Lattice field theory

Up to now we talked about Quantum Field Theory in a continuum space-time; ~~the~~ quatum fields were labeled by a variable x^μ, that was continuous.

Within this framework, Green's functions can be related to path integral

$$\langle 0| T\,\phi(x_1)\ldots\phi(x_n)|0\rangle = Z^{-1}\int[\mathcal{D}\varphi]\,\varphi(x_1)\ldots\varphi(x_n)\,e^{iS[\phi]},$$

with $Z = \int [\mathcal{D}\varphi]\, e^{iS[\varphi]}$, but we know of no good way to evaluate it beyond perturbation theory (note that even the <u>precise meaning of the measure</u> $[\mathcal{D}\varphi]$ is obscure).

In 1974 K. Wilson suggested to define the Eucledian version of the path integral <u>on a space-time lattice</u>. This means the following. <u>Define</u> the Eucledian action by taking $t \to i\tau$, so that the Lagrange density becomes

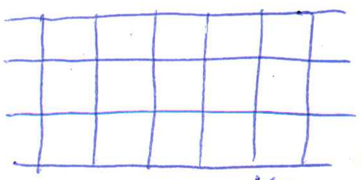
$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu\varphi\,\partial^\mu\varphi - m^2\varphi^2\right) - V(\varphi)$$

$$\to \frac{1}{2}\left[\partial_{\bar\mu}\varphi\,\partial^{\bar\mu}\varphi + m^2\varphi^2 + V(\varphi)\right]$$

& the metric $g_{\mu\nu} \equiv (1,1,1,1)$.

<u>Assume</u> that fields are only defined

on a space-time lattice (hypercube), with sites separated by spacing a.



$\}a$ $\overset{\leftrightarrow}{a}$

There are N sites in each direction; a site is identified by a collection of 4 integer numbers $\{n_1, n_2, \dots, n_4\}$, so that

$$x = a \sum_{\mu=1}^{4} n_\mu e_\mu$$ where $\{e_\mu\}$ are four unit vector that form the basis of the 4-dim. space. We take $1 \le n_\mu \le N$, for all "μ".

What happens to the path integral if we do that? The measure $[\mathcal{D}\varphi]$ turns into ordinary integration measure:

$$[\mathcal{D}\varphi] \to \prod_{\vec{n}} d\phi_{\vec{n}}$$, where $\phi_{\vec{n}}$ is the value of the field at the $\vec{n}$'th site. The action that we had in the continuum needs to be generalized to describe lattice.

We take $$\partial_\mu \phi \equiv \frac{\phi_{n+\mu} - \phi_n}{a}$$, so that

$$S_{lattice} = a^4 \sum_{\vec{n}} \sum_{\mu} \left\{ \frac{1}{2a^2} (\phi_{n+\mu} - \phi_n)^2 + \frac{m^2}{2}\phi_n^2 + V(\phi_n) \right\}$$

We need to define what happens for $\phi_{\vec{N}+\mu}$ & we assume that fields are periodic $$\boxed{\phi(\vec{n} + \vec{N}) = \phi(\vec{n})}$$

with these clarifications, the action is fully defined & the path integral average reads

$$\langle \mathcal{O}(\phi) \rangle_{lattice} = Z^{-1} \int \prod_{\{n^\mu\}} d\phi_{n_\mu} \, e^{-S_{lattice}}$$

Given the periodicity of the fields, it is useful to write them as Fourier series:

$$\phi_{\vec{n}} = N^{-4} \sum_{\{q_\mu\}} \phi_{\{q\}} \, e^{-i q_\mu \cdot n^\mu}, \quad \text{where } q_\mu = \frac{2\pi k_\mu}{N},$$

$$k_\mu \in \left\{ -\frac{(N-1)}{2}, \dots \frac{(N-1)}{2} \right\}$$

with the constraint $\phi_q^* = \phi_{-q}$, since $\phi_{\vec{n}}$ is real.

