Lecture 3
Kinks and fermions

We will consider a theory of a scalar field coupled to a fermion in 1+1-dimensional space-time. The Lagrangian is

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{4}g^2(\phi^2-v^2)^2 + \bar\psi\, i\hat\partial\psi + \lambda\phi\bar\psi\psi.$$

Here ϕ is a scalar field; the scalar part of the Lagrangian admits kink-like solutions. Fermion field ψ is, initially, massless. The Dirac γ-matrices in 2 dimensions are

$$\gamma^0 = \sigma_2 \qquad \gamma^1 = i\sigma_3 \qquad \gamma^5 = \gamma^0\gamma^1 = -\sigma_1.$$

As usual, $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$, $\{\gamma^5, \gamma^\mu\} = 0.$

The theory with the Lagrangian $\mathcal{L}$ ~~requires~~ possesses quite a few symmetries.

First, there is a <u>$U(1)$ symmetry</u>; the Lagrangian $\mathcal{L}$ is invariant under $\psi \to e^{i\alpha}\psi$, where α is constant. This symmetry leads to a conserved vector

$j^\mu = \bar\psi\gamma^\mu\psi$; $\partial_\mu j^\mu = 0$. Once we have the conserved current, we can introduce a <u>charge</u>

$$Q = \int_{-\infty}^{\infty} dx\, j^0(t,x) \quad \text{such that} \quad \frac{dQ}{dt} = 0.$$

There is also a Z_2 symmetry related to the scalar field: $\phi \to -\phi$.

The fermion part of the Lagrangian $\lambda \phi \bar{\psi} \psi$ can be also made invariant under this transformation if we change $\psi \to \gamma_5 \psi$. Indeed $\bar{\psi} \to \psi^+ \gamma_5^+ = \psi^+ \gamma_5 \Rightarrow$ $\bar{\psi} \to -\bar{\psi}\gamma_5$ and $\bar{\psi}\psi \to -\bar{\psi}\psi$. Hence, combining transformations for a scalar and fermion fields we obtain

$\phi \bar{\psi} \psi \to \phi \bar{\psi} \psi$, which is Z_2-symmetric.

In the standard way of dealing with the theory, the Z_2 symmetry is spontaneously broken by the choice of a vacuum state. Choosing $\langle \phi \rangle = v$, we select a state that isn't Z_2-symmetric. For $\langle \phi \rangle = \pm v$, the fermion gets the mass term

$\mp m_\psi = \mp \lambda v$; so the mass terms have different-sign masses for the two vacua.

The kink solution interpolates between the two vacua: $\phi = -v \big|_{x=-\infty} \to \phi = +v \big|_{x=\infty}$ and the fermion mass term "changes the sign" along the way.

This leads to interesting effects ~~along the way~~ that we'll discuss.

Let us first discuss the standard quantization procedure. Choose $\langle \phi \rangle = -v$. The fermion part of the Lagrangian becomes

$$\mathcal{L}_F = \bar{\psi} i \hat{\partial} \psi - \lambda v \bar{\psi} \psi = \bar{\psi} i \hat{\partial} \psi - m_\psi \bar{\psi} \psi.$$

The field ψ is decomposed into solutions of the Dirac equation

$$\psi = \sum_p \frac{1}{\sqrt{2E_p L}} \left(a_p u_p e^{-i(E_p t - px)} + b_p^+ v_p e^{i(E_p t - px)} \right),$$

where $E_p = \sqrt{p^2 + m^2}$ and the spinors are

$$u_p = \begin{pmatrix} \sqrt{E_p} \\ \frac{(-p + im)}{\sqrt{E_p}} \end{pmatrix} \qquad v_p = \begin{pmatrix} \sqrt{E_p} \\ \frac{(-p - im)}{\sqrt{E_p}} \end{pmatrix}.$$

Note that $\bar{u}_p u_p = \bar{v}_p v_p = 2m$ and that $u_p^* = v_p$. The sum over "p" corresponds to a standard quantization in a box of size L with e.g. periodic boundary conditions. $\left(p = \frac{2\pi n}{L},\ n = \ldots -1, 0, 1 \ldots \right)$

The operators a_p & b_p^+ are standard annihilation operators for particles and anti-particles. They satisfy

$$\{ a_p, a_{p'}^+ \} = \delta_{pp'} \qquad \{ b_p, b_{p'}^+ \} = \delta_{pp'}$$

$$\{\psi_\alpha(t,x),\ \psi^+_\beta(t,x')\} = \delta_{\alpha\beta}\,\delta(x-x').$$

The vacuum state is defined as a state that is annihilated by a_p & b_p:
$a_p|vac\rangle = b_p|vac\rangle = \emptyset$. At the same time, $a^+_p|vac\rangle$ gives a state with a fermion & $b^+_p|vac\rangle$ – a state with an anti-fermion.

