More on the anomaly

We will now discuss a connection between a more traditional calculation of the anomaly and its infra-red nature using the Schwinger model as an example. We want to prove an equation

$$\partial_\mu j^{\mu,5} = -\frac{1}{2\pi} \varepsilon^{\mu\nu} F_{\mu\nu}$$

To see how this equation appears, let us first define the <u>regularized axial current</u>

$$j^{\mu,5} \Rightarrow j^{\mu,5}_{Reg} = \bar{\psi} \gamma^\mu \gamma_5 \psi - \bar{R} \gamma^\mu \gamma_5 R. \quad (\text{note sign!})$$

Here $(\psi, \bar{\psi})$ are the massless fermion fields and $(R, \bar{R})$ are the massive fermion fields. The mass M_R is supposed to be taken to infinity at the end of the calculation. We would like to calculate the divergence of the current directly.

The fermion R is supposed to regularize ~~generalize~~ the current, so that we can apply equations of motion:

$$\partial_\mu j^{\mu,5}_{reg} = \bar{\psi} \vec{\hat{\partial}} \gamma_5 \psi - \bar{R} \hat{\partial} \gamma_5 R + \bar{\psi} \overleftarrow{\hat{\partial}} \gamma_5 \psi + \bar{R} \overleftarrow{\hat{\partial}} \gamma_5 R = -i \{ \bar{\psi} i \vec{\hat{\partial}} \gamma_5 \psi - \bar{R} i \vec{\hat{\partial}} \gamma_5 R + \bar{\psi} i \overleftarrow{\hat{\partial}} \gamma_5 \psi + \bar{R} i \overleftarrow{\hat{\partial}} \gamma_5 R \} = -2i M_R \bar{R} \gamma_5 R,$$

(Same result holds if ψ & R couple to gauge fields).

~~Where~~ Have we used the Dirac equations, e.g.

$i\hat{D}\psi = 0$, $i\hat{D}R = M_R\psi$, ~~Hence~~ we ~~would~~ obtain the same result:

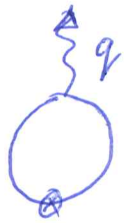
$$\boxed{\partial_\mu j^{\mu,5}_{reg} = -2iM_R \bar{R}\gamma_5 R}$$

This is an operator equation and we want to calculate the matrix elements of both sides of the previous equation.

There are several matrix elements that we can imagine:

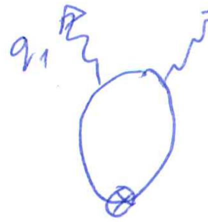
$= \langle 0| -2iM_R\bar{R}\gamma_5 R|0\rangle$;

a)



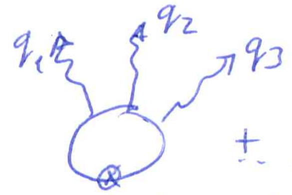
$= \langle \gamma_q| -2iM_R\bar{R}\gamma_5 R|0\rangle$

b)



$= \langle \gamma_{q_1}\gamma_{q_2}| -2iM_R\bar{R}\gamma_5 R|0\rangle$;

c)



$+ \ldots$

d)

We need to understand which contributions survive $M_R \to \infty$ limit. To this end, we will need a couple of formulas for ~~.~~ traces with γ_5. They are $[\gamma^0 = \sigma_2, \gamma^1 = -i\sigma_1, \gamma^5 = \sigma_3]$

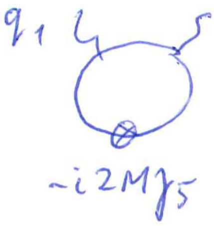
$Tr(\gamma_5) = 0$ $\quad$ $Tr(\hat{a}\gamma_5) = 0$ $\quad$ $Tr(\hat{a}\hat{b}\gamma_5) = 2\varepsilon^{\mu\nu}a_\mu b_\nu$,

with $\varepsilon^{01} = 1$ & $\varepsilon^{\mu\nu} = -\varepsilon^{\nu\mu}$.

The first diagram (a) is proportional to $Tr(\gamma_5(\hat{k} + M_R)) \equiv \emptyset$. The fourth & all higher multiplicity diagrams vanish by power counting in $M_R \to \infty$ limit. (Example (d): 4 propagators $\sim \frac{1}{M_R^4}$, the integration measure $d^2k \sim M_R^2$, and the vertex for $-2iM_R \bar{R}\gamma_5 R$; M_R; all together, $\lim_{M_R\to\infty} M_R^3/M_R^4 \to 0$). The third diagram (c) requires some consideration. Naive power counting gives

$$\underbrace{M}_{\text{vertex}} \cdot \underbrace{M^2}_{\text{measure}} \cdot \underbrace{\frac{1}{M^3}}_{\text{props}} \to 1$$

and so doesn't show any decoupling. However, since coupling of the external vector field to heavy loop must be gauge invariant, the matrix element must be proportional to $\varepsilon^\mu_{1,2} q^\nu_{1,2} - \varepsilon^\nu_{1,2} q^\mu_{1,2}$ for each of the photons. Therefore, the diagram is proportional to

