{1}{2} n^j F(r).$$

It is possible to show that this "Anzats" is consistent with the equations of motion for static fields. The general argument about the consistency is as follows. The Lagrangian for static fields is invariant ~~for~~ ~~the~~ under field $SO(3)$ & spatial $SO(3)$ rotations. The monopole solutions have a ~~mor~~ symmetry that combines both of them: It is easy to check that for monopole solutions

$$\begin{cases} (\Lambda^{-1})^a{}_b\, \varphi^b(\Lambda^i_j n^j) = \varphi^a(n_i) & \left(\Lambda \text{ is an } SO(3) \text{ matrix}\right) \\ (\Lambda^{-1})^a{}_b (\Lambda^{-1})^i_j\, A^b_j(\Lambda^k_m n^m) = A^a_i(n^m). & (*) \end{cases}$$

This is a combination of $SU(2)$ & $SO(3)$ rotations and, therefore a valid symmetry of the Lagrangian. It should therefore be a symmetry of equations of motion. The only question we need to address is if ~~our~~ Anzats general enough. For φ^a one can't do anything else that is consistent with the above symmetry. For the gauge field, one can add two more terms: $A^a_i \sim n^a n^i$ and $A^a_i \sim \delta^{ai}$ so that the most general Anzats is

$$A^a_i = f_1(r)\, n^a n^i + f_2(r)\, \delta^{ai} + f_3(r)\, \varepsilon^{aij} n^j.$$

~~We~~ However, the first two terms can be disregarded if - in addition to invariance under $(*)$ - we require the antisymmetry under parity, i.e.

$$\varphi^a(\vec{r}) \to -\varphi^a(-\vec{r}) \quad \text{and} \quad A^a_i(-\vec{r}) \to -A^a_i(\vec{r}).$$

This and $(*)$ shows that the monopole anzats must separately "go through" equations of motion.

It is instructive to estimate the mass of the monopole. To this end, ~~to~~ choose variables $\vec{r} = v g \vec{\rho}$, and use the Anzats for the fields ϕ^a and A_i^a to find the energy. The result reads

$$E = \frac{4\pi v}{g} \int_0^\infty d\rho \, \rho^2 \left[\frac{(F')^2}{\rho^2} + \frac{(2F - F^2)^2}{2\rho^4} + \frac{H^2(1-F)^2}{\rho^2} + \frac{(H')^2}{2} + \frac{\lambda}{g^2}(H^2-1)^2 \right].$$

($F' = dF/d\rho$, $H' = \frac{dH}{d\rho}$ as usual)

Hence, the monopole mass is

$$\boxed{E_m = \frac{4\pi v}{g} \cdot f\left(\frac{\lambda}{g^2}\right)},$$ so that $\frac{4\pi v}{g}$ sets the scale for the monopole mass.

On the other hand for vector boson masses we have $m_W = gv$ and for the Higgs mass $m_H^2 = 8\lambda v^2 \Rightarrow$

$$\frac{\lambda}{g^2} = \frac{m_H^2}{8 m_W^2} \Rightarrow E_m = \frac{4\pi v}{g} f\left(\frac{m_H^2}{8 m_W^2}\right).$$

Note that $m_H \approx m_W$, so the non-trivial function is of order 1 but $\frac{4\pi v}{g} = \frac{4\pi v g}{g^2} \simeq \frac{m_W}{\alpha}$, where α is the "fine structure constant". It follows that - as long $\alpha \ll 1$, i.e.

we have perturbative theory, the monopole is much heavier than the gauge boson masses.

— Now we will go back to the discussion of the calculation of the magnetic flux and we'll discuss it from a somewhat different perspective. The question is - if it is possible to map the Higgs field $\varphi^a = v n^a$ at $|\vec{r}| = \infty$ to a "perturbative" vacuum, say $\varphi = v\delta^{a3}$. We know that it shouldn't be possible ~~by~~ to do so by a regular (non-singular) gauge transformation but let us see this explicitely.

Let us take the gauge transformation matrix to be: $U_N = e^{-i\frac{\varphi}{2}\tau_3} e^{i\frac{\theta}{2}\tau_2} e^{+i\frac{\varphi}{2}\tau_3}$.