Now we can calculate the current and the charge. Since $j^0 = \bar\psi\gamma^0\psi = \psi^+\gamma^0\gamma^0\psi = \psi^+\psi$, we write

$$j^0(t,x) = \sum_{p,p'} \frac{1}{\sqrt{2E_pL\,2E_{p'}L}}\left(a^+_p u^+_p e^{i(E_pt-px)} + b_p v_p e^{-i(E_pt-px)}\right)\left(a_{p'}u_{p'}e^{-i(E_{p'}t-p'x)} + b^+_{p'}v_{p'}e^{i(E_{p'}t-p'x)}\right)$$

The charge is obtained by integration over x $\Rightarrow$

$$\int_{-L}^{L} dx\, e^{i(p'-p)x} = 2L\,\delta_{p'p}.$$

When brackets in the previous equation for $j^0(t,x)$ are expanded, we obtain four terms: a^+a, bb^+, a^+b^+ and ba.
Those last ones involve $u^+_p v_{-p}$ and $v_p u_{-p}$ spinor products. Explicitely: e.g

$$u^+_p v_{-p} = \left(\sqrt{E}\quad \frac{(-p\mp im)}{\sqrt{E}}\right)\begin{pmatrix}\sqrt{E}\\ \frac{(p-im)}{\sqrt{E}}\end{pmatrix} = E - \frac{p^2+m^2}{E} =$$

$$= \frac{E^2-p^2-m^2}{E} \equiv \emptyset.$$

The same applies to $v_p u_{-p}$.

Hence, no cross-terms survive and we obtain

$$Q = \int_{-L}^{L} dx\, j^0(t,x) = \sum_p \left[a_p^+ a_p - (b_p^+ b_p - 1)\right]$$

Using properties of the vacuum, we find

$$Q|vac\rangle = \sum_p (+1)|vac\rangle \equiv \infty |vac\rangle = Q_{vac}|vac\rangle$$

and

$$Q\, a_p^+|vac\rangle = (1+\infty)\, a_p^+|vac\rangle = Q_{vac+f}|vac+f\rangle$$

$\Rightarrow$ $Q_{vac+f} - Q_{vac} = 1$ is the fermion charge.

A similar calculation fixes the charge of an anti-fermion to -1.

We can change the definition of a current to remove the infinite charge of the vacuum. To do that, we use the charge conjugation. Charge conjugation changes a fermion to an antifermion

$$a_p \to b_p \quad , \quad b_p \to a_p \quad , \quad \text{etc.}$$

Upon this transformation we have

$$\psi \to \psi_c = \sum_p \frac{1}{\sqrt{2E_p L}} \left(b_p u_p e^{-i(Et-px)} + a_p^+ v_p e^{i(E_p t - px)} \right)$$

Note that $u_p = v_p^*$ $\Rightarrow$

$$\underline{\psi_c} = \sum_p \frac{1}{\sqrt{2E_p L}} \left(a_p^+ \left(u_p e^{-i(Et-px)}\right)^* + b_p \left(v_p e^{i(Et-px)}\right)^* \right) \equiv \underline{\psi^*}$$

We can easily check that the Lagrangian $\mathcal{L}$ is invariant under C-parity transformation. We can also use it to rewrite the current:

$$j^\mu = \frac{1}{2}\left(\bar{\psi}\gamma^\mu\psi - \bar{\psi}^c\gamma^\mu\psi^c\right).$$

Formally, the two terms are the same up to the fact that when their identity is attempted to be shown, one disregards the a/commutation relation between ψ & $\bar{\psi}$ taken at the same point. This is equivalent to removing the infinite charge from the vacuum.

Recalculation of a charge now gives

$$Q = \sum_p \left(a_p^+ a_p - b_p^+ b_p\right)$$ and the hamiltonian reads $H = \sum_p E(p)\left(a_p^+ a_p + b_p^+ b_p\right).$

— This is a familiar story. We will now repeat it considering the kink background. Note that we will consider the kink field fixed. In general, this is not true since fermions act on the scalar field and force it to back-react. However, since the mass/energy of the kink is $M_K \sim g v^3$ and a typical energy/mass of a fermion is $m_\psi \sim \lambda v$, as long as $M_K \gg m_\psi$ or

$v^2 \gg \frac{\lambda}{g}$, the kink can be thought of as ~~an~~ a fixed background for the purpose of fermion quantization.