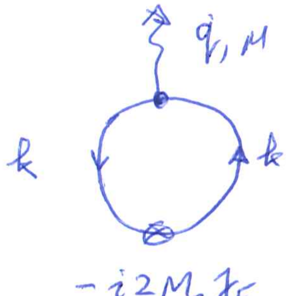


$$\sim \frac{M_R q_1 q_2}{M_R^3} \to 0, \text{ as } M_R \to \infty.$$

The last diagram to check is (b):

The power counting works like this: the mass dimension of this diagram is one. Thanks to gauge-invariance, it must be proportional to $[\varepsilon^\mu q^\nu - \varepsilon^\nu q^\mu]$ the dimensions match, which means that the diagram (b) is mass (M_R) independent.

Let us check this by explicit calculation:



$$= (-1)_F(-2iM_R)\int \frac{d^2k}{(2\pi)^2}\, \frac{Tr[\hat\gamma_5(\hat k + M_R)\hat\varepsilon(\hat k + M_R + \hat q)]}{(k^2-M_R^2)((k+q)^2-M_R^2)}\, i^3 =$$

$$= +2M_R\int \frac{d^2k}{(2\pi)^2}\, \frac{Tr[\hat\gamma_5(\hat k + M_R)\hat\varepsilon(\hat k + M_R + \hat q)]}{(k^2-M_R^2)((k+q)^2-M_R^2)}$$

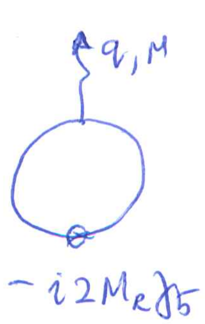
$$Tr[\hat\gamma_5(\hat k + M_R)\hat\varepsilon_\gamma(\hat k + \hat q + \hat M_R)] = M_R^2\, \cancel{Tr(\hat\gamma_5\hat\varepsilon_\gamma)} + M_R\left[Tr[\hat\gamma_5\hat\varepsilon_\gamma(\hat k + \hat q)] + Tr(\hat\gamma_5\hat k\hat\varepsilon_\gamma)\right] + Tr(\hat\gamma_5\hat k\hat\varepsilon_\gamma(\hat k + \hat q)) = M_R\, Tr(\hat\gamma_5\hat\varepsilon_\gamma\hat q) + Tr(\hat\gamma_5\hat k\hat\varepsilon\hat q)$$

$\to 2M_R\, \varepsilon^{\mu\nu}\varepsilon_{\gamma\mu}q_\nu$, and the last term can be neglected because it vanishes in $M_R\to\infty$ limit. Hence, we obtain ($f^{\mu\nu} = \varepsilon^\mu q^\nu - \varepsilon^\nu q^\mu$)

$$\underset{M_R\to\infty}{\Rightarrow} + \int \frac{d^2k}{(2\pi)^2}\, \frac{2M_R\, 2M_R\, \varepsilon^{\mu\nu}\varepsilon_{\gamma\mu}q_\nu}{(k^2-M_R^2)^2} =$$

$$= +4M_R^2\, \frac{\varepsilon^{\mu\nu}}{2} f_{\mu\nu}(q) \int \frac{d^2k}{(2\pi)^2}\, \frac{1}{(k^2-M_R^2)^2}$$

Perform the Wick rotation and integrate; the result is $\int \frac{d^2k}{(2\pi)^2}\frac{1}{(k^2-M_R^2)^2} = \frac{i}{4\pi M_R^2} \Rightarrow$



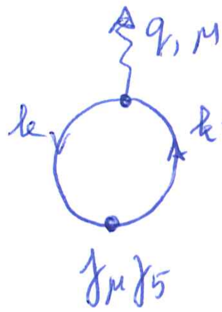
$$= \frac{+i}{2\pi}\varepsilon^{\mu\nu}\cdot f_{\mu\nu}(q).$$

In position space, this equation implies the anomaly equation:

$$\boxed{\lim_{M_R\to\infty} \partial_\mu J^{\mu,5}_{reg} = -\frac{1}{2\pi}\varepsilon^{\mu\nu}F_{\mu\nu}}$$

This anomaly equation is now obtained from UV properties of the divergence of the current (non-decoupling of the UV regulator; the UV regulator is introduced in a way that keeps the gauge invariance intact).

Next, we consider the calculation of the matrix element of the current itself:



$\equiv \langle 0 | j_5^\mu | q, \varepsilon \rangle$. This matrix element should be transversal w.r.t. $[\varepsilon^\mu \to q^\mu]$ & parity odd.