This matrix can be written in a compact form using the following equations:

$$e^{i\theta/2\,\tau_2} = \cos\frac{\theta}{2} + i\tau^2 \sin\frac{\theta}{2}$$

and

$$e^{-i\varphi/2\,\tau_3}\, \tau^a\, e^{+i\varphi/2\,\tau_3} = \begin{cases} \cos\varphi\, \tau_1 + \sin\varphi\, \tau_2 & a=1 \\ -\sin\varphi\, \tau_1 + \cos\varphi\, \tau_2 & a=2 \\ \tau_3 & a=3 \end{cases}$$

With these equations, it is easy to see that

$$U_N = \cos\theta/2 + i\sin\frac{\theta}{2}\left(\cos\varphi\,\tau_2 - \sin\varphi\,\tau_1\right)$$

and $U_N^{-1} = \cos\frac{\theta}{2} - i\sin\frac{\theta}{2}\left(\cos\varphi\,\tau_2 - \sin\varphi\,\tau_1\right)$.

It is easy to check that the monopole Higgs vacuum becomes a "perturbative" one.

$$\hat{\Phi}_M = v n^a \tau^a \to v\, U_N n^a \tau^a U_N^{-1} = v\,\delta^{a3}.$$

It appears that we found a gauge transformation that changes a topologically non-trivial vacuum into a trivial one and this shouldn't be happening. So somehow our gauge transformation should be pathological.

Indeed, it is. To see this, consider what happens around $\theta \simeq \pi$, i.e at the <u>south</u> <u>pole</u> of a sphere. Then

$U_N \to i(\cos\varphi\,\tau_2 - \sin\varphi\,\tau_3)$ which means that

~~$\lim_{\theta\to\frac{\pi}{2}} U(\theta \ne \frac{\pi}{2})$ doesn't have a limit~~

$$\lim_{\theta\to\frac{\pi}{2}} U_N(\theta,\varphi) \quad \text{doesn't exist.}$$

This is the meaning of a singular gauge transformation and the absence of the limit implies that the gauge potential that one obtains with this gauge transformation is, again, <u>singular</u> at $\theta = \pi/2$ $\left(A \to U_N \partial_i U_N^{-1} \Rightarrow A \sim \frac{1}{(\theta - \frac{\pi}{2})}\right)$.

On the other hand, it is easy to construct a gauge transformation that will not be singular at the South pole. To do that, we note two identities:

$$i\tau_2\, e^{i\frac{\varphi}{2}\tau_3}\, e^{i\left(\frac{\theta-\pi}{2}\right)\tau_2}\, e^{i\frac{\varphi}{2}\tau_3} = \mathcal{U}_N$$

and $\tau_2\, e^{i\alpha\tau_3} = e^{-i\alpha\tau_3}\,\tau_2$ (anti-commutation $\{\tau_2,\tau_3\}=0$).

Using these equations, we write

$$\mathcal{U}_N = i\tau_2\, e^{i\frac{\varphi}{2}\tau_3}\, e^{i\left(\frac{\theta}{2}-\pi\right)\tau_2}\, e^{i\frac{\varphi}{2}\tau_3} =$$

$$= i\tau_2\, e^{i\varphi\tau_3}\, e^{-i\frac{\varphi}{2}\tau_3}\, e^{i\left(\frac{\theta}{2}-\pi\tau_2\right)}\, e^{i\frac{\varphi}{2}\tau_3} \equiv$$

$$\equiv e^{-i\varphi\tau_3}\, i\tau_2\, e^{-i\frac{\varphi}{2}\tau_3}\, e^{i\left(\frac{\theta}{2}-\pi\tau_2\right)}\, e^{i\frac{\varphi}{2}\tau_3} \Rightarrow$$

$$\boxed{\mathcal{U}_N \equiv e^{-i\varphi\tau_3}\,\hat{\mathcal{U}}_S}. \Rightarrow \boxed{\mathcal{U}_S = i\tau_2\, e^{-i\frac{\varphi}{2}\tau_3}\, e^{i\left(\frac{\theta}{2}-\pi\right)\tau_2}\, e^{i\frac{\varphi}{2}\tau_3}}$$

The matrix $\hat{\mathcal{U}}_S$ transform the monopole vacuum field in the right way

$$\mathcal{U}_S\, \hat{\varphi}_M\, \mathcal{U}_S^{-1} = e^{i\varphi\tau_3}\, \hat{\mathcal{U}}_N\, \hat{\varphi}_M\, \hat{\mathcal{U}}_N^{-1}\, e^{-i\varphi\tau_3} =$$

$$= e^{i\varphi\tau_3}\, \hat{\varphi}_{vac}\, e^{-i\varphi\tau_3} \equiv \hat{\varphi}_{vac} \quad \text{and}$$

is obviously non-singular at $\theta=\pi$, the South pole. Hence, it is ~~indeed~~ the transformation that we were looking for.

Given the existence of <u>two</u> types of gauge transformation matrices that overlap on a sphere, we ~~note~~ can use them at the same point.