We write the equation of motion

$$(i\hat{\partial} + \lambda\phi_K)\psi = \begin{bmatrix} -\partial_x + \lambda\phi_K & \partial_t \\ -\partial_t & \partial_x + \lambda\phi_K \end{bmatrix}\begin{pmatrix}\psi_1 \\ \psi_2\end{pmatrix} = \phi.$$

Here $\phi_K \equiv \phi_K(x)$ is a kink field. Since the kink field is static, we can talk about solutions to the Dirac equation with fixed energy. So we write

$$\psi_1 = e^{-i\omega t}\tilde{\chi} \qquad \psi_2 = e^{-i\omega t}\chi.$$

The equation for $\tilde{\chi}$ and χ becomes

$$\begin{pmatrix} -\partial_x + \lambda\phi_K & -i\omega \\ i\omega & \partial_x + \lambda\phi_K \end{pmatrix}\begin{pmatrix}\tilde{\chi} \\ \chi\end{pmatrix} = 0 \quad \text{or}$$

$$\begin{cases} (-\partial_x + \lambda\phi_K)\tilde{\chi} - i\omega\chi = 0 \\ (\partial_x + \lambda\phi_K)\chi + i\omega\tilde{\chi} = 0 \end{cases}$$

We introduce two differential operators

$$\boxed{P = \partial_x + \lambda\phi_K \quad \text{and} \quad P^+ = -\partial_x + \lambda\phi_K}$$

The two operators P^+ & P are Hermitian conjugates of each other

$$\Big[\ \langle g|\hat{P}|f\rangle = \int dx\, g^*(x)\,[\partial_x f + \lambda\phi_K]f =$$

$$= \int dx\,[-(\partial_x g^*) f + g^* f\lambda\phi_K] = \int dx\, f\,[-\partial_x + \lambda\phi]g^*$$

$$= [\langle f|P^+|g\rangle]^*\ \Big]$$

Now, the two Eqs. become

$$\begin{cases} p^+ \tilde{\chi} - i\omega \chi = 0 \\ p \chi + i\omega \tilde{\chi} = 0 \end{cases}$$

Starting from these two equations, we can easily obtain Eqs. for χ & $\tilde{\chi}$ by acting on a second (first) equation with p^+ (p). We find:

$$\begin{cases} p^+ p \chi - \omega^2 \chi = 0 \\ p p^+ \tilde{\chi} - \omega^2 \tilde{\chi} = 0 \end{cases} \Rightarrow \begin{matrix} p^+ p = L_2 \\ p p^+ = \tilde{L}_2 \end{matrix} \Rightarrow \begin{matrix} L_2 \chi = \omega^2 \chi \\ \tilde{L}_2 \tilde{\chi} = \omega^2 \tilde{\chi} . \end{matrix}$$

Note that both L_2 and $\tilde{L}_2$ are hermitian which means that their eigenfunctions form complete set of states (also orthonormal).

Now, suppose we found all the eigenfunction for the operator L_2. Hence, we know the solutions of the equation
$L_2 \chi_n = \omega_n^2 \chi_n$. Suppose χ_n is normalized $\langle \chi_n \chi_n \rangle = \int_{-L}^{L} dx \, \chi_n^+ \chi_n = 1$.

We will now show that $\tilde{\chi}_n = \frac{p_n \chi_n}{\omega_n}$ is a normalized solution for $\tilde{L}_2 \tilde{\chi}_n = \omega_n^2 \tilde{\chi}_n$

Indeed $\tilde{L}_2 = p p^+ \Rightarrow$

$$\tilde{L}_2 \tilde{\chi}_n = p p^+ \frac{p_n \chi_n}{\omega_n} = p \frac{L_2 \chi_n}{\omega_n} = \omega_n^2 \frac{p \chi_n}{\omega_n} = \omega_n^2 \tilde{\chi}_n .$$

$$\langle \tilde{\chi}_n^* \tilde{\chi}_n \rangle = \frac{\langle \chi_n p_n^+ p_n \chi_n \rangle}{\omega_n^2} = \frac{\langle \chi_n L_2 \chi_n \rangle}{\omega_n^2} = 1 .$$

We conclude, therefore, that the spectrum of L_2 and $\tilde{L}_2$ is the same. The only problem can occur if L_2 contains a zero mode (solution with $\omega_n = 0$).

