Kink mass at one-loop

We will now discuss how quantum effects ~~and~~ - inherent to all field theories - and the classical solution that we found, should be combined. We will do this by considering small fluctuations around the kink solution and quantizing them. We already did something similar at the end of the previous lecture, but we'll do it from a slightly different perspective now.

Again, $S[\varphi] = \int d^2x \, \mathcal{L}(\partial_\mu\varphi, \varphi)$

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{g^2}{4}(\varphi^2 - v^2)^2 \equiv \frac{1}{2}(\partial_\mu\varphi)(\partial^\mu\varphi) - V(\varphi).$$

We assume that a static kink-like solution is found. We denote such a solution φ_K.

We write, for small fluctuations,

$\varphi(t,x) = \varphi_K(x) + \chi(t,x)$ and $\chi(t,x) = \sum_n a_n(t)\chi_n(x)$.

We will specify what the summation means shortly. For the time being, we assume that $\{\chi_n(x)\}$ ~~are~~ an orthonormal set of basis functions, so that $\int_{-\infty}^{\infty}\chi_n(x)\chi_m(x)\,dx = \delta_{nm}$

We will construct those functions specifically later. Now:

$$S[\varphi_K + \chi] = S[\varphi_K] + \int d^2x \left.\frac{\delta S}{\delta\varphi(x)}\right|_{\varphi=\varphi_K}\chi(x) + \int d^2x \times \left[\frac{1}{2}\partial_\mu\chi\partial^\mu\chi - \left.\frac{\delta^2 V}{2\delta\varphi^2}\right|_{\varphi=\varphi_K(x)}\cdot\chi^2\right]$$

The linear term in χ vanishes since φ_K is the solution of the equation of motion; so that we obtain

$$S[\phi] = S[\varphi_K] + \int d^2x \left[\frac{1}{2}\partial_\mu \chi \partial^\mu \chi - \frac{1}{2} \frac{\partial^2 V}{\partial \varphi^2}\Big|_{\varphi=\varphi_K} \chi^2 \right].$$

We use the representation of V in terms of "superpotential": $V(\varphi) = \frac{1}{2}\left(\frac{dW}{d\varphi}\right)^2$. Then

$$\frac{dV}{d\varphi} = \frac{dW}{d\varphi}\cdot\frac{d^2W}{d\varphi^2} \quad \text{and} \quad \frac{d^2V}{d\varphi^2} = \frac{dW}{d\varphi}\frac{d^3W}{d\varphi^3} + \left(\frac{d^2W}{d\varphi^2}\right)^2.$$

We therefore write

$$S[\phi] = S[\phi_K(\cancel{x})] + \int d^2x \left[\frac{1}{2}\left(\frac{\partial \chi}{\partial t}\right)^2 - \frac{1}{2}\chi \hat{L}_2 \chi \right],$$

where $\boxed{\hat{L}_2 = -\frac{d^2}{dx^2} + \left(\frac{d^2W}{d\varphi^2}\right)^2 + \frac{dW}{d\varphi}\frac{d^3W}{d\varphi^3}}$, where $\varphi \to \varphi_K(x)$.

Now we want to compute the operator $\hat{L}_2$.

We know from the previous lecture that

$\phi_K(x) = \frac{m}{\sqrt{2}g} th\left(\frac{m\cancel{x}x}{2}\right)$, where $m = vg\sqrt{2}$ is the mass of a small excitation in a ~~broken~~ Higgs phase.

Also $W = \frac{g}{\sqrt{2}}\left(\frac{\phi^3}{3} - v^2\phi\right)$, so that

$$\frac{dW}{d\phi} = \frac{g}{\sqrt{2}}(\phi^2 - v^2) \quad \frac{d^2W}{d\varphi^2} = \frac{g}{\sqrt{2}}\cdot 2\phi \quad \frac{d^3W}{d\varphi^3} = \frac{g}{\sqrt{2}}\cdot 2$$