As the result, there is just one term that we can write down:

$$\boxed{\langle 0 | j_5^\mu | q, \varepsilon \rangle = F(q^2) \frac{q^\mu}{q^2} \varepsilon^{\alpha\beta} f_{\alpha\beta} ,}$$

where $f_{\mu\nu} = \varepsilon_\mu q_\nu - \varepsilon_\nu q_\mu$. [Show that e.g. $\varepsilon^{\mu\alpha} f_{\alpha\beta} q_\beta$ can be written as $\frac{1}{2} q^\mu \epsilon^{\alpha\beta} f_{\alpha\beta}$ using Schouten identities].

The importance feature of this equation for $\langle 0 | j_5^\mu | q, \varepsilon \rangle$ is the appearance of the pole at $q^2 = 0$. The $q^2 \to 0$ singularities appear only because of the infra-red behavior of the diagram. The infrared behavior of a diagram is, however, not ambiguous.

We find:

$$\langle j_5^\mu \rangle = i^2 (-1)_F \int \frac{d^2k}{(2\pi)^2} \,\mathrm{Tr}\left[\gamma^\mu \gamma_5 \frac{i\hat{k}}{k^2} \gamma^\rho \frac{i(\hat{k}+\hat{q})}{(q+k)^2} \right] \epsilon_\rho$$

$$= \int \frac{d^2k}{(2\pi)^2} \frac{\mathrm{Tr}\left[\gamma^\mu \gamma_5 \hat{k} \gamma^\rho (\hat{k}+\hat{q}) \right]}{k^2(k+q)^2} \epsilon_\rho$$

Consider the tensor integral:

$$\int \frac{d^2k}{(2\pi)^2} \frac{k^\alpha (k+q)^\beta}{k^2(k+q)^2};$$ To extract the $1/q^2$ pole from this integral, write

$$\frac{1}{k^2(k+q)^2} = \int_0^1 \frac{dx}{\left[k^2(1-x) + (k+q)^2 x \right]^2} = \int_0^1 \frac{dx}{\left[(k+qx)^2 + q^2 x(1-x) \right]^2}$$

$$\int \frac{d^2k}{(2\pi)^2} \frac{k^\alpha (k+q)^\beta}{k^2(k+q)^2} = \int_0^1 dx \int \frac{d^2k}{(2\pi)^2} \frac{k^\alpha (k+q)^\beta}{\left[(k+qx)^2 + q^2 x(1-x) \right]^2}$$

$$= (k \to k - qx) = \int_0^1 dx \int \frac{d^2k}{(2\pi)^2} \frac{k^\alpha k^\beta - q^\alpha q^\beta x(1-x)}{\left[k^2 + q^2 x(1-x) \right]^2}$$

The first ($k^\alpha k^\beta$) term doesn't produce $1/q^2$ and, in fact, is UV divergent. The second term is what we want to find. Performing the Wick rotation and integrating, we find

$$\boxed{\int \frac{d^2k}{(2\pi)^2} \frac{k^\alpha (k+q)^\beta}{k^2(k+q)^2} \longrightarrow \frac{i q^\alpha q^\beta}{4\pi q^2} + \text{non-pole}}$$

Using this in the matrix element, we find

$$\langle j_5^\mu \rangle = -\mathrm{Tr}\left[\gamma^\mu \gamma_5 \hat{q} \gamma^\rho \hat{q} \right] \epsilon_\rho \frac{i}{4\pi q^2} + \text{n/pole} =$$

$$= -\mathrm{Tr}\left[\gamma^\mu \gamma_5 \hat{q}\right] 2q_\rho \cdot \varepsilon^\rho \frac{i}{4\pi q^2} + n/pole$$

$$= +2\epsilon^{\mu\alpha} q_\alpha \, 2q_\rho \epsilon_\rho \frac{i}{4\pi q^2} = \frac{+i}{\pi q^2} \varepsilon^{\mu\alpha} q_\alpha (q_\rho \epsilon_\rho) + n/pole.$$

We use Schouten identities

$$q^\mu \varepsilon^{\alpha\rho} + q^\alpha \varepsilon^{\rho\mu} + q^\rho \varepsilon^{\mu\alpha} = 0$$, to write $\begin{pmatrix} \text{contr. w/} \\ q_\alpha \varepsilon_\rho \end{pmatrix}$

$$\varepsilon^{\mu\alpha} q_\alpha (q \cdot \varepsilon) = q^\mu \varepsilon^{\alpha\beta} \varepsilon^\alpha q^\beta + n\text{-pole.} \Rightarrow$$

$$\langle j_5{}^\mu \rangle = \frac{+i}{2\pi q^2} q^\mu \varepsilon^{\alpha\rho} f_{\alpha\rho}$$

Taking the divergence and going back to the position space, we again obtain

$$\boxed{\partial_\mu j_5^\mu = -\frac{1}{2\pi} \varepsilon^{\alpha\rho} F_{\alpha\rho}}$$

- Anomaly is as much UV phenomenon (regulator) as it is IR (level crossing, non-decoupling) as it is IR (level crossing, poles in q^2). This ~~way~~ connection is particularly useful in confining theories (QCD) through t' Hooft matching condition.