Now, let us use the Anzatz for $\phi_{\vec{n}}$ in the action, neglecting the potential energy.

We write:

$$\sum_{n^\mu} \phi_n^2 = N^{-8} \sum_{\vec{n}} \sum_{\vec{q}, \vec{q}'} \phi_{\vec{q}} \phi_{\vec{q}'} e^{-i\vec{q}\cdot\vec{n}} e^{-i\vec{q}'\cdot\vec{n}} =$$

$$= N^{-8} \sum_{\vec{q}, \vec{q}'} \phi_{\vec{q}} \cdot \phi_{\vec{q}'} \left[\sum_n e^{-i(\vec{q}+\vec{q}')\cdot n} \right] =$$

$$= N^{-8} \sum_{\vec{q}, \vec{q}'} \phi_{\vec{q}} \phi_{\vec{q}'} [N \delta_{q_1 q_1'}][N \delta_{q_2 q_2'}][N \delta_{q_3 q_3'}] [N \delta_{q_4 q_4'}] \equiv$$

$$\equiv N^{-4} \sum_q \phi_{\vec{q}} \cdot \phi_{-\vec{q}} = N^{-4} \sum_q |\phi_q|^2.$$

The kinetic term is similar:

$$\sum_{n,\mu} \frac{1}{2a^2} (\phi_{\vec{n}+\mu} - \phi_{\vec{n}})^2 = N^{-8} \sum_{n,\mu} \frac{1}{2a^2} (\phi_{n+\mu} \phi_{n+\mu} + \phi_n \phi_n - 2 \phi_{n+\mu} \phi_n) =$$

$$= N^{-8} \sum_{n,\mu} \frac{1}{2a^2} \phi_q \phi_{q'} e^{-iqn-iqn'} \Big(e^{-i\mu\cdot q} e^{-i\mu q'} + 1 - e^{-i\vec{\mu}\cdot\vec{q}} - e^{-i\vec{\mu}\cdot\vec{q}\,'} \Big) =$$

$$= \frac{N^{-4}}{2a^2} \sum_{q,\mu} |\phi_{\vec{q}}|^2 \big(2 - 2\cos(\vec{\mu}\cdot\vec{q}) \big).$$

The action for the free scalar field becomes

$$\boxed{S = \frac{a^4}{N^{+4}} \sum_{q} \Big\{ \frac{1}{2a^2} \sum_{\mu} \big(2 - 2\cos(\mu\cdot q)\big) |\phi_{\vec{q}}|^2 + \frac{m^2}{2} |\phi_{\vec{q}}|^2 \Big\}.}$$

The scalar propagatr is the inverse quadratic form:

$$\Delta(q) = \frac{1}{\Big[m^2 + \frac{2}{a^2} \sum_{\mu} \big(1 - \cos(q\cdot\mu)\big)\Big]}$$

The propagatr in the position space is

$$\langle \phi_n \phi_m \rangle = \frac{1}{a^4 N^4} \sum_q \Delta(q)\, e^{iq\cdot(m-n)}$$

Now, writing $q = \frac{2\pi k}{N}$, $\Delta q = \frac{2\pi}{N}$ & we obtain, taking the $N \to \infty$ limit

$$\langle \phi_n \phi_m \rangle = \frac{1}{a^4} \int_{-\pi}^{\pi} \frac{d^4 q}{(2\pi)^4} \frac{e^{iq\cdot(m-n)}}{m^2 + \frac{2}{a^2} \sum_{\mu} \big(1 - \cos(q\cdot\mu)\big)} =$$

$$= (q \Rightarrow aq) = (an = x_n,\ a\cdot m = x_m) \Rightarrow$$

$$\boxed{\langle \phi_{x_m} \phi_{x_n} \rangle = \int_{-\pi/a}^{\pi/a} \frac{d^4 q}{(2\pi)^4} \frac{e^{iq(x_m - x_n)}}{m^2 + \frac{2}{a^2} \sum_{\mu=1}^{4} \big[1 - \cos(a\cdot q\cdot\mu)\big]}}$$

