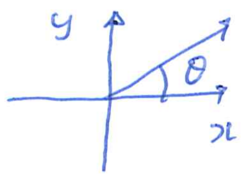
Lecture 4 Vortex solutions

We will now discuss an example of a non-trivial solution in <u>2 + 1 dimensions.</u> Consider a field theory defined by the Lagrangian

$$\mathcal{L} = (\partial_\mu \phi)(\partial^\mu \phi)^* - \mathcal{U}(\phi), \quad \mathcal{U}(\phi) = \lambda(\phi\phi^* - v^2)^2.$$

A vacuum state corresponds to $|\phi| = v$, but the phase of the field can be arbitrary. Now, let us imagine that we want to find the solution with finite energy. We must have $|\phi| \to v$ as $r \to \infty$. Suppose we write also $\phi = v e^{in\theta}$, $r \to \infty$, where n is an integer number and θ is ~~an~~ azimuthal angle on the (x,y)-plane



. Our Lagrangian, $\mathcal{L}$, has the $\mathcal{U}(1)$ symmetry: $\phi \to e^{i\alpha}\phi$ and so the equation $\phi = v e^{in\theta}$ can be viewed as the mapping of the "space" $\mathcal{U}(1)$ [rotation in the xy plane] onto "field" $\mathcal{U}(1)$ [phase of the field φ at $r \to \infty$].

This mapping is non-trivial: as we rotate once in the physical x-y plane (θ goes from 0 to 2π), the phase in the "field" space changes from 0 to $2\pi n$.

~~Characteristically~~ The mappings with different members "n" are distinct: they can not be changed into one another by continuous field transformations. Mathematically, this is said in the following way: mappings $U(1)\to U(1)$ are characterized by different topological classes, labelled by integers $\pi_1[U(1)]=Z$. An integer -"n"- counts how many times the field phase changes ~~for~~ in units of 2π as we make one circle at spatial infinity.

Now let us check what happens to the energy of a field with such a phase at $r=\infty$. At $r\to\infty$ $\phi(r,\theta)=v e^{in\theta}$, so that $\partial_t\phi(r,\theta)=0$ and $\partial_i\phi(r,\theta)=\phi(r,\theta)\, in\, \partial_i\theta$

$$\partial_i\theta = \partial_i \operatorname{atan}\left(\frac{y}{x}\right) = \frac{1}{1+\left(\frac{y}{x}\right)^2}\,\partial_i\left(\frac{y}{x}\right) =$$

$$= \frac{x^2}{x^2+y^2}\left\{\frac{\partial_i y}{x} - \frac{y}{x^2}\partial_i x\right\} \Rightarrow$$

$$\partial_i\theta = \partial_i y\,\frac{x}{x^2+y^2} - \partial_i x\,\frac{y}{x^2+y^2}$$

Then:

$$\partial_x\theta = -\frac{y}{x^2+y^2} = -\frac{1}{r}\frac{y}{r} = -\frac{1}{r}\frac{y}{r}$$

$$\partial_y\theta = \frac{x}{x^2+y^2} = \frac{1}{r}\frac{x}{r}$$

This can be written in a compact way by introducing the Levi-Cevita tensor ε_{ij}: $\varepsilon_{xy}=1$, $\varepsilon_{ij}=-\varepsilon_{ji}$

Then: $$\boxed{\partial_i\theta = -\frac{\varepsilon_{ij}x_j}{r^2}}$$

Therefore $\partial_i \phi(r,\theta) = v\, e^{in\theta}\left(-in \frac{\varepsilon_{ij} x_j}{r^2}\right)$ and

$(\partial_i \phi)(\partial_i \phi)^* = \frac{v^2 n^2}{r^2}$; $d^2x = d\theta\, r dr \Rightarrow$

the energy of such field configuration is

$$E = \int d\theta\, r dr\, (\partial_i \phi)(\partial_i \phi)^* \approx n^2 \int d\theta \frac{dr}{r} \to \infty$$. unless $n=0$

So it seems that the only finite energy field configuration that we can have is the one with the vanishing winding number — the "standard" vacuum.

This problem can be solved by introducing the gauge field. Consider "gauging" the global $U(1)$ symmetry. Then $\mathcal{L} \to \mathcal{L}_{gauge} + \mathcal{L}_{matter}$, where

$$\boxed{\mathcal{L}_{gauge} + \mathcal{L}_{matter} = (D_\mu \phi)^*(D_\mu \phi) - \frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} - U(\phi)}$$

Here $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $D_\mu = \partial_\mu - i n_e A_\mu$, where n_e is the electric charge of the field ϕ, in units of the electric charge "e".