This is indeed what happens. L_2 has a normalizable solution with $\omega_n = 0$ and $\tilde{L}_2$ doesn't.

To prove this, consider equation

$$L_2 \chi_0 = \phi \Rightarrow P\chi_0 = 0 \quad P = \partial_x + \lambda\phi_K \Rightarrow$$

$$(\partial_x + \lambda\phi_K)\chi_0 = 0 \Rightarrow \chi_0 \propto \exp\left(-\lambda\int_0^x dx\,\phi_K\right)$$

Since $\phi_K = v\,\mathrm{th}\left[\frac{gvx}{\sqrt{2}}\right]$, we have $\phi_K \to \pm v$ as $x \to \pm\infty$. Therefore

$$\int_0^x dx\,\phi_K(x) \to vx, \; x \to +\infty \quad \text{and}$$

$$\int_0^{-|x|} dx\,\phi_K = -\int_{-|x|}^0 dx\,\phi_K(x) \underset{x\to-\infty}{\longrightarrow} -(-|x|)(-v) = -v|x|,$$

$|-x|$

Hence, we see that $\chi_0 \underset{|x|\to\infty}{\longrightarrow} \exp(-\lambda v|x|)$

and, hence, χ_0 is normalizable.

Suppose we repeat the same exercise for $\tilde{L}_2$. Then we'll need $P^+\tilde{\chi}_0 = \phi$. One can find solutions for this equation

But, it will asymptote to $\chi_0 \to e^{\lambda v |\vec{x}|}$ and, therefore, it wouldn't be normalizable.

We therefore conclude that L_2 has a zero-mode and $\tilde{L}_2$ does not.

Now, it is easy to check that the zero-mode solution is invariant under C-parity transformation (it is real).

We can use the orthonormal set of functions χ, to express $\psi(t,x)$ through creation and annihilation operators:

$$\psi(t,x) = a_0 \begin{pmatrix} 0 \\ \chi_0(x) \end{pmatrix} + \sum_{n \neq 0} \Big[e^{-i\omega_n t} \frac{a_n}{\sqrt{2}} \begin{pmatrix} \tilde{\chi}_n(x) \\ -i\chi_n(x) \end{pmatrix} + e^{i\omega_n t} \frac{b_n^+}{\sqrt{2}} \begin{pmatrix} \tilde{\chi}_n(x) \\ i\chi_n(x) \end{pmatrix} \Big]$$

with $\{a_n, a^+_{n'}\} = \{b_n, b_{n'}\} = \delta_{nn'}$

We can use exactly the same logic as in the vacuum case to find the expression for the Hamilton operator:

$$H = \sum_{n \neq 0} \omega_n (a_n^+ a_n - b_n^+ b_n).$$

The important point is that the <u>zero-mode doesn't contribute to H.</u>

Suppose now that we construct a vacuum state $|vac\rangle$. It's energy is $\emptyset$; $H|vac\rangle = 0$.

Suppose that we constructed a state where the zero mode is filled:

$a_0^+|vac\rangle$. In this case $H a_0^+|vac\rangle = 0$, and so $|vac\rangle$ and $a_0^+|vac\rangle$ are degenerate. At the same time, when their charges are computed, one obtains Q_{vac} and $Q_{vac+zero}$

Recall that $\hat{Q} = \frac{1}{2}\left(a_0^+ a_0 - a_0 a_0^+\right) +$

$$+ \frac{1}{2}\sum_{n\neq 0}\left(a_n^+ a_n + b_n b_n^+ - b_n^+ b_n - a_n a_n^+\right)$$

$$= \frac{1}{2}\left(a_0^+ a_0 - a_0 a_0^+\right) + \sum_{n\neq 0}\left(a_n^+ a_n - b_n^+ b_n\right)$$

$$Q_{vac} = \langle vac|\hat{Q}|vac\rangle \equiv -\frac{1}{2}.$$

$$Q_{vac+zero} = \langle vac+Z|\hat{Q}|vac+Z\rangle \equiv \frac{1}{2}\left(\underbrace{a_0^+ a_0^+|vac\rangle}_{=0}\right)$$

We emphasize that $|vac+Z\rangle$ state is yet another ground state of the kink, with the same energy.

Hence, we found two ground states of the kink with charges $-1/2$ and $1/2$, in spite of the fact that all "elementary" excitations of a fermion field ψ all have integer charges.

An interesting question of course is how this could have happened.