The continuum limit of the lattice propagator follows if one takes $a \to 0$ limit at fixed q. Then $1-\cos(a \cdot q \cdot \mu) \approx \frac{a^2 q_\mu^2}{2}$, so that ($\pi/a \to \infty$)

$$\langle \phi_{x_m} \phi_{x_n} \rangle \approx \int_{-\infty}^{\infty} \frac{d^4 q}{(2\pi)^4} \frac{e^{i q_\mu (x_m - x_n)^\mu}}{m^2 + \sum_{\mu=1..4} q_\mu^2} .$$

The important point about the previous equation is that the integration over q is restricted to $|q_\mu| < \pi/a$, so that the inverse lattice spacing is an ultraviolet regulator. Another aspect of it, is that <u>Lorentz symmetry is violated by the lattice</u> (homework, examples).

Suppose we want to do something similar for fermions. The action is

$$S_E = \int \bar{\psi} (\hat{\partial} + m) \psi \, d^4x .$$

To discretize this action, we write

$$S_E = \int \bar{\psi} (\hat{\partial} + m) \psi \, d^4x = \int d^4x \left[-\frac{1}{2} (\partial_\mu \bar{\psi}) \gamma^\mu \psi + \frac{1}{2} \bar{\psi} \gamma^\mu \partial_\mu \psi + m \bar{\psi} \psi \right].$$

Now, the discretization can be performed in a symmetric manner and one obtains:

$$S_E = a^4 \sum_n \left(-\frac{1}{2} \bar{\psi}_{n+\mu} \gamma^\mu \psi_n + \frac{1}{2} \bar{\psi}_n \gamma^\mu \psi_{n+\mu} + m \bar{\psi}_n \psi_n \right).$$

This form of action, when written in momentum space, gives the quadratic form $\sum_k \bar{\psi}_k M_k \psi_k$ with $M_k = m + \frac{i}{a} \sum_\mu \gamma_\mu \sin\left(\frac{2\pi k}{N}\right)$ so that the fermion propagator reads:

$$\boxed{\langle \psi_m \bar{\psi}_n \rangle = \int_{-\pi/a}^{\pi/a} \frac{d^4 q}{(2\pi)^4} \frac{1}{m + i/a \sum_\mu \gamma_\mu \sin(a q^\mu)}}$$

Although this propagator look innocent, its continuum limit shows bizzare features.

To see this, note that the integration ~~measure~~ volume can be rewritten as:

$$\int_{-\pi/a}^{\pi/a} dq = \int_{-\pi/2a}^{\pi/2a} dq + \int_{\pi/2a}^{\pi/a} dq + \int_{-\pi/a}^{-\pi/2a} dq =$$

(change $q = \pi/a + r$ in the second & $q = -\frac{\pi}{a} + r$ in the third), so that

$$\int_{-\pi/a}^{\pi/a} dq = \int_{-\pi/2a}^{\pi/2a} dq + \int_{-\pi/2a}^{\pi/2a} dr,$$

What happens to M_q under this transformation?

$$M_k = m + \frac{i}{a} \sum_\mu \gamma_\mu \sin\left(a\left(\pm\pi/a + r_\mu\right)\right) \Rightarrow$$

$$M_k \equiv m + \frac{i}{a} \sum_\mu \gamma_\mu \sin(a z_\mu) (-1)^{\pm} ;$$

this implies that in the $a \to 0$ limit, $\frac{1}{M_k}$ has a pole for small $z_\mu \Rightarrow q_\mu \sim \frac{\pi}{a}$!

Therefore, for each components of momentum q, there are 2 contributions to the continuum limit which implies that

1 lattice fermion produces 16 continuum fermions. This is known as the "fermion doubling problem". The deep reason for the problem is the need to cancel the chiral anomaly since the lattice theory is a theory with finite ~~deg~~ number of degrees of freedom where anomaly can't occur. In general, however, getting 16 copies of what we want to get isn't ideal.