As the result, substituting $\phi_K = \frac{m}{\sqrt{2}g} th\left(\frac{m\cancel{x}}{2}\right)$ we obtain

$$\frac{dW}{d\phi}\Big|_{\phi=\phi_K} = \frac{g}{\sqrt{2}}\left(\frac{m^2}{2g^2} th^2\left(\frac{mx}{2}\right) - v^2\right) = \frac{gv^2}{\sqrt{2}}\left(\frac{sh^2\left(\frac{mx}{2}\right)}{ch^2\left(\frac{mx}{2}\right)} - 1\right)$$

$$\equiv -\frac{gv^2}{\sqrt{2}\, ch^2\left(\frac{mx}{2}\right)} = -\frac{m^2}{2^{3/2} g\, ch^2\left(\frac{mx}{2}\right)} = \frac{-m^2}{2^{3/2} g\, ch^2\left(\frac{mx}{2}\right)}$$

$$\frac{d^2W}{d\varphi^2} = \frac{g}{\sqrt{2}}\frac{2m}{\sqrt{2}g}\, th\left(\frac{m\cancel{x}}{2}\right) \equiv m\, th\left(\frac{m\cancel{x}}{2}\right) \Rightarrow$$

$$\left(\frac{d^2W}{d\varphi^2}\right)^2 + \frac{dW}{d\varphi}\frac{d^3W}{d\varphi^3} \equiv \frac{g}{\sqrt{2}}\cdot 2\frac{(-m^2)}{2^{3/2}g\, ch^2\left(\frac{mx}{2}\right)} + m^2 th^2\left(\frac{mx}{2}\right)$$

$$= m^2 th^2\left(\frac{mx}{2}\right) - \frac{1}{2}\frac{m^2}{ch^2\left(\frac{mx}{2}\right)} = m^2 - \frac{3m^2}{2ch^2\left(\frac{mx}{2}\right)} \Rightarrow$$

$$\boxed{\hat{L}_2 = -\frac{d^2}{dx^2} + m^2\left(1 - \frac{3}{2ch^2\left(\frac{mx}{2}\right)}\right)}$$

We would like to diagonalize $\hat{L}_2$. the part of the action that involves the X-fields. To do this, we can find all the eigenfunctions of $\hat{L}_2$ and, since this is a hermitian operator, the set of its eigenfunctions is complete and orthonormal. So, we look for

$L_2 X_n(z) = \omega_n^2 X_n(z)$. As we said, we can interpret this as an eigenvalue problem in Quantum Mechanic with the potential $V(x) = m^2\left[1 - \frac{3}{2ch^2\left(\frac{mx}{2}\right)}\right]$.

Since L_2 is hermitian the action becomes diagonal:

$$S[X] = \int dt\,dx\left(\sum_{n,m}\frac{1}{2}\dot{a}_n\dot{a}_m X_n(x)X_m(x) - \frac{1}{2}\sum_{n,m}\omega_m^2 a_n a_m X_n(x)X_m(x)\right) \Rightarrow$$

$S[x]=\sum_n \int dt \left(\frac{1}{2}\dot{a}_n^2-\frac{1}{2}\omega_n^2 a_n^2\right)$. The Hamiltonian is simply obtained by changing the sign of the last term:

$$H[x]=\sum_n\left[\frac{1}{2}\dot{a}_n^2+\frac{\omega_n^2}{2}a_n^2\right].$$

What we have done is the standard thing we do in QFT: we mapped the problem onto a collection of "quantum oscillators" that we can now quantize. The difference is that these "oscillators" do not describe oscillations around "empty space" (vacuum) but around a classical kink-like solution.

So, the quantization condition $[a_n,\dot{a}_m]=i\delta_{nm}$.

If we now write $a_n, \dot{a}_n$ in terms of creation and annihilation operators and express $H[x]$ through them, the ~~conta~~ energy for zero-occupation number will be

$H[x\to 0]=\sum_n\frac{\omega_n}{2}$. This zero-point energy is interpreted as a quantum correction to the kink mass,

$$\boxed{\delta M_k = H[x\to 0]=\sum_n\frac{\omega_n}{2},}$$

that we would like to calculate.