The Lagrangian is invariant under Gauge transformations: $\phi \to e^{i\beta(x)}\phi$, $A_\mu \to A_\mu + \frac{1}{n_e}\partial_\mu \beta$

The standard way of treating this theory is to choose the vacuum $\phi = v$, $A_\mu = 0$ and consider small fluctuations around this vacuum : $\phi = v + \frac{h}{\sqrt{2}}$, $A_\mu \to A_\mu$.

The result is a theory of a single real scalar field (higgs) and a massive gauge boson. The mass of the gauge boson is $m_V = \sqrt{2}\, e n_e v$; the mass of the Higgs field is $m_H = 2\sqrt{\lambda}\, v$. This is a familiar "perturbative" story. We would like to see if non-perturbative solutions with finite energy exist. To this end we choose the gauge, $A_0 = 0$ and consider static, i.e. time-independent, fields. Then we obtain the energy functional of the gauge & matter fields:

$$\mathcal{E}\left[\vec{A}(\vec{x}), \phi(\vec{x})\right] = \int d^2\vec{x} \left[\frac{1}{4e^2} F_{ij} F_{ij} + (\mathcal{D}_i \varphi)^* (\mathcal{D}_i \varphi) + \mathcal{U}(\phi) \right]$$

We want the energy to be finite. For the potential energy, this implies $|\varphi| \to v$, $|\vec{x}| \to \infty$. Consider $\varphi \underset{|\vec{x}| \to \infty}{\longrightarrow} v e^{in\theta}$, so the asymptotics from one of the topological classes. Then, as we know

$\partial_i \varphi = v e^{in\theta}\left(-in \frac{\varepsilon_{ij} x_j}{r^2}\right)$. To have $(\mathcal{D}_i \varphi)^* (\mathcal{D}_i \varphi)$ give the convergent contribution, we need

$\partial_i \varphi - i n_e A_i \varphi \longrightarrow 0$, as $r \to \infty$ $\Rightarrow$

$\varphi\left(-in \frac{\varepsilon_{ij} x_j}{r^2}\right) - i n_e A_i \varphi \underset{r \to \infty}{\longrightarrow} 0$ $\Rightarrow$

$$\boxed{A_i \to -\frac{n}{n_e} \frac{\varepsilon_{ij} x_j}{r^2}}$$

Let us calculate the gauge field at infinity.

We have $A_i \sim \partial_i \theta \Rightarrow F_{ij} = \partial_i A_j - \partial_j A_i = \partial_i \partial_j \theta - \partial_j \partial_i \theta = 0$. Hence the asymptotic form of the vector potential is a pure gauge and for this reason there is no gauge field at infinity.

One interesting aspect of this construction is the existence of the flux of the gauge field through the xy plane. Indeed, consider the integral $\oint A_i \cdot dx_i$ over a circle of a very large radius. Using explicit form of the gauge potential we find

$$\oint A_i \cdot dx_i = -\frac{n}{n_e} \int \frac{x_j \varepsilon_{ij}}{r^2} r\, d\theta (\vec{e}_\theta)_i$$

, where the tangential vector $\vec{e}_\theta \equiv (-\sin\theta, \cos\theta) = -\frac{\varepsilon_{ij} x_j}{r}$.

Hence

$$\oint A_i dx_i = -\frac{n}{n_e} \oint d\theta \frac{x_j \varepsilon_{ij}}{r^2} (-\varepsilon_{ik} x_k) = (\varepsilon_{ij} \varepsilon_{ik} = \delta_{jk})$$

$$= \frac{n}{n_e} \oint d\theta \frac{x^2+y^2}{r^2} = \frac{2\pi n}{n_e}.$$

But:

$$\oint A_i dx_i \equiv \int B \cdot dS, \text{ where } B = \frac{1}{2} \varepsilon_{ij} F^{ij} = F_{12}.$$

Hence, the flux of the magnetic field through the surface is given by

$$\boxed{\int B \cdot dS = \frac{2\pi n}{n_e}}$$

and is, therefore, quantized.

Before we continue, we will comment on ~~the~~ our choice of the asymptotic of the φ-field $\varphi \xrightarrow[|\vec{x}|\to\infty]{} v e^{in\theta}$. This, of course, is not the most general asymptotic; the most general form is $\varphi = v e^{if(\theta)}$. The <u>winding number</u> is given by $\frac{f(2\pi)-f(0)}{2\pi} = \frac{1}{2\pi v^2} \oint dx^i \underbrace{\varphi^* \partial_i \varphi}_{i \cdot r}$.