A way to deal with this problem was suggested by ~~[illegible]~~ Wilson and it, essentially, gives an example of how one can change ~~[illegible]~~ the lattice action to change the continuum limit. ["Designer" action]

Wilson suggested to change the action $S = \int \bar{\psi}(\hat{\partial} + m)\psi \, d^4x$ by adding ~~to~~ it a term

$S_W = -ar \int \bar{\psi}\, \partial_\mu \partial^\mu \psi \, d^4x$, where r is a parameter. Since π/a is the UV cutoff of the theory, in the IR, we are changing our theory by negligible amount.

But, since the doublers come from momenta regions where $p \sim \pi/a$, the Wilson operator may help to get rid of them.

It is easy to see what Wilson operator will do. Since $\bar{\psi}\partial^2\psi$ doesn't distinguish between fermions and scalars, ~~the~~ it adds the "scalar" propagator to the fermion propagator. With proper normalization

$$M_K \longrightarrow m + \frac{i}{a}\sum_\mu \gamma_\mu \sin(aq_\mu) + \frac{r}{a}\sum_\mu (1 - \cos aq_\mu)$$

~~Since~~ If $q \ll 1/a$,

$$M_K \simeq m + i\hat{q} + O(aq^2) \xrightarrow[a \to 0]{} m + i\hat{q}$$

If, on the other hand, we look at a doubler, whose momentum is $q_\mu \sim \frac{\pm\pi}{a}$,

$1 - \cos(aq_\mu) \to 2$, for each q_μ close to a cut off.

Since the $\sin(aq_\mu)$ term isn't affected, the Wilson addition to the action ~~splits the~~ adds to the mass of a doubler:

$m_{doubler} \simeq m + \frac{2r}{a}\#$, where $\#$ is the $\#$ of ~~dour~~ components of doubler's momenta that are close to $\pm \pi/a$.

Suppose that we want to put gauge fields on the lattice. In principle we can try to do this in the same way as what we do in the continuum, but the following consideration suggests that it might be better to do it differently. Indeed, ~~the~~ lattice actions for the matter fields aren't local ($\phi_{n+\mu} \cdot \phi_n$) and if matter fields transform in the standard way under gauge transformations $\phi_n \to \Omega_{n} \phi_n$, $\Omega \in SU(N)$, the lattice action isn't gauge invariant. There is a way to make products of matter fields being invariant & this is to connect them by a Wilson line:

$$\phi^\dagger(x_1)\,\phi(x_2) \longrightarrow \phi^\dagger(x_1)\, \underbrace{P\, e^{ig\int_{x_2}^{x_1} A_\mu dx^\mu}}\, \phi(x_2),$$

where $P\, e^{ig\int_{x_2}^{x_1} A_\mu dx^\mu}$ is the path-ordered exponential. To see that this expression is gauge invariant, consider the two points being close to each other. ~~Then~~ ~F

We find

$$\phi^+(x_1)\, P\, e^{ig\int_{x_2}^{x_1}\hat{A}_\mu dx^\mu}\phi(x_2) \to \phi^+(x_1)\Omega^+_{x_1} P e^{ig\int_{x_2}^{x_1}\hat{A}_\mu(\Omega)dx^\mu}\Omega_{x_2}\phi,$$

where $\hat{A}_\mu(\Omega) = \Omega\hat{A}_\mu\Omega^+ + \frac{i}{g}\Omega\partial_\mu\Omega^+$.

Expanding in Δx_μ, we find that $\phi^+_{x_1} e^{ig\int_{x_2}^{x_1}\hat{A}_\mu dx^\mu}\phi_{x_2}$ does not change under gauge transformations.