There are a few things that we will have to do in connection with all this and one such thing is to understand the implications of the existence of the zero-mode. The zero-mode can be obtained from the kink solution by taking the derivative:

$$\chi_0(x) \sim \frac{\partial \phi_K}{\partial x} \equiv \frac{m^2}{2^{3/2} g \, ch^2\left(\frac{mx}{2}\right)} .$$

The effect of the zero mode on the kink energy is easy to understand. A ~~slow~~ slow motion of the kink can be described by a t-dependent "center" : $\phi(t,x) \equiv \phi_K(x - x_0(t))$. The energy of this solution is

$$E = \int dx \left[\frac{1}{2}\left(\frac{\partial \varphi}{\partial t}\right)^2 + \frac{1}{2}\left(\frac{\partial \varphi}{\partial x}\right)^2 + V(\varphi)\right].$$

$$\frac{\partial \varphi}{\partial t} = \frac{d\phi_K}{dz}(-\dot{x}_0) \; ; \quad \frac{\partial \varphi}{\partial x} = \frac{d\phi_K}{dz} \; ; \quad V(\phi_K) = V(\phi_K(z))$$

Now $dx \equiv dz$ and

$$E = \int dz \left[\frac{1}{2}\left(\frac{d\phi_K}{dz}\right)^2 \dot{x}_0^2 + \frac{1}{2}\left(\frac{d\phi_K}{dz}\right)^2 + V(\phi_K)\right]$$

$M_K = \int dx \left[\frac{1}{2}\left(\frac{d\phi_K}{dx}\right)^2 + V(\phi_K)\right]$ is the kink mass.

Hence $\quad E = M_K + \int dz \, \frac{1}{2}\left(\frac{d\phi_K}{dz}\right)^2 \dot{x}_0^2$

One can use the explicit solution ~~of~~ for the kink to simplify this further: as we saw the kink solution satisfies $\boxed{\frac{d\phi_K}{dx} = \frac{dW}{d\phi}}$ and $V(\phi_K) = \frac{1}{2}\left(\frac{dW}{d\phi_K}\right)^2$. Hence,

$$\int dx \left[\frac{1}{2}\left(\frac{d\phi_K}{dx}\right)^2 + \frac{1}{2}\left(\frac{dW}{d\phi_K}\right)^2\right] = M_K \Rightarrow$$

$$\boxed{\int dx \, \frac{1}{2}\left(\frac{d\phi_K}{dx}\right)^2 = \frac{M_K}{2}.}$$

Hence, we find:

$$E = M_K + \int dz \frac{1}{2}\left(\frac{d\phi_K}{dz}\right)^2 \dot{x}_0^2 = M_K + \frac{M_K}{2}\dot{x}_0^2$$

As you see, the zero mode gives the kinetic energy contribution to the kink total energy. Hence, the ~~energy~~ mass of the kink is

$$\boxed{M_K^{1-loop} \equiv M_K + \sum_{n\neq 0}^{\infty} \frac{1}{2}\omega_n}$$

, where $\{\omega_n\}$ are the eigenvalues of the operator L_2:

$$\hat{L}_2 = -\frac{d^2}{dx^2} + m^2\left(1 - \frac{3}{2ch^2\left(\frac{mx}{2}\right)}\right); \quad L_2 \chi_n(x) = \omega_n^2 \chi_n(x)$$

The spectrum of this operator will be studied in the HW1; there are two discrete eigenvalues $\omega^2 = 0$ and $\omega^2 = \frac{3m^2}{4}$ and then continuous eigenvalues from $\omega^2 = m^2$ up to $\omega^2 = \infty$

It is easy to ~~add~~ include discrete eigenvalues into the sum; we need to discuss what to do with the continuum part of the spectrum. Continuous solutions are labeled by an index $p \equiv \sqrt{\omega_p^2 - m^2}$; $0 < p < \infty$.

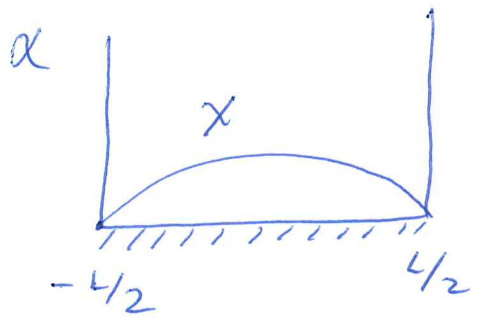
The important feature of the potential $m^2\left(1 - \frac{3}{2ch^2\left(\frac{mx}{2}\right)}\right)$ is that it is reflection-less (see textbooks in QM).