Although we fixed the gauge $A_0 = 0$, we can still do gauge transformations that are independent of time t. Under those transformations – provided that they are <u>non-singular</u> – the winding number doesn't change. It is then possible to write

$f(\theta) = n\theta + (f(\theta) - n\theta) = n\theta + \Delta f(\theta)$.

Since $\Delta f(2\pi) - \Delta f(0) = 0$, $\Delta f(\theta)$ <u>can be removed by a non-singular gauge transform.</u>

Hence, $f(\theta) = n\theta$ is the most general form of the phase that we need to consider.

Now, it is interesting to find the energy ~~construct explicit~~ ~~solutions of~~ vortex solutions of the field equations ~~that gives rise ~~vortex~~.~~ <u>It is not possible to do this in general</u>. However, it is possible ~~to~~ to do that for a particular case called the "critical" vortex; the critical vortex is defined by the condition that the masses of the gauge boson m_V & the Higgs boson m_H are equal.

This implies $\boxed{\lambda^2 = \frac{e^2 n_e^2}{2}}$, i.e. a relation between the gauge coupling & the self-coupling of the field φ. To see how the relation between couplings helps, let us rewrite the energy functional

$$E[\vec{A}, \phi] = \int d^2\vec{x} \left[\frac{1}{4e^2} F_{ij} F^{ij} + (\mathcal{D}_i\varphi)^* \mathcal{D}_i \varphi + \frac{n_e^2 e^2}{2} \left(|\varphi|^2 - v^2 \right)^2 \right]$$

F_{ij}, of course is F_{12} or F_{21}, nothing else is possible. Let's say F_{12} is B, the magnetic field. Then $F_{ij} F^{ij} = 2B^2$. Then

$$\frac{1}{4e^2} F_{ij} F^{ij} + \frac{n_e^2 e^2}{2} \left(|\varphi|^2 - v^2 \right)^2 = \frac{1}{2e^2} B^2 + \frac{n_e^2 e^2}{2} \left(|\varphi|^2 - v^2 \right)^2$$

$$\equiv \frac{1}{2} \left(\frac{B}{e} + n_e e \left(|\varphi|^2 - v^2 \right) \right)^2 - B n_e \left(|\varphi|^2 - v^2 \right).$$

Next, consider the kinetic energy term for the field φ and write it as

$$\mathcal{D}_i \varphi \mathcal{D}_i \varphi^* = \mathcal{D}_1 \varphi (\mathcal{D}_1 \varphi)^* + \mathcal{D}_2 \varphi (\mathcal{D}_2 \varphi)^* =$$

$$= (\mathcal{D}_1 + i \mathcal{D}_2) \varphi \times [\mathcal{D}_1 \varphi + i \mathcal{D}_2 \varphi]^* + \Delta[A, \varphi]$$

To find $\Delta[A, \varphi]$, write

$$\Delta[A, \varphi] = \mathcal{D}_1 \varphi (\mathcal{D}_1 \varphi)^* + (\mathcal{D}_2 \varphi)(\mathcal{D}_2 \varphi)^* - |(\mathcal{D}_1 + i \mathcal{D}_2)\varphi|^2$$

$$\equiv i (\mathcal{D}_1 \varphi)(\mathcal{D}_2 \varphi)^* - i (\mathcal{D}_2 \varphi)(\mathcal{D}_1 \varphi)^*$$

Consider a term without A:

$$i(\partial_1 \varphi)(\partial_2 \varphi^*) - i(\partial_2 \varphi)(\partial_1 \varphi)^* = i \partial_1 (\varphi \partial_2 \varphi^*) - i \partial_2 (\varphi \partial_1 \varphi^*)$$

$$- i \varphi \partial_1 \partial_2 \varphi^* + i \varphi \partial_1 \partial_2 \varphi^* =$$

$$= i \partial_1 (\varphi \partial_2 \varphi^*) - i \partial_2 (\varphi \partial_1 \varphi^*).$$

This is the <u>total derivative</u> ~~that~~ of the "current"

$$j_i = \varepsilon_{ij} \varphi \partial_j \varphi^*.$$

Hence, an addition to the energy functional is $\int d^2\vec{x}\, \partial_i \cdot J^i = \int_0^{2\pi} r d\theta \, \frac{\vec{r}}{|r|} \cdot \vec{J} \Big|_{r\to\infty} = \int_0^{2\pi} d\theta \lim_{r\to\infty} \vec{r}\cdot\vec{J}$.

Since $\vec{J}_i = \varepsilon_{ij}\, \varphi \partial_j \overset{*}{\varphi}$ and $\varphi \to v e^{in\theta}$, as $r\to\infty$, the above integral can be computed in a straightforw. way. Let us denote is value by $\underline{\underline{N}}$.