For this reason, Wilson suggested to place matter fields on lattice sites (as we ~~saw~~ did) and use Wilson lines

$$U(\Delta x) = P e^{ig\int_{x_2}^{x_1} A_\mu dx^\mu}$$ between sites

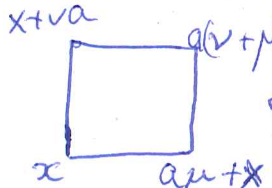
as degrees of freedom for gauge fields.

We have then, as degrees of freedom for gauge fields, the links of the form.

$$U_{x,\mu} = P\exp\left[ig\int_0^a dt\, \hat{A}_\mu(x+\mu t)\right]$$

Under gauge transformations, links transform as $U'_{x,\mu} \to \Omega_x U_{x,\mu}\Omega^+_{x+\mu}$

Now, we can notice that taking a product of the links that form a plaquette and Tracing the products is gauge-invariant.



$$\Longleftrightarrow U^{(P)}_{\mu\nu} = \mathrm{Tr}\left(U_{x,\mu}\, U_{x+\mu,\nu}\, U_{x+\mu+\nu,-\mu}\, U_{x+\nu,-\nu}\right)$$

We will now show that in the continuum we recover the kinetic term of the gauge field from "plaquett action".

We'll show this for the $\underline{\text{abelian}}$ case

Then
$$U^{(P)}_{\mu\nu} = \exp\Big[ig\int_0^a A_\mu(x+\mu t)\,dt + ig\int_0^a A_\nu(x+\mu a+\nu t)\,dt$$
$$- ig\int_0^a A_\mu(x+\mu a+\nu a-\mu t)\,dt - ig\int_0^a A_\nu(x+\nu a-\nu t)\,dt\Big]$$

We expand each integral taking the field A_μ at the center of the interval, e.g.
$$\int_0^a A_\mu(x+\mu t)\,dt \approx A_\mu\left(x+\frac{\mu a}{2}\right)\cdot a + \frac{1}{12}\frac{\partial^2 A_\mu}{\partial x_\mu^2}\Big|_{x+\frac{\mu a}{2}}\cdot a^3 + O(a^5)$$

Then
$$U^{(P)}_{\mu\nu} = \exp\Big[ig\Big(A_\mu\left(x+\frac{\mu a}{2}\right)\cdot a + \frac{a^3}{12}\frac{\partial^2 A_\mu}{\partial x_\mu^2}\left(x+\frac{\mu a}{2}\right)$$
$$+ A_\nu\left(x+\frac{\mu a}{2}+\frac{\nu a}{2}\right)a + \frac{a^3}{12}\frac{\partial^2 A_\nu}{\partial x_\nu^2}\left(x+\frac{\mu a}{2}+\frac{\nu a}{2}\right)$$
$$- A_\mu\left(x+\nu a+\frac{\mu a}{2}\right)a - \frac{a^3}{12}\frac{\partial^2 A_\mu}{\partial x_\mu^2}\left(x+\nu a+\frac{\mu a}{2}\right)$$
$$- A_\nu\left(x+\frac{\nu a}{2}\right)a - \frac{a^3}{12}\frac{\partial^2 A_\nu}{\partial x_\nu^2}\left(x+\frac{\nu a}{2}\right)\Big)\Big] \simeq$$
$$\simeq \exp\left[ig\left[\left(-\frac{\partial A_\mu}{\partial x^\nu} + \frac{\partial A_\nu}{\partial x^\mu}\right)a^2 - \frac{a^4}{12}\left(\frac{\partial^3 A_\mu}{\partial^2 x_\mu \partial x_\nu} - \frac{\partial^3 A_\nu}{\partial^2 x_\nu \partial x_\mu}\right)\right]\right]$$
$$\simeq \exp\left[ig F_{\mu\nu} a^2 + \mathcal{O}(a^4)\right] \simeq$$
$$\simeq 1 + ig F_{\mu\nu} a^2 - \frac{g^2 F_{\mu\nu} F^{\mu\nu} a^4}{2}.$$