This means that if the continuous spectrum solution is defined through a boundary condition at $x \to \infty$ as $\chi_p^+(x) = e^{ipx}$, then this solution should be $\chi_p^+(x) = e^{ipx + i\delta_p}$ at $x \to -\infty$ (i.e. no reflected wave).

The phase δ_p is determined from the following equation

$$\boxed{e^{i\delta_p} = \left(\frac{1+ip/m}{1-ip/m}\right)\left(\frac{1+2ip/m}{1-2ip/m}\right)}$$

An independent solution describes a ~~photo~~ particle moving in the opposite direction, i.e. e^{-ipx}
It can be obtained as $\chi_p^-(x) = \chi_p^+(-x)$
Now, to have a well-defined calculation, we can put the system in a large box and require that the wave functions vanish at the boundaries



In general,

$$\chi(x) = A\chi_p^+(x) + B\chi_p^-(x) \Rightarrow$$

$$\chi\left(\tfrac{L}{2}\right)=0 \Rightarrow A\chi_p^+\left(\tfrac{L}{2}\right)+B\chi_p^-\left(\tfrac{L}{2}\right)=0$$

$$\chi\left(-\tfrac{L}{2}\right)=0 \Rightarrow A\chi_p^+\left(-\tfrac{L}{2}\right)+B\chi_p^-\left(-\tfrac{L}{2}\right)=0$$

$$A\chi_p^+\left(\tfrac{L}{2}\right)+B\chi_p^-\left(\tfrac{L}{2}\right) = A e^{ipL/2} + B\chi^+\left(-\tfrac{L}{2}\right) =$$

$$= A e^{ipL/2} + B e^{-ipL/2+i\delta_p} = 0$$

$$0 = A\chi^+(-L/2) + B\chi_p^-\left(-\tfrac{L}{2}\right) = A e^{-ipL/2+i\delta_p} + B e^{ipL/2} \Rightarrow$$

$$\begin{pmatrix} e^{ipL/2} & e^{-ipL/2+i\delta_p} \\ e^{-ipL/2+i\delta_p} & e^{ipL/2} \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = 0 \Rightarrow$$

$$e^{ipL} - e^{-ipL+2i\delta_p} = 0 \Rightarrow \boxed{e^{ipL-i\delta_p} = \pm 1}$$

This equation of course implies that

$$\boxed{pL - \delta_p = \pi n \,, \quad \text{with} \quad n = 0, 1, \dots \text{ etc.}}$$

Suppose we repeat the same calculation but with the vacuum solution; the result will be the same but for $\delta_p \equiv \phi$., so $\tilde{p}_n L = \pi n$.

The increase in the kink mass should be "renormalized", i.e. all contributions from the vacuum fluctuations to any energy density should be subtracted.

Then

$$\delta M_K = \frac{\sqrt{3}m}{4} + \sum_n \left(\frac{\omega_n}{2} - \frac{\omega_n^{vac}}{2} \right) = \frac{\sqrt{3}\,m}{4} + \delta \tilde{M}_K$$

$$\delta \tilde{M}_K = \sum_n \left(\frac{\omega_n}{2} - \frac{\omega_n^{vac}}{2} \right) = \frac{1}{2} \sum_n \left(\sqrt{m^2 + p_n^2} - \sqrt{m^2 + \tilde{p}_n^2} \right)$$