Next, we need to consider terms with two vector potentiales, $O(\vec{A}^2)$. It is easy to see that $i(\mathcal{D}_1\varphi)(\mathcal{D}_2\varphi)^* - i(\mathcal{D}_2\varphi)(\mathcal{D}_1\varphi)^* \xrightarrow[O(\vec{A}^2)]{} 0$.

Hence, the only remaining terms we need to consider are terms that are linear in $\vec{A}$.

For these we find:

$$i(\mathcal{D}_1\varphi)(\mathcal{D}_2\varphi)^* - i(\mathcal{D}_2\varphi)(\mathcal{D}_1\varphi)^* \xrightarrow[O(\vec{A})]{}$$

$$i\Big[(\partial_1\varphi)(i n_e A_2 \overset{*}{\varphi}) + \partial_2\overset{*}{\varphi}(-i n_e A_1\varphi)$$
$$- (\partial_2\varphi)(i n_e A_1\overset{*}{\varphi}) - (\partial_1\varphi)^*(-i n_e A_2\varphi)\Big]$$

$$= i^2 [n_e] \Big[A_2(\overset{*}{\varphi}\partial_1\varphi + (\partial_1\varphi)^*\varphi) - A_1\big((\partial_2\overset{*}{\varphi})\varphi + \overset{*}{\varphi}\partial_2\varphi\big)\Big]$$

$$= -n_e\left[A_2(\overset{*}{\varphi}\partial_1\varphi + (\partial_1\varphi)^*\varphi) - A_1((\partial_2\overset{*}{\varphi})\varphi + \overset{*}{\varphi}\partial_2\varphi)\right]$$

$$= -n_e\left[A_2\,\partial_1(\overset{*}{\varphi}\varphi) - A_1\,\partial_2(\overset{*}{\varphi}\varphi)\right] =$$

$$= + n_e\left[(\partial_1 A_2 - \partial_2 A_1)\overset{*}{\varphi}\varphi\right] - n_e\left[\partial_1(A_2\overset{*}{\varphi}\varphi) - \partial_2(A_1\overset{*}{\varphi}\varphi)\right].$$

The last term is written as $\partial_i(\varepsilon_{ij} A_j \overset{*}{\varphi}\varphi)$, i.e. total derivative of the current. It can be integrated in the energy functional and gives $\underline{\underline{-N}}$.

Hence, we find:

$\Delta[A,\varphi] = n_e F_{12}(\overset{*}{\varphi}\varphi) = n_e B(\overset{*}{\varphi}\varphi)$, so that

$$D_i\varphi(D_i\varphi)^* = |D_1\varphi + iD_2\varphi|^2 + n_e B(\varphi^*\varphi)$$

This allows us to write

$$E[\vec{A},\varphi] = \int d^2\vec{x}\left\{\frac{1}{2}\left(\frac{B}{e} + n_e e\left(|\varphi|^2 - v^2\right)\right)^2 - Bn_e|\varphi|^2 + Bn_e v^2 + |D_1\varphi + iD_2\varphi|^2 + n_e B|\varphi|^2\right\} \equiv$$

$$E = 2\pi n v^2 + \int d^2\vec{x}\left\{\frac{1}{2}\left(\frac{B}{e} + n_e e\left(|\varphi|^2 - v^2\right)\right)^2 + |D_1\varphi + iD_2\varphi|^2\right\}.$$

Both of the terms that contribute to the integrand are positive definite. To minimise the energy, we therefore need

$$\begin{cases} \frac{B}{e} + n_e e\left(|\varphi|^2 - v^2\right) = 0 \\ D_1\varphi + iD_2\varphi = 0 \end{cases}$$

Now, to solve those equations, we use the anzats: for the simplest $n=1$ vortex

$$\boxed{\begin{cases} \phi = v\varphi(r)\,e^{i\theta} \\ A_i(x) = -\frac{1}{n_e}\,\varepsilon_{ij}\frac{x_j}{r^2}\left(1 - f(r)\right) \end{cases}}$$

Taking $\boxed{\rho = n_e e v r}$, we find that this anzats goes through the field equations & we find:

$$\boxed{-\frac{1}{\rho}\frac{df}{d\rho} + \varphi^2 - 1 = 0\,; \quad \rho\frac{d\varphi}{d\rho} - f\varphi = 0}$$

The boundary conditions for this equations are: at $\rho = \infty$: $\varphi(\rho) \to 1$ $f(\rho) \to 0$; this is consistent with our previous analysis; at $\rho = 0$, $\varphi(0) \to 0$ $f(0) \to 1$, as required by the fact that fields should be non-singular at the origin.