To get rid of imaginary part, we can take the real part of the plaquette action, subtract 1 and find: $-\mathrm{Re}(\mathcal{U}^P_{\mu\nu}-1) = \dfrac{g^2 F_{\mu\nu} F^{\mu\nu} a^4}{2}$

The full action is

$$S^{AB} = \frac{1}{g^2} \sum_{\substack{x \\ \text{(all sites)}}} \sum_{\mu<\nu} \mathrm{Re}\left(1-\mathcal{U}^P_{\mu\nu}\right) = \frac{1}{4}\int d^4x\, F_{\mu\nu}F^{\mu\nu} + \mathcal{O}(a^2)$$

In the non-abelian case with the gauge group $SU(N)$, we have to take the trace of $\mathcal{U}^P_{\mu\nu}$, and normalize it. ~~We~~ ~~fo~~ The action reads:

$$\boxed{S^{(SU(N))} = \sum_{x,\{\mu\nu\}} \frac{2N}{g^2}\left[1-\frac{1}{2N}\mathrm{Re}\left(\mathrm{Tr}\left[\mathcal{U}^P_{\mu\nu}\right]\right)\right].}$$

If we were to use this action in the path integral, we ~~will~~ need a measure of integration. The degrees of freedom we have are values of the elements $\mathcal{U}_{x,\mu}$, for each lattice link. Hence, we want to integrate over $\mathcal{U}$'s for all lattice links

$$\int [dA_\mu] \longrightarrow \int [\mathcal{D}\mathcal{U}_{x,\mu}]$$

Let's construct the integration measure:

The link element $\mathcal{U}_{x,\mu}$ is an $SU(N)$ matrix. Consider $SU(N) \to \mathcal{U}(1)$

$$\mathcal{U}_{x,\mu} = e^{ig\int_0^a A_\mu(x+\mu t)\,dt} \in \mathcal{U}(1)$$

Hence, we should write $ig\int_0^A A_\mu(x+\mu t)\,dt \equiv \theta,$

$-\pi \leq \theta \leq \pi$ & $\int dU_{x,\mu} \equiv \int_{-\pi}^{\pi} d\theta_{x,\mu}$.

For $SU(2)$, any $su(2)$ matrix can be written as $U = \cos\theta \cdot \mathbb{1} + \vec{n}\cdot\vec{\tau}\cdot\sin(\theta)$, so that if we write $U = a_0 \mathbb{1} + \vec{a}\cdot\vec{\tau}$, then for $su(2) \equiv a_0^2 + \vec{a}^2 = 1$. Hence, we can take $\int dU^{su(2)}_{x,\mu} = \int \frac{d^4 a}{\pi^2}\, \delta(\vec{a}^2 - 1) = \int \frac{d\vec{\Omega}_4}{2\pi^2}$, where $d\vec{\Omega}_4$ is a solid angle in 4-dimensions.

It reads then $\int \frac{d\vec{\Omega}_4}{2\pi^2} = \int_0^{\pi} d\theta \sin^2\theta \int_0^{\pi} d\varphi \sin\varphi \int_0^{2\pi} d\alpha$

$\Rightarrow$ $a_0 = \cos\theta$ $a_1 = \sin\theta\cos\varphi$ $a_2 = \sin\theta\sin\varphi\cos\alpha$ $a_3 = \sin\theta\sin\varphi\sin\alpha$.

Fine. Now it is easy to see that

*) $\int dU_{x,\mu} \cdot U_{x,\mu} = 0$ (this explicitly follows from the integrals that we have written).

It also follows that

**) $\int dU_{x,\mu} \cdot U_{x,\mu} U^{+}_{x,\mu} \equiv 1$

These results are manifestations of ~~the~~ a more general integration rules, valid for $\forall$ $SU(N)$'s, but these properties will be sufficient for us.

We do not have time for that but one can ~~ease~~ use these results to construct the strong coupling expansion $g^2 \to \infty$ of the lattice theory & do prove that such a theory "confines".