To calculate this sum we note that p_n and $\tilde{p}_n$ get close as $L \to \infty$. So we write

$$\delta \tilde{M}_K = \frac{1}{2} \sum_n \left(\sqrt{m^2 + \tilde{p}_n^2 + (p_n^2 - \tilde{p}_n^2)} - \sqrt{m^2 + \tilde{p}_n^2} \right)$$

$$= \frac{1}{2} \sum_n \left(\frac{1}{2} \frac{p_n^2 - \tilde{p}_n^2}{\sqrt{m^2 + \tilde{p}_n^2}} \right) = \frac{1}{2} \sum_n \frac{1}{2} \frac{(p_n - \tilde{p}_n)(p_n + \tilde{p}_n)}{\sqrt{m^2 + \tilde{p}_n^2}}$$

$$(p_n - \tilde{p}_n)(p_n + \tilde{p}_n) \approx 2\tilde{p}_n \frac{\delta_{p_n}}{L}. \quad \text{Now}$$

$$\sum_n \equiv \frac{L}{\pi} \int dp \qquad \delta \tilde{M}_K = \frac{L}{2\pi} \int_0^\infty d\tilde{p} \, \frac{1}{2} \, \frac{2\tilde{p}\, \delta_{\tilde{p}}}{L \sqrt{m^2 + p^2}} \Rightarrow$$

$$\delta \tilde{M}_K = \frac{1}{2\pi} \int_0^\infty dp \, \delta_p \cdot \frac{\partial}{\partial p} \left[\sqrt{m^2 + p^2} \right]$$

We now integrate by parts and use the fact – that follows from the exact expression for $e^{i\delta_p}$ on page 7 — that $\delta_p = 0$ for $p=0$ & $p=\infty$. Hence, we obtain

$$\delta\tilde{M}_k = \left(-\frac{1}{2\pi}\right)\int_0^\infty dp\,\sqrt{m^2+p^2}\,\frac{\partial \delta_p}{\partial p}.$$

We will try to pick up the logarithmic integration. To this end, calculate $\frac{d}{dp}e^{i\delta_p} = i\frac{d\delta_p}{dp}e^{i\delta_p} \Rightarrow$

$$i\frac{d\delta_p}{dp} = \frac{\frac{d}{dp}e^{i\delta_p}}{e^{i\delta_p}} = \frac{(1-ip/m)(1-2ip/m)}{(1+ip/m)(1+\frac{2ip}{m})}\times\frac{d}{dp}\left[\frac{1+i\frac{p}{m}}{1-i\frac{p}{m}}\times\frac{1+\frac{2ip}{m}}{1-\frac{2ip}{m}}\right]$$

It is straight-forward to calculate the derivative, in the limit $p \gg m$. We find ($p\gg m$)

$$\frac{d\delta_p}{dp} = \frac{3m}{p^2}.$$

Then, the shift in the mass

$$\delta\tilde{M}_k = -\frac{1}{2m}\int_0^\infty dp\,p\,\frac{3m^2}{p^2} = -\frac{3m}{2\pi}\int_m^{M_{uv}}\frac{dp}{p} = -\frac{3m}{2\pi}\ln\left(\frac{M_{uv}}{m}\right)$$

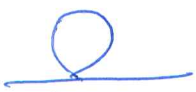
The one-loop shift of the mass becomes:

$$\delta M_k = \frac{\sqrt{3}\,m}{4} - \frac{3m}{2\pi}\ln\frac{M_{uv}}{m} + \begin{pmatrix}\text{uncalculated}\\ \text{non-logarithmic}\\ \text{continuum terms}\end{pmatrix}.$$

The full mass of the kink at one-loop is

$$M_k^{\text{one-loop}} = M_k + \delta M_k = \frac{m^3}{3g^2} + \frac{\sqrt{3}\,m}{4} - \frac{3m}{2\pi}\ln\frac{M_{uv}}{m} + \dots$$

To get to the very finite result, we need to account for the fact that ~~the~~ there is a one-loop renormalization of the mass parameter in QFT

 $\Leftrightarrow \quad m_R^2 = m^2 - \frac{3g^2}{2\pi}\ln\left(\frac{M_{uv}^2}{m^2}\right)$

This allows us to absorb the $\ln \frac{M_{uv}}{m}$ divergence into renormalized mass.

Thus we have $M_K^{\text{one-loop}} = \frac{m_R^3}{3g^2} + \frac{\sqrt{3}\, m_R}{4} + \dots$

The complete one-loop result (i.e. including non-logarithmic correction) reads:

$$M_K^{\text{one-loop}} = \frac{m_R^3}{3g^2} - m_R\left(\frac{3}{2\pi} - \frac{\sqrt{3}}{12}\right).$$

(Dashen, Hasslacher, Neveu)