That point can't be North or South pole but the equator for example is completely fine.

We then have at those points

$$\mathcal{U}_N \varphi_M \mathcal{U}_N^{-1} = \varphi_{vac} \qquad \hat{\mathcal{U}}_S \varphi_M \mathcal{U}_S^{-1} = \varphi_{vac} \Rightarrow$$

$$\varphi_M = \mathcal{U}_N^{-1} \varphi_{vac} \mathcal{U}_N \Rightarrow \mathcal{U}_S \mathcal{U}_N^{-1} \varphi_{vac} \mathcal{U}_N \mathcal{U}_S^{-1} = \varphi_{vac}$$

$\Rightarrow$ $\Omega = \mathcal{U}_S \mathcal{U}_N^{-1}$ is an element of $SU(2)$ that belongs to a $U(1)$ subgroup that rotates around the third axis ($\varphi_{vac}^a = v\delta^{a3}$).

From our previous construction we find explicitely: $\mathcal{U}_S \mathcal{U}_N^{-1} = e^{i\varphi\tau_3} \equiv \Omega(\varphi)$ everywhere in the interior of a sphere.

Note that - by writing $e^{i\varphi\tau_3} \equiv e^{if(\varphi)\tau_3}$, we have $f(\varphi) = \varphi \Rightarrow \dfrac{f(2\pi) - f(0)}{2\pi} = 1$

which tells us that the winding number of the monopole solution is $n=1$ and that it is topologically non-trivial.

To see how this is related to the magnetic flux and the magnetic charge, we need to consider the vector potentials. Similar to the previous case, we apply $\mathcal{U}_N$ & $\mathcal{U}_S$ to $\hat{A}_i$. We find

$$\hat{A}_i^N = \mathcal{U}_N \hat{A}_i \mathcal{U}_N^{-1} + \mathcal{U}_N \partial_i \mathcal{U}_N^{-1}$$

$$\hat{A}_i^S = \mathcal{U}_S \hat{A}_i \mathcal{U}_S^{-1} + \mathcal{U}_S \partial_i \mathcal{U}_S^{-1}.$$

What can be said about the gauge potentials in the "perturbative" gauge (where $\varphi \to v\tau_3$) ?

In the "perturbative" gauge, we have the standard picture of ~~ellec~~ symmetry breaking with two components of the gauge field becoming massive and one (aligned with the third axis) - massless. The massive gauge potentials decrease exponentially as $r \to \infty$, which means that

$$\hat{A}_i^N = \frac{g\tau^3}{2i} A_i^N \qquad \hat{A}_i^S = \frac{g\tau^3}{2i} A_i^S$$

It is easy to show that gauge potentials satisfy:

$$\hat{A}_i^S = \Omega \hat{A}_i^N \Omega^{-1} + \Omega \partial_i \Omega^{-1}, \text{ where}$$

$\Omega = U_S U_N^{-1}$, as before. $\Omega = e^{+i\tau_3 f(\varphi)}$

$f(\varphi) = \varphi$

which implies that

$$\frac{g\tau^3}{2i} A_i^S = \frac{g\tau^3}{2i} A_i^N - i\tau_3 \partial_i f(\varphi) \Rightarrow$$

$$\boxed{A_i^S = A_i^N + \frac{2}{g} \partial_i f(\varphi)}$$

This is the relations between vector potentials in the interior of a sphere.

We find the flux ~~by~~ ~~integrating~~ of the magnetic field using the standard formula:

$$\text{flux} = \int \vec{B}\cdot d\vec{S} = -\int(\vec{\nabla}\times\vec{A})\cdot d\vec{S} =$$

$$= -\int\limits_{\substack{\text{North}\\ \text{semi-sphere}}}(\vec{\nabla}\times\vec{A}_N)\cdot d\vec{S} - \int\limits_{\substack{\text{South}\\ \text{semi-sphere}}}(\vec{\nabla}\times\vec{A}_S)\cdot d\vec{S} =$$

$$= -\int_{S_1}\vec{A}_N\cdot d\vec{l} + \int_{S_1}\vec{A}_S\cdot d\vec{l} = \int_{S_1}(\vec{A}_S-\vec{A}_N)\cdot d\vec{l}$$

$$= \frac{2}{g}\int[\partial_i f(\varphi)]\cdot d\vec{l} = \frac{2}{g}\left[f(2\pi)-f(0)\right] = \frac{4\pi}{g}.$$

Hence, we find the same result as before for the flux of the magnetic field but using different arguments that provide ~~clear~~ a clear connection to non-trivial topology of the monopole solution.