The equations can be solved numerically. The result is that the energy of the critical ~~soliton~~ vortex solution is

$$\boxed{E = 2\pi n v^2} = \pi n \frac{m_V^2}{e^2 n_e^2} \gg m_V^2, m_H^2 \text{ ; since } \underline{e \ll 1}$$

Hence, it is a heavy object.

The fields of the vortex extend all the way up to $\rho \sim 1$ and then decrease exponentially : $f, \varphi \sim e^{-\rho}$. Since $\rho = n_e e v r = \frac{m_V}{\sqrt{2}} r$, the vortex fields extend to $r \sim 1/m_V$.

In case of a <u>non-critical</u> vortex, analytic calculation of the energy is impossible. However, one can still write equations for the fields and try to solve them using the same Anzats as before. The equations read:

$$\frac{d}{dr}\left(\frac{1}{r}\frac{df}{dr}\right) - 2 n_e^2 e^2 v^2 \frac{\varphi^2}{r} f = 0$$

$$-\frac{d}{dr}\left(r \frac{d\varphi}{dr}\right) + 2\lambda v^2 r \varphi (\varphi^2 - 1) + \frac{\varphi}{r} f^2 = 0$$

Let us write the approximate version of those equations at large r. The first equation has small f and $\varphi \to 1$, so it becomes

$$\frac{d}{dr}\left(\frac{1}{r}\frac{df}{dr}\right) - 2 n_e^2 e^2 v^2 \frac{f}{r} = 0$$

The second has $\varphi(r) = 1 - h(r)$ and, assuming $h(r) \sim f(r)$, we find

$$-\frac{d}{dr}\left(r \frac{dh}{dr}\right) + 4\lambda v^2 r h(r) = 0$$

Since $m_V^2 = 2 e^2 n_e^2 v^2$ and $m_H^2 = 4\lambda v^2$, we

have
$$\begin{cases} r\frac{d}{dr}\left(\frac{1}{r}\frac{df}{dr}\right) - m_V^2 f = 0 \\ \frac{1}{r}\frac{d}{dr}\left(r\frac{dh}{dr}\right) - m_H^2 h(r) = 0 \end{cases}$$

To find the asymptotic behavior of the first equation, write $f(r) = g(r)\, e^{-m_V r}$

$$\frac{df}{dr} = g' e^{-m_V r} - m_V g(r) e^{-m_V r}$$

$$\frac{d^2f}{dr^2} = g'' e^{-m_V r} - 2g' m_V e^{-m_V r} + m_V^2 g e^{-m_V r}$$

$$\frac{d}{dr}\left(\frac{1}{r}\frac{df}{dr}\right) = \frac{1}{r}\frac{d^2f}{dr^2} - \frac{1}{r^2}\frac{df}{dr}$$ $\Rightarrow$ The equation for g reads

$$\left[g'' - \frac{g'}{r}\right] - 2m_V\left[g' - \frac{g}{2r}\right] = 0.$$ Since we are interested in the limit $r\cdot m_V \gg 1$, the last term in equation needs to be considered. Then

$$g' - \frac{g}{2r} \to 0 \Rightarrow g(r) = \sqrt{r}. \Rightarrow \underline{f(r) = C\sqrt{r}\, e^{-m_V r}} \text{ is}$$

the asymptotic solution at $r \to \infty$

Similarly for $h(r)$ we find

$$\lim_{r\to\infty} h(r) \Rightarrow \frac{C'}{\sqrt{r}}\, e^{-m_H r}.$$

As the result, we conclude that the scalar field ϕ reaches its asymptotic form at $r \gtrsim 1/m_H$ and the gauge potential reaches its asymptotic form at $r \gtrsim \frac{1}{m_V}$

Now, consider a situation when $\underline{m_H \gg m_V}$.

Then at $r \gtrsim m_H^{-1}$ the scalar field is already asymptotic $\phi = ve^{i\theta}$ and the vector potential is <u>small</u> ($r \sim m_H^{-1}$ is $r \to 0$ from the point of view of gauge field).

In this situation, the energy functional becomes that of a global vortex, so that

$$E = v^2 \int d\theta \frac{r\,dr}{r^2} = 2\pi v^2 \int \frac{dr}{r} .$$

The integral has to be cut ar $r \sim 1/m_H$ from below at at $r \sim 1/m_V$ from above since this is where the vector potential kicks in. The result

$$E = 2\pi v^2 \ln \frac{m_H}{m_V}$$

is the energy of the vortex for $m_H \gg m_